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THE USE OF THE FINITE ENERGETIC SUM RULES FOR THE
CALCULATION OF THE LIGHT QUARK MASSES ^{*)}

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A B S T R A C T

The light quark masses are calculated on the basis of finite energy sum rules. The obtained result $((m_u + m_d)(1 \text{ GeV}) \sim 20 \text{ MeV})$ allows us to estimate the ratio $m_b/m_s \sim 20$ which should be compared with the prediction of the SU(5) model $m_b/m_s \approx 17$. We estimate a value of the π' meson residue and propose a model for the π meson radial excitations' mass spectrum. In particular, we obtain a mass of the second π meson excitation $m_\pi^2(2) = 2m_\pi^2$, in agreement with experimental data.

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1. - INTRODUCTION

The deep Euclidean character of perturbative QCD predictions necessitates the use of the analytical continuation to the physical region. The finite energy sum rules (FESR)^{1),2)}, analogous to the strong interaction dual sum rules³⁾, allow us to solve the above problem. From a phenomenological point of view, they have already given some interesting information about hadron properties. Thus, in Ref. 6), FESR and $1/N$ expansion⁵⁾ have been exploited to study the excited states of the meson resonances within QCD. A satisfactory agreement of the obtained results with the existing experimental data stimulated an application of this approach to the perturbative estimate of the η' meson residue⁶⁾. This paper, being a continuation of the previous investigations^{4),6)}, is devoted to the calculation of the invariant light quark masses from FESR.

The problem of the precise determination of the light quark masses does not lose its topicality. In spite of the fact that during recent years a number of different estimates has been obtained [for a review of the current status of the problem and further references, see Ref. 7)], there is no final answer to the question of what the values of these fundamental parameters are. To see this, it is sufficient to compare the contradictory results of the recent papers^{7),8)} on this subject.

In this paper, to determine the light quark masses, we shall use, on the one hand, the QCD information about the two-point function of the π meson current

$$F(-q^2) \equiv i \int e^{iqx} \langle 0 | T(\partial_\mu A^\mu(x), (\partial_\mu A^\mu(0))^+) | 0 \rangle dx, \quad (1.1)$$

$$A_\mu = \bar{u} \gamma_\mu \gamma_5 d, \quad \partial_\mu A^\mu = i(m_u + m_d) \bar{u} \gamma_5 d,$$

and, on the other hand, the experimental value of the first π meson radial excitation mass $m_{\pi^*} = (1.205 \pm 0.007) \text{ GeV}$ ^{9),10)}. Within the renormalization group approach [see, e.g., Ref. 11)] the concept of quark masses has several meanings. To be precise, there exist different types of masses. Thus, in the 't Hooft-Weinberg approach, the renormalized current quark mass m is related to the bare mass m_B as

$$m_B = Z_m(\alpha) m \quad (1.2)$$

while the running mass $m(\mu)$ is determined by the equation

$$f_m(\alpha) \equiv -\mu^2 \frac{d}{d\mu^2} \ln Z_m = f_0 \alpha + f_1 \alpha^2 + \dots, \quad (1.3)$$

$$\mu^2 \frac{dm}{d\mu^2} = f_m(\alpha) m \quad (1.3) \text{ cont.}$$

Since we prefer to perform all calculations in the $\overline{\text{MS}}$ scheme¹²⁾, the renormalization constant $Z_m(\alpha)$ depends only on the QCD coupling constant $\alpha = g^2/4\pi$ ¹³⁾. It is convenient to introduce the renormalization group invariant mass $\hat{m}$, defined as follows¹⁴⁾

$$\hat{m} = m \exp \left[\frac{d_0}{\beta_0} \ln(2\beta_0 \alpha) + \left(\frac{d_1}{\beta_0} - \frac{\beta_1}{\beta_0} \right) \alpha + \dots \right], \quad (1.4)$$

$$\mu^2 \frac{d\alpha}{d\mu^2} \equiv \beta(\alpha) = -\beta_0 \alpha^2 - \beta_1 \alpha^3 + \dots$$

It is the determination of the above quantity (1.4) that will be the subject of this paper. Using the obtained estimate of the invariant light quark masses, we shall extract the value of the ratio m_b/m_s and compare it with the prediction of the SU(5) grand unified model. We shall also find a value of the π' meson pole residue and propose a model for the π meson radial excitation spectrum

2. - FESR: ONE RESONANCE APPROXIMATION

Let us consider the two-point function (1.1). Having calculated its perturbative two-loop approximation, we re-obtained the result of Ref. 15). In the leading order on quark masses and within the $\overline{\text{MS}}$ scheme¹²⁾ it reads

$$F_{\text{p.t.}}(Q^2) = \frac{3}{8\pi^2} \hat{m}^2 \psi^{-2}(\alpha) Q^2 \ln \frac{Q^2}{\mu^2} \left[1 + \frac{\alpha}{\pi} \left(\frac{17}{3} - \ln \frac{Q^2}{\mu^2} \right) \right],$$

$$\hat{m} \equiv \hat{m}_u + \hat{m}_d = (m_u + m_d) \psi(\alpha),$$

$$\psi(\alpha) = \left(\frac{9}{2\pi} \alpha \right)^{-4/9} \left[1 + \frac{16}{9} \frac{\alpha}{\pi} \right]^{-\frac{145}{288}} \quad (2.1)$$

To estimate the numerical value of the invariant mass we shall use a one-resonance approximation for the spectral density and FESR

$$\int_0^{s_0} \rho \exp(s) ds = \int_0^{s_0} \rho^{\text{th}}(s) ds \quad (2.2)$$

where $\rho^{\text{th}}(s)$ is the spectral density defined by the perturbative result (2.1)

$$\rho^{\text{th}}(s) = \frac{1}{\pi} \text{Im} F_{\rho\pi}(-s - i\varepsilon) \quad (2.3)$$

while $\rho^{\text{exp}}(s)$ is the physical value of the same quantity defined by the PCAC relation

$$\rho^{\text{exp}}(s) = 2 f_{\pi}^2 m_{\pi}^4 \delta(s - m_{\pi}^2), \quad (2.4)$$

$$f_{\pi} = 93 \text{ MeV}$$

Note that the existence of resonances and threshold effects does not allow us to substantiate properly the procedure of the renormalization group improvement of the spectral density. Moreover, it was shown that the zero moment $\int_0^{S_0} \rho^{\text{th}}(s) ds$ in the considered region of energies behaves rather smoothly²⁾. Thus, we shall apply the renormalization group method¹¹⁾ not to the spectral density itself, but to its zero moment.

After the integration of (2.2) and summation of the log terms in accordance with the above remark, we obtain our final equation for the determination of the invariant mass $\hat{m} = \hat{m}_u + \hat{m}_d$

$$\frac{3}{8\pi^2} \psi^{-2}(\alpha(S_0)) \left[1 + \frac{20}{3} \frac{\alpha(S_0)}{\pi} \right] \frac{S_0^2}{2} = \frac{2 f_{\pi}^2 m_{\pi}^4}{\hat{m}^2},$$

$$\alpha(s) = \frac{1}{\beta_0 \ln \frac{s}{\Lambda_{\overline{\text{MS}}}^2}} - \frac{\beta_1 \ln \ln \frac{s}{\Lambda_{\overline{\text{MS}}}^2}}{\beta_0^3 \ln^2 \frac{s}{\Lambda_{\overline{\text{MS}}}^2}}, \quad (2.5)$$

where $\beta_0 = g/4\pi$ and $\beta_1 = 4/\pi^2$ [for a review see, e.g., Ref. 16)].

The solution of Eq. (2.5) depends on two parameters $\Lambda_{\overline{\text{MS}}}$ and S_0 . As for the first one, its value somewhere in the interval $100 \text{ MeV} \leq \Lambda_{\overline{\text{MS}}} \leq 200 \text{ MeV}$ appears to be consistent with experimental data [see, e.g., Ref. 17)], while the value of the duality interval is more ambiguous. The recent experimental information on the position of the first π meson radial excitation $m_{\pi^*} = (1.205 \pm 0.07) \text{ GeV}^{9),10)}$ indicates that the self-consistency of the one-resonance approximation enables us to choose a smaller value of S_0 than that of

Ref. 18). Thus, to define the duality interval, we shall use the prescription

$$S_0 = \frac{m_\pi^2 + m_{\pi'}^2}{2} \quad (2.6)$$

which by now is a standard procedure (and from our point of view a most natural one)^{17),4),20)}. One should remember, however, that from the point of view of local duality, also acceptable are the values of S_0 , rather remote from the physical poles π and π' , say $S_0 = 1 \text{ GeV}^2$. Substituting the numerical values of S_0 and $\Lambda_{\overline{\text{MS}}}$ into (2.5), we obtain the following estimates on $\hat{m}$

$$\hat{m} = \begin{cases} (33 \div 27) \text{ Mev}, & S_0 = 0.73 \text{ GeV}^2 \\ (27 \div 22) \text{ Mev}, & S_0 = 1 \text{ GeV}^2 \end{cases} \quad (2.7)$$

Taking into account the leading non-perturbative contributions²¹⁾ to the two-point function (1.1)

$$F_{N.P.}(Q^2) = \frac{\hat{m}^2}{Q^2} \psi^{-2}(\alpha) \left[\frac{1}{8} \langle \frac{\alpha}{\pi} G_{\mu\nu}^2 \rangle - \right. \\ \left. - \frac{88}{27} \frac{\pi}{Q^2} (\langle \sqrt{\alpha} \bar{q}q \rangle)^2 + O\left(\frac{1}{Q^2}\right) \right] \quad (2.8)$$

slightly decreases the invariant mass estimate (2.7). In fact, taking into account the appropriate correction to the FESR (2.2), and substituting the numerical values of the quark and gluon condensates parameters $\langle \alpha/\pi G_{\mu\nu}^2 \rangle = 0.012 \text{ GeV}^4$, $\langle \sqrt{\alpha} \bar{q}q \rangle = -(0.25 \text{ GeV})^3$ ²¹⁾, we obtain our final estimates of the invariant light quark mass

$$\hat{m} = \begin{cases} (30 \div 24) \text{ Mev}, & S_0 = 0.73 \text{ GeV}^2 \\ (25 \div 20) \text{ Mev}, & S_0 = 1 \text{ GeV}^2 \end{cases} \quad (2.9)$$

and the corresponding running mass values

$$m(1 \text{ GeV}^2) = \begin{cases} 20 \text{ Mev}, & S_0 = 0.73 \text{ GeV}^2 \\ 16 \text{ Mev}, & S_0 = 1 \text{ GeV}^2 \end{cases} \quad (2.10)$$

We would like to stress that our results (2.9) and (2.10) are consistent with the bounds obtained in Ref. 15) and slightly widen the interval of modern light quark mass values deduced in Ref. 7).

3. - FESR: TWO-RESONANCE APPROXIMATION

As has already been mentioned, the ambiguity of the method previously used leads to strong dependence of the invariant mass estimates (2.9) on the parameter S_0 , the basic criterion of its choice having been intuition rather than some first "principles". In order to check whether the ad hoc definition of the energy averaging interval (2.6) is reasonable enough, we shall compare the estimates (2.9) and (2.10) with the result of the independent approach based on the same starting point (FESR and local duality) as the method described above.

Let us consider the two-resonance approximation

$$\rho^{\text{exp}}(s) = 2 f_{\pi}^2 m_{\pi}^4 \delta(s - m_{\pi}^2) + 2 f_{\pi'}^2 \delta(s - m_{\pi'}^2) m_{\pi'}^4 \quad (3.1)$$

If we now recall that the theoretical quantity $\rho^{\text{th}}(s)$ is determined by means of Eqs. (2.1), (2.3) and (2.8) and use the FESR

$$\int_0^{S_0} \rho^{\text{th}}(s) s^K ds = \int_0^{S_0} \rho^{\text{exp}}(s) ds s^K, \quad (3.2)$$

$$K = 0, 1$$

we shall obtain the system of two equations

$$\begin{aligned} 2 f_{\pi}^2 m_{\pi}^4 + 2 m_{\pi'}^4 f_{\pi'}^2 &= \hat{m}^2 \psi^{-2}(\alpha(S_0)) \left\{ \frac{3}{8\pi^2} \right. \\ &\left. \frac{S_0^2}{2} \left[1 + \frac{20}{3} \frac{\alpha(S_0)}{\pi} \right] + \frac{1}{8} \left\langle \frac{\alpha}{\pi} G_{\mu\nu}^2 \right\rangle \right\}, \\ 2 f_{\pi}^2 m_{\pi}^6 + 2 f_{\pi'}^2 m_{\pi'}^6 &= \hat{m}^2 \psi^{-2}(\alpha(S_0)) \left\{ \frac{3}{8\pi^2} \right. \\ &\left. \frac{S_0^3}{3} \left[1 + \frac{19}{3} \frac{\alpha(S_0)}{\pi} \right] - \frac{88}{27} \pi \alpha(S_0) \left(\langle \bar{q}q \rangle \right)^2 \right\} \end{aligned} \quad (3.3)$$

which do not depend on the unknown unperturbative terms $o(1/Q^2)$ in expansion (2.8). The moments with $K \geq 2$ do depend on these unknown power corrections and therefore their application is not adequately motivated.

It is obvious that to find the values of the three independent parameters $(f_{\pi}, \hat{m}, S_0)$, one needs additional equation. To circumvent this difficulty we should estimate the dependence of the physical quantities $f_{\pi}, \hat{m}$ on the parameter S_0 , the latter being the vestige of the method used. Indeed, the Stevenson criterion²²⁾ determines its "optimum value" S_0^* by requiring that the mass of the π excitation be least sensitive to the change of the energy averaging interval S_0 , i.e., we choose an "optimum value" S_0^* by demanding

$$\left. \frac{d m_{\pi'}^2(S_0)}{d S_0} \right|_{S_0=S_0^*} = 0 \quad (3.4)$$

Numerically, the solution of Eqs. (3.3) and (3.4) is as follows

$$\begin{aligned} S_0^* &= 1.85 \text{ GeV}^2, \\ f_{\pi'} &\approx 3.7 \text{ Mev}, \\ \hat{m} &= (132 \div 27) \text{ Mev}, \\ 100 \text{ Mev} &\leq \Lambda_{\overline{m}s} \leq 200 \text{ Mev} \end{aligned} \quad (3.5)$$

The estimation (3.5) is consistent with the estimations (2.9) and (2.10) based on the duality interval (2.6).

The problem outlined in this section was also examined by the authors of Refs. 7) and 23). However, they have proceeded from other mass values of the π' meson state and have considered its residue as a parameter, allowing it to vary within a large interval of values. Their final estimates do not agree with Eqs. (3.5). While writing this work we became aware of a paper²⁴⁾ on the same subject in which, to determine the values of $f_{\pi}, \hat{m}, S_0$, the system of three equations (3.2) with $K = 0, 1, 2$ has been used within the assumption that all the power corrections $o(1/Q^2)$ are irrelevant. Taking into account the leading non-perturbative terms (2.8) one obtains the estimates which are consistent with (3.5). The agreement of our results with those recounted from the similar ones of Ref. 24) demonstrates that the contribution of the $O(1/Q^4)$ terms is a small one (the moment $K = 2$ a priori does depend on the unknown $1/Q^4$ power correction. We conclude this section with a remark concerning the use of the ITEP-like sum rules²¹⁾. They are known to be independent of the main ambiguity of FESR - the duality interval S_0 . However, they are sensitive to the choice of another unknown quantity - the continuum threshold. Moreover, as was shown in Ref. 24), allowing the continuum threshold to start from the point S_0 , one can obtain the results which agree with our estimates (3.5).

4. - COMPARISON WITH THE SU(5) PREDICTION

The quark mass ratios are well-defined physical quantities. Their values can be extracted not only from QCD sum rules or current algebra, but from the predictions of grand unified models as well, and specifically the SU(5) model²⁵⁾.

Let us first find the ratio m_b/m_s with the help of the estimates (2.9) and (3.5). As an additional input parameter we shall choose the running mass of the b quark $m_b(m_b) = 4.3 \text{ GeV}$ ⁷⁾ (accuracy 10%), and the value of the light quark mass ratio from Ref. 26)

$$\frac{m_d + m_u}{m_s} = \frac{2 m_{\pi^0}^2}{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2} \equiv a = 0.077 \quad (4.1)$$

Using the renormalization invariance of the mass ratios and expressing the results (2.9), (3.5) in terms of the running masses at the scale $m_b \approx 4.3 \text{ GeV}$, one can get that

$$\frac{m_b}{m_s} = \frac{m_b a}{m_u + m_d} = \begin{cases} 18 \div 25, S_0 = 0.73 \text{ GeV}^2 \\ 23 \div 30, S_0 = 1 \text{ GeV}^2 \\ 17 \div 23, \text{ eq. (3.5)} \end{cases} \quad (4.2)$$

$$100 \text{ MeV} \leq \Lambda_{\overline{MS}} \leq 200 \text{ MeV}$$

If we recall that the SU(5) model predicts the equality of quark and lepton masses ($m_d = m_e, m_s = m_\mu, m_b = m_\tau$) at the energy of grand unification $M \sim 10^{14} - 10^{15} \text{ GeV}$ ²⁵⁾, we shall obtain the result²⁵⁾

$$\frac{m_b}{m_s} = \frac{m_\tau}{m_\mu} = 17$$

which is in "quasiquantitative" agreement with our estimate (4.1).

5. - THE π MESON EXCITATION SPECTRUM

In this section we shall use the main ideas of Ref. 4) to propose a model for the π meson excitation mass spectrum $m_\pi(n)$ ($n = 1, 2, \dots$). Let us first consider the FESR

$$M_K^{\text{exp}} = M_K^{\text{th}}, \quad M_K = \int_{S_{n-1}}^{S_n} \rho(s) s^K ds, \quad (5.1)$$

where the physical spectral density is approximated by the spectrum of the infinitely narrow resonances, analogous to the QCD spectrum obtained in the leading order of the $1/N$ expansion⁵⁾

$$\rho^{\text{exp}}(s) = \sum_n 2 f_{\pi}^2(n) M_{\pi}^4(n) \delta(s - m_{\pi}^2(n)) \quad (5.2)$$

and the theoretical quantity is defined via Eq. (2.1). Neglecting its slight log-dependence, one can see that

$$\frac{M_0^{\text{th}}}{M_{-1}^{\text{th}}} = \frac{S_n + S_{n-1}}{2} = m_{\pi}^2(n) \quad (5.3)$$

If we now choose the bounds of integration in the form

$$S_n = \frac{m_{\pi}^2(n) + m_{\pi}^2(n+1)}{2} \quad (5.4)$$

with $S_0 = 0.5 m_{\pi}^2$, defined previously [see Eq. (2.6) and discussions after it]. We find a mass formula for the π meson excitations

$$m_{\pi}^2(n) = m_{\pi}^2 n, \quad n = 1, 2 \dots \quad (5.5)$$

Of course, our result (5.5) can be considered only as a model spectrum, but a rather encouraging one. In the framework of our approach we can also estimate the π excitations' residues

$$f_{\pi}(n) = 2\sqrt{2} \left(\frac{m_{\pi}}{m_{\pi}(n)} \right)^2 f_{\pi}, \quad (5.6)$$

$$f_{\pi}(1) = 3.3 \text{ MeV}$$

The reasonable agreement of its value with the independent estimate (3.5) can be considered as an argument in favour of this approach. Moreover, the obtained value of the second excitation $m_{\pi}(2) \approx 1.75 \text{ GeV}$ is in fair agreement with the observation of the O^- resonance with mass $m = 1.77 \pm 0.03 \text{ GeV}$ ²⁶⁾, which can correspond to a second radial π meson excitation²⁷⁾. Encouraged by the above fact we propose to search for the third radial π excitation candidate near $m_{\pi}(3) \approx 2.15 \text{ GeV}$.

6. - CONCLUSION

In this paper we have calculated the light quark masses with the help of two independent approaches based on FESR and one- and two-resonance approximations. They both gave similar results. Our calculation of the light quark mass value is in a "quasiquantitative" agreement with the SU(5) model prediction $m_b/m_s = m_t/m_\mu$. We also proposed a model for the π meson excitations' mass spectrum which is in agreement with existing experimental data.

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