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A NEW FORMULATION OF YANG-MILLS THEORY

IN $d = 3$ DIMENSIONS

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ABSTRACT

It is shown that the weighted random surfaces that determine the correlations of fermionic loops in $d=3$ Z_2 gauge theory may be generated by a Grassmann action that is locally $U(1)$ invariant. By a modification of the theory one obtains a new formulation of $U(N)$ or $SU(N)$ Yang-Mills theory in which confinement is essentially obvious.

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1. Recently, it has been shown [1,2] that mixed order disorder variables in $d = 3$ and $d = 4$ Z_2 gauge theory obey linear equations of motion^{*)} that appear to be those of non-interacting random surfaces. It has further been argued [4] that the existence of a continuum limit in $d = 4$ Z_2 gauge theory might imply the existence of a new formulation of $d = 4$ $U(N)$ Yang Mills theory. In this note I want to make the latter claim more explicit by constructing such a formulation of $U(N)$ Yang Mills theory in $d = 3$ dimensions.

The strategy is to find a quartic Grassmann action in terms of which the correlations of fermionic loops in $d = 3$ Z_2 gauge theory may be calculated. This Grassmann action will be seen to possess a local $U(1)$ invariance^{**)} which presumably reflects the orientability of the random surfaces that determine the loop correlations in $d = 3$ Z_2 gauge theory. A modification of the action leads to a new kind of $U(N)$ Yang Mills theory. It will be argued that the continuum limit of this theory should be similar to the conventional theory by appealing to the idea that Yang Mills gluons are the Goldstone bosons of spontaneously broken gauge invariance.

This paper is intended to clarify the relation between random surfaces and Yang Mills theory and to prepare the ground for effective numerical and analytical calculations. Such calculations must be preceded, however, by an extension of the present work from $d = 3$ to $d = 4$ dimensions.

2. We want to extract a $U(1)$ gauge theory from $d = 3$ Z_2 gauge theory which is dual to the $d = 3$ Ising model and is defined by

$$Z = \sum_{\{\sigma(\ell)\}} e^{\beta \sum_p \sigma(p)} \quad (1)$$

where the $\sigma(\ell)$ are located on links ℓ of a $d = 3$ cubic lattice and $\sigma(p) = \prod_{\ell \in \partial p} \sigma(\ell)$, $\sigma = \pm 1$. Essentially linear loop equations were found in this theory in Ref. [1] by following these steps:

*) Equations of this type had previously been mentioned, but not published, by A. Polyakov [3].

**) This local $U(1)$ invariance was absent in the Grassmann action constructed by E. Fradkin et al. [5].

i) select as fundamental dynamical variables Wilson loops with each segment of the loop multiplied by a disorder variable*) at the centre of one of the four adjoining cubes;

ii) insert the tautology

$$1 = \frac{1}{\cosh 2\beta} e^{-2\beta \sigma(p)} + \tanh 2\beta \sigma(p) \quad (2)$$

into the partition function where it states a relation between a loop on the plaquette p and a product of disorder variables at the two cubes touching p ;

iii) for each segment of the Wilson loop with associated disorder variable, one may multiply by 1, in the form above in two distinct ways, each of which (using the Z_2 property of order and disorder variables) gives an equation displacing or modifying the order-disorder loop (Fig. 1);

iv) to resolve the ambiguity of which equation to use, one gives an orientation to the Wilson loop and specifies a sense of orientation for the disorder variables to rotate around the loop.

For further details the reader is referred to Ref. [1]. It is introducing an orientation in step (iv) that gives rise to a local $U(1)$ invariance as we shall see below.

It is illuminating, before considering the $d = 3$ theory, to first consider the Grassmann action that corresponds to the $d = 2$ Ising model which is defined by

$$Z = \sum_{\{\sigma(r)\}} e^{\beta \sum_{r,\mu} \sigma(r) \sigma(r+\mu)} \quad (3)$$

with $\sigma = \pm 1$ on a square lattice in $d = 2$. The equations of motion are generated by the tautology

$$1 = \frac{1}{\cosh 2\beta} e^{-2\beta \sigma(r) \sigma(r+\mu)} + \tanh 2\beta \sigma(r) \sigma(r+\mu) \quad (4)$$

*) A disorder variable is realised by flipping the sign of the couplings across a line starting at the centre of a cube.

The dynamical variables are spins multiplied by nearby disorder variables^{*)} and the choice of equations (there are two of them for each pair) is done by giving ± 1 charges and specifying a correlated sense of orientation in the motion of order and disorder variables (see Fig. 2). With the notation $\Psi_i(x,y)$, $i = 1, 2, 3, 4$ for an order variable at (x,y) multiplied by, respectively, a disorder variable at $(x + \frac{1}{2}, y + \frac{1}{2})$, $(x - \frac{1}{2}, y - \frac{1}{2})$, $(x - \frac{1}{2}, y + \frac{1}{2})$, $(x + \frac{1}{2}, y - \frac{1}{2})$ one finds the following equation of motion:

$$\sum_{r_2'} (1 - \Gamma)_{r_1 r_2'} \langle \bar{\Psi}_{r_1} \Psi_{r_2'} \rangle = 0, \quad r_1 \neq r_2 \quad (5)$$

with $r = [(x,y), i]$ a compound index and

$$\Gamma = \begin{pmatrix} 0 & 0 & ST_x & -C \\ 0 & 0 & -C & -ST_x \\ C & -ST_y & 0 & 0 \\ -ST_y & -C & 0 & 0 \end{pmatrix} \quad (6)$$

where T_x, T_y stand for translation by one unit in the x , resp. y direction and with

$$C = \frac{1}{\cosh 2\beta}, \quad S = \tanh 2\beta, \quad C^2 + S^2 = 1 \quad (7)$$

$$C \leftrightarrow S \quad \text{under} \quad \beta \leftrightarrow \beta^*, \quad \sinh 2\beta \sinh 2\beta^* = 1$$

The equation above, Eq. (5), is actually only true modulo non-local minus signs due to semi-infinite lines of flipped bonds emanating from each disorder variable [in Eq. (5) these lines are taken to run in the $-\hat{y}$ direction]. These lines act like cuts in the complex plane in the sense that whenever a spin crosses such a line there is an extra minus sign in Eq. (5). One may solve Eq. (5) iteratively which amounts to summing over weighted paths from r_2 to r_1 . It turns out that all paths from r_2 to r_1 (with an exception to be discussed below) carry the same number of non-local minus signs due to a spin crossing a line of flipped bonds (the local minus signs due to a spin crossing its own line of flipped bonds must be kept). In other words, the non-local effects determine only the over-all sign of the correlation in Eq. (5) and may therefore be ignored. From Fig. 3 we see that we must exclude paths in our sum that touch r_1 more than once. Summing over this

*) A disorder variable in the $d = 2$ Ising model is realized by flipping the sign of the couplings across a line starting at the centre of a square.

restricted set of paths and using $\langle \bar{\psi}_r \psi_r \rangle = 1$ we may easily solve Eq. (5) and find*):

$$\langle \bar{\psi}_i(x, y) \psi_j(x', y') \rangle = \frac{2}{(2\pi)^2} \int d^2 p \left(\frac{1}{1-\Gamma(p)} \right)_{ij} e^{i[P_x(x-x') + P_y(y-y')]}$$

with

$$\frac{1}{1-\Gamma(p)} = \frac{1}{\det(p)} \begin{pmatrix} 1 - \cos(\tau_y + \tau_x) & c^2 - \tau_x \tau_y & c - S \tau_x & c - S \tau_y \\ S^2 \tau_x \tau_y - c^2 & 1 - \cos(\tau_x + \tau_y) & c - S \tau_y & -c + S \tau_x \\ -c + S \tau_x & -c + S \tau_y & 1 - \cos(\tau_y + \tau_x) & S^2 \tau_y \tau_x - c^2 \\ -c + S \tau_y & c - S \tau_x & c^2 - S^2 \tau_x \tau_y & 1 - \cos(\tau_x + \tau_y) \end{pmatrix}$$

$$\tau_{\pm x} = e^{\pm i p_x}, \quad \tau_{\pm y} = e^{\pm i p_y} \quad (8)$$

$$\det(p) = 2 (1 - \cos(\cos p_x + \cos p_y))$$

So it turned out that

$$\langle \bar{\psi}_{r_1} \psi_{r_2} \rangle = 2 \left(\frac{1}{1-\Gamma} \right)_{r_1 r_2} \quad (9)$$

which is, apart from a factor 2, what one would get from an action

$$S = \bar{\psi}(1-\Gamma)\psi \quad (10)$$

This action would have resulted in the equation of motion

$$(1-\Gamma)_{r_2 r_2'} \langle \bar{\psi}_{r_1} \psi_{r_2'} \rangle = \delta_{r_1 r_2} \quad (5')$$

where the δ function in Eq. (5') may be considered to be due to the zero length path's contribution to an *unrestricted* sum over paths.

*) By specializing this result one finds the internal energy of the Ising model and, after an integration, Onsager's partition function.

This suggests that we should be able to remove the restriction in the sum over paths that went into the solution of Eq. (5). This is easily done by exploiting the condition $\langle \bar{\psi}_r \psi_r \rangle = 1$ and by flipping the weight of a path just before the collision of two order-disorder pairs, see Fig. 4. In Fig. 4 one replaces

$$\langle \bar{\Psi}_3(x,y) \Psi_2(x,y) \rangle = C \langle \bar{\Psi}_3(x,y) \Psi_3(x,y) \rangle + \dots \quad (11)$$

by

$$\langle \bar{\Psi}_3(x,y) \Psi_2(x,y) \rangle = -C \langle \bar{\Psi}_3(x,y) \Psi_3(x,y) \rangle + 2C + \dots \quad (11')$$

In other words, we have added, exploiting the condition $\langle \bar{\psi}_r \psi_r \rangle = 1$, an inhomogeneous term to the equation in a way that repairs the phase for the homogeneous part. We now have an amplitude 2 for annihilation and may ignore all non-local effects (they only influence the over-all sign of a correlation)*).

3. Having filled in this lacuna of Ref. [1] we may now return to $d = 3$ Z_2 gauge theory. There also we must introduce an inhomogeneous term +2 for annihilation of oppositely oriented bits of loop and the difficulties with the phases noticed in Ref. [1] disappear**).

In Ref. [1] the phase for a closed surface M was given as

$$\text{phase}(M) = (-1)^{\ell(M)} \quad (12)$$

where ℓ stands for the total length of self-intersections (lattice surfaces in $d = 3$ may have open lines of self-intersections). This is *formally* similar to the result for closed paths C in $d = 2$ which carry a phase

$$\text{phase}(C) = (-1)^{n(C)} \quad (12')$$

where $n(C)$ stands for the number of self-intersection points. The question is now how to render the apparently non-local information (12) local. This is certainly a precondition for generating the weight (12) from a local field theory as we

*) The main lesson to be learnt from the $d = 2$ Ising model is the global $U(1)$ invariance of its associated Grassmann Lagrangian.

**) In $d = 2$ one may also verify $\psi_r \psi_r = 0$ which trivially allows to flip the sign in a collision of two ψ 's. This holds also in $d = 3$.

intend to do. Now in Ref. [1] it was essentially described how to find such a weight, i.e. it was noticed that endpoints of open self-intersection lines can be determined locally by a rotation being a 2π or 4π rotation^{*}). For notation, give a topological charge $Q = 1 \pmod{2}$ to an endpoint of an open line of self-intersection and let Q_{even} be the total topological charge of all even points on the surface (on the type of lattice we are using, points may be classified as even or odd according to $x + y + z$). Then, clearly,

$$\ell(M) \equiv Q_{\text{even}} \pmod{2} \quad (13)$$

In this paper we will use a different construction due to Dotsenko and Polyakov and quoted in Ref. [6]. The observation is that one may intersect the surface with planes $x = \frac{1}{2} + \ell$, $y = \frac{1}{2} + m$, $z = \frac{1}{2} + n$ (ℓ, m, n integer) and watch the winding numbers of the intersection lines (Fig. 5). For every plane

$$\# (\text{Selfintersections}) \equiv \text{Windingnumber} + 1 \pmod{2} \quad (14)$$

and

$$\ell(M) \equiv \sum_{\text{planes}} \# (\text{Selfintersections in a plane}) \pmod{2}$$

A direct geometric relation between Eqs. (13) and (14) suggests itself when one observes that the even points on a surface all get encircled by the intersection lines. [To complete the argument, it remains to "abelianise" the integrals needed in Eq. (13) which involve integrating in $SU(2)$ by distorting the surface until one may replace Eq. (13) by Eq. (14) and then deform the surface back to its original shape. (See fig. 6.)]

To measure the winding number needed in Eq. (14) we just follow the orientation of order-disorder pairs on the surface and give phases $e^{\pm i\pi/4}$ per $\pm\pi/2$ rotation. The extra minus sign we used in Eq. (14) should result from Fermi statistics. The proposed action then reads

$$S = \sum_{\ell i} \bar{\Psi}_{\ell i} \Psi_{\ell i} + c e^{i\pi/4} \sum_{\ell i} \bar{\Psi}_{\ell i} \Psi_{R(\ell i)} + 2is \sum_{\text{oriented } p} \psi_{\ell_1 i_1} \dots \psi_{\ell_4 i_4} \quad (15)$$

^{*}) The rotation referred to here is that of a certain orthogonal frame of vectors (as constructed in Ref. [1]) upon encircling a vertex.

where we have associated one complex Grassmann variable with every oriented order-disorder pair, ' ℓ ' labelling the segment and ' i ' the associated disorder variable. $R(\ell i)$ means a $\pi/2$ rotation of ℓ, i in an anticlockwise (say) direction around the oriented segment ℓ . Finally, $\psi_{\ell_1 i_1} \dots \psi_{\ell_4 i_4}$ is a product of fermion variables along an oriented plaquette p with the disorder variable corresponding to $i_1 \dots i_4$ in a cube touched by p and pointed at by the normal of p (Fig. 7). The factor i in the quartic term is meant to reflect the $\pi/2$ rotation that the quartic term gives to $\{(\ell_1 i_1), (\ell_3 i_3)\}$ and $\{(\ell_2 i_2), (\ell_4 i_4)\}$ i.e. to both pairs of order-disorder variables attached to p^*).

The loop equations are now derived via**)

$$\langle e^{-S} \frac{d}{d\bar{\psi}_{\ell i}} \left[\left(\frac{d}{d\psi_{\ell i}} \psi(c) \right) e^S \right] \rangle = 0 \quad (16)$$

where S is the action and $\psi_{\ell i}$ the order-disorder pair considered in the loop operator $\Psi(C)$ [$\Psi(C)$ is simply a product of individual fermion operators along C].

It is easy to verify that the loop equations (16) agree with the old loop equations directly computed in $d = 3$ Z_2 gauge theory modulo phases.

To check whether the phases agree for closed surfaces (which is all we need) we shall consider directly all the interactions that "tile" a given surface. It is then possible to avoid extra unwanted phases by correlating the sign of oppositely oriented plaquette terms and by specifying an order of writing $(+\hat{x}, +\hat{y}, +\hat{z})$ precedes $(-\hat{x}, -\hat{y}, -\hat{z})$ for $\bar{\psi}_{\ell i} \psi_{\ell i}$ and $\bar{\psi}_{\ell i} \psi_{R(\ell i)}$. Out of all these quartic and quadratic interactions one then pulls out closed fermion loops in the xy, xz, yz planes and which contain the extra minus sign needed in Eq. (14) ***). This concludes our discussion of the local Grassmann action that generates the random surfaces of $d = 3$ Z_2 gauge theory.

*) The factor 2 in Eq. (15) is due to the amplitude 2 per annihilation.

**) Equation (16) is a Schwinger Dyson type equation derived from the translation invariance of the Grassmann measure. The fact that quartic lagrangians give rise to equations for loops that sweep out surfaces was first noticed by Weingarten [7].

***) The fermion loops in the planes $z(y, x) = \text{half integer}$, carry phases which measure their winding numbers due to Eq. (15).

4. Since the Grassmann theory (15) serves to calculate correlations in the $d = 3$ Z_2 gauge theory which are known to be long range at a critical point β_c (they are known to be long range in the $d = 3$ Ising model which is dual to the gauge theory) we are guaranteed to have long range correlations in the Grassmann theory also and some kind of continuum string theory should exist at $\beta = \beta_c$ (for a very interesting attempt to define continuum string models in $d=4$, see Ref. [8]). Now we want to replace the local $U(1)$ invariance of Eq. (15) by a local $U(N)$ invariance through a modification of the action. We will see below that this just suppresses the surfaces of higher topology in the usual way by $1/N^2$ per handle [9] and it is therefore plausible that the new theory, with the suppression of higher topologies, should also have a continuum limit, i.e. ∞ range correlations at a critical point β'_c (possibly: $\beta'_c \neq \beta_c$). To find the new action just replace the complex variables $\Psi, \bar{\Psi}$ by complex matrices ψ_{ij} and $\bar{\psi}_{ij}$:

$$S = N \left[\sum_{\ell i} \text{Tr} \bar{\psi}_{\ell i} \psi_{\ell i} + c e^{i\pi/4} \text{Tr} \sum_{\ell i} \bar{\psi}_{\ell i} \psi_{R(\ell i)} + 2is \text{Tr} \sum_{p \text{ oriented}} \psi_{\ell_1 i_1} \dots \psi_{\ell_4 i_4} \right] \quad (17)$$

With this action, the process of attaching a handle via $\mathcal{S} \rightarrow \mathcal{O} \rightarrow \mathcal{S}$ carries a suppression factor $1/N^2$ as desired. In the old $U(1)$ invariant Grassmann action one had the identity $\langle \bar{\psi}_{\ell i} \psi_{\ell i} \rangle = 1$ by the relation with the $d = 3$ Z_2 gauge theory. However, it is not clear whether $\langle \bar{\psi}_{\ell i_{ab}} \psi_{\ell i_{bc}} \rangle = \delta_{ac}$ in the $U(N)$ invariant theory*).

To compare the present theory with $d = 3$ QCD one may have to divide out the $U(1)$ factor by introducing a non-dynamical higher gauge field (antisymmetric tensor field) that would be defined on plaquettes with curvature on cubes. The question would be mainly whether such a field might become dynamical as it happens with an analogous vector field in the $d = 2$ CP^n models [10].

*) The coefficient in this "unitarity relation" might be β -dependent. The form of this relation is dictated by gauge invariance.

Another matter of interest is that the loops are already fermionic (it would seem) so that quarks seem to be automatically included in a closed loop in our new theory.

Finally, we need an argument that our theory is related to $d = 3$ $U(N)$ or $SU(N)$ Yang Mills theory in the usual language. It has been argued by R. Ferrari and L.E. Picasso [11] that the photon may be considered a Goldstone boson associated with the breakdown of the subgroup of linear gauge transformations $\delta A_\mu = \partial_\mu \phi$, $\phi = C_{\mu\mu} X_\mu$. Similar arguments have also been given by Brandt and Ng and by Ivanov and Ogievetsky [11]. One might hope that these arguments also apply to non-Abelian gauge theories since in such theories the coupling usually tends to zero at small distances and the theory becomes almost Abelian.

For the moment this is the only argument linking our $U(N)$ gauge theory with the usual one. (The reader might also consult ref. [12] on related questions.)

5. In summary, we have constructed a theory which might serve as a new definition of $d = 3$ Yang Mills theory and in which confinement is obvious. Recent analytical and numerical work [2,4] suggests that it might be possible to extract a corresponding version of $d = 4$ Yang Mills theory from the Grassmann transcription of $d = 4$ Z_2 gauge theory^{*)}. In the end this might give us a new version of $d = 4$ Yang Mills theory that might be more suited to numerical and analytical calculations than the old version of the theory that is used now.

Note added:

After this work was completed, V. Dotsenko kindly showed me his thesis on his work with A. Polyakov on the $d = 3$ Ising model (in Russian). Their approach has many interesting features but differs from that of ref. [1] by leading to sums over surfaces that also include non-orientable ones. Therefore there is no local $U(1)$ invariance in their approach.

*) A fermionic Hamiltonian for $d = 4$ Z_2 gauge theory without local $U(1)$ invariance has already been given [13]. A locally $U(1)$ invariant Grassmann transcriptions of the $d = 4$ theory and a corresponding reformulation of $d = 4$ QCD will be reported elsewhere.

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Figure captions

- Fig. 1 : A loop equation in $d = 3$ Z_2 gauge theory.
- Fig. 2 : Two point equations of opposite handedness in the $d = 2$ Ising model.
- Fig. 3 : A path that must be excluded (at first) in the sum over paths $r_2 \rightarrow r_1$. After explicitly including the inhomogeneous term in the point equations this path will be permitted.
- Fig. 4 : Generating an extra inhomogeneous term for the collision between order-disorder variables of opposite handedness.
- Fig. 5 : Polyakov and Dotsenko's way of measuring the total length of self-intersection.
- Fig. 6 : An attempt at establishing a direct geometric relation between the two different ways of counting the total length of self-intersection. The even points on this surface (full dots) are encircled by the intersection lines with planes $x(y,z) = \text{half integer}$. The regions surrounding even points and bounded by intersection lines are shaded.
- Fig. 7 : The 4 fermion interaction in Eq. (15) associated with every oriented plaquette P .

Fig. 1

$$\begin{aligned}\Psi_1(x,y) &= [\Psi_4(x,y) - S\Psi_2(x+1,y)] \\ \Psi_1(x,y) &= -[\Psi_2(x,y) + S\Psi_4(x,y+1)]\end{aligned}$$

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7