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THE EGUCHI-KAWAI MODEL IN TWO DIMENSIONS:
AN ACCURATE ANALYSIS

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ABSTRACT

The strong coupling expansion of the free energy for the Eguchi-Kawai model in two dimensions is evaluated by the use of the character expansion for the two-matrix model. The analyticity domain of the formal series in the complex β plane is a circle of radius $\beta = 1$ showing quite a pathological behaviour. The analysis of the weak coupling regime by the saddlepoint method gives further support in favour of the possible "breakdown" of the reduced model, which is equivalent to the quenched version for $d = 2$.

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1. - INTRODUCTION

A recent development of the large N limit of lattice gauge theory has emerged in connection with the observation, due to Eguchi and Kawai¹⁾ (EK), that under some assumptions the space dependence of the link variables can be suppressed and the model reduced to a new one defined on a single space-time site. This reduced model should give predictions identical to the original one in the $N \rightarrow \infty$ limit.

The partition function for the EK model is

$$Z = \int \prod_u^d \mathcal{D}u_u \exp N\beta \sum_{u \neq v} \text{Tr} [u_u u_v u_u^\dagger u_v^\dagger] \quad (1)$$

and the Wilson loops are defined as products of U variables chosen in such a way that the orientation of the links is the same as for the corresponding links in the original spatial loop. The open Wilson loops present in the loop equation are taken to be zero due to the $U^d(1)$ symmetry of Eq. (1). However, this procedure gives reliable predictions only in the strong coupling (small β) region where $U^d(1)$ cannot be spontaneously broken²⁾. In order to avoid the (possible) breaking of $U^d(1)$ for $\beta \rightarrow \infty$, a quenched version of the model has been proposed which seems to reproduce (at least) the planar approximation for both the lattice and continuum version of the theory³⁾⁻⁵⁾. It has been noticed that the instability of the EK model is active for $d > 2$ ²⁾, and the quenched version³⁾ eliminates the attractive factor for $d \geq 2$, being identical to Z_{EK} for $d = 2$. Therefore, the subsequent analysis is also relevant for the quenched model.

Since pathologies of a lattice gauge theory [like phase transition⁶⁾ or non-convergence of the expansions] often have a purely kinematical origin due to the properties of the large N limit of group integration, we decided to investigate the simplest of the group integrals involved in the EK problem in order to identify carefully the origin of the eventual differences between the two models and/or the limit of validity of the EK reduced model.

We shall study the large N limit of the group integral

$$Z_2(\beta) = \int \mathcal{D}u_1 \mathcal{D}u_2 \exp N\beta [\text{Tr}(u_1 u_2 u_1^\dagger u_2^\dagger + u_2 u_1 u_2^\dagger u_1^\dagger)] \quad (2)$$

i.e., the two-dimensional version of the EK model whose large N limit is well known.

We shall show that a relevant phenomenon occurs which is a dramatic restriction of the analyticity domain for the strong coupling results, indicating a much more violent transition from the strong to the weak coupling regime than in the standard model. The series expansion for $\beta \rightarrow \infty$ of Z_{EK} is obtained and reproduces exactly the large N limit of the exact expansion [see Eq. (17)].

The free energy for $\beta \rightarrow \infty$ has been evaluated using the straightforward saddle point method. The first term is reproduced, but a correction term proportional to $1/N$ is present which indicates that the model in the weak coupling region has unusual properties.

We have also found it quite hard to envisage a systematic way of evaluating $1/N$ correction terms corroborating the considerations of Ref. 4).

2. - STRONG COUPLING

We found it convenient to solve first the simpler problem of evaluating

$$W(\beta) = \int \mathcal{D}u_1 \mathcal{D}u_2 \exp N\beta T_2(u_1 u_2 u_1^\dagger u_2^\dagger) \quad (3)$$

According to Itzykson and Zuber⁷⁾

$$\begin{aligned} \int \mathcal{D}u_2 \exp N\beta T_2(u_1 u_2 u_1^\dagger u_2^\dagger) = \\ = \sum_{\{r\}} \left\{ \frac{(N\beta)^{|\mu|}}{m!} \frac{\tilde{\sigma}_{\{r\}}}{d_{\{r\}}} \chi_{\{r\}}(u_1) \chi_{\{r\}}(u_1^\dagger) \right\} \end{aligned} \quad (4)$$

where the sum is over the irreducible representation of $U(N)$, characterized by the sequence of non-decreasing numbers $n_0 \leq n_1, \dots, \leq n_{N-1}$ and we need only the polynomial representations. $n_0 \geq 0$ and $\chi_{\{r\}}$ are the corresponding characters

$$\chi_{\{r\}}(V) = \frac{\det \{ e^{i \Phi_i \cdot (n_j + j)} \}}{\det \{ e^{i \Phi_i \cdot j} \}} \quad (5)$$

where $\exp i \Phi_i$ are the eigenvalues of V , $d_{\{r\}}$ is the dimension of the representation expressible through Weyl's formula

$$d_{\{2\}} \equiv \chi_{\{2\}}(E) = \frac{\det[(n+j)^i]}{\prod_{p=0}^{N-1} p!} \quad (6)$$

and $\sigma_{\{r\}n}$ is the number of times the representation $\{r\}$ occurs in the tensor product V , that is the number of distinct ways of constructing piece by piece the Young tableaux for $\{r\}$ while respecting the rules for such tableaux. Therefore,

$$\sigma_{\{2\}} = |n|! d_{\{2\}} \frac{\prod_{p=0}^{N-1} p!}{\prod_{p=0}^{N-1} (n_p + p)!} = \frac{|n|!}{(N\beta)^{|n|}} \int dV \chi_{\{2\}}^*(V) e^{N\beta T_2 V} \quad (7)$$

and since

$$\int dV \chi_{\{2\}}(V) \chi_{\{2\}}^*(V) = 1$$

$$W(\beta) = \sum_{\{2\}} (N\beta)^{|n|} \frac{\prod_{p=0}^{N-1} p!}{\prod_{p=0}^{N-1} (p + n_p)!} \quad (8)$$

For the $N \rightarrow \infty$ limit, and in the strong coupling limit, the only non-zero terms correspond to $p \approx N$, and therefore the $1/N$ expansion amounts to

$$\begin{aligned} \sum_{\{2\}} (N\beta)^{|n|} \prod_{\substack{n_{N-q} \neq 0 \\ N \gg q}} \frac{1}{(N-q+n_{N-q}) \dots (N-q+1)} &= \\ &= \sum_{\{2\}} \frac{(N\beta)^{|n|}}{N^{|n|}} \prod_{\substack{n_{N-q} \neq 0 \\ N \gg q}} \frac{1}{(1 + \frac{n_{N-q}-q}{N}) \dots (1 + \frac{1-q}{N})} = \\ &= \sum_{\{2\}} (\beta)^{|n|} (1 + \mathcal{O}(1/N)) \end{aligned} \quad (9)$$

Our problem has been reduced in the large N limit to that of counting the representations with fixed $|n|$ (fixed number of boxes in the Young tableaux).

We have found that the generating function for this number can be expressed in closed form in the following way:

$$\begin{aligned}
 \sum_{\{2\}} \beta^{|\mu|} &= \sum_{n_0 \leq \dots \leq n_{N-1}} \beta^{n_0 + n_1 + \dots + n_{N-1}} = \\
 &= \sum_{n_0} \sum_{n_1 \leq \dots \leq n_{N-1}} \beta^{n_0 + (n_0 + n_1) + \dots + (n_0 + n_{N-1})} = \\
 &= \frac{1}{1-\beta^N} \sum_{n_1 \leq \dots \leq n_{N-1}} \beta^{n_1 + \dots + n_{N-1}}
 \end{aligned} \tag{10}$$

and by recursion

$$\sum_{\{2\}} \beta^{|\mu|} = \prod_{p=1}^N \frac{1}{1-\beta^p}$$

It is easy to check that the series expansion

$$W(\beta) = 1 + \beta + 2\beta^2 + 3\beta^3 + 5\beta^4 + 7\beta^5 \tag{11}$$

coincides with the first few explicit terms in the summation.

For the sake of comparison, let us consider the corresponding integral in the standard theory

$$\int \mathcal{D}V \exp N\beta T_2 V = 1 \tag{12}$$

Let us now turn to the full integral $Z_2\{\beta\}$; according to Eguchi and Kawai we can use the properties of the δ function over the group to turn our integral into

$$Z_2(\beta) = \int \mathcal{D}V \sum_{\{2\}} \frac{\chi_{\{2\}}^*}{d_{\{2\}}} \exp N\beta T_2 (V + V^+) \tag{13}$$

We must now pay attention to the fact that all irreducible representations must be counted, that is also representations with $n_i < 0$ do in fact contribute.

It is easy to show that the sum over characters with $n_i < 0$ can be replaced by the sum over characters of $SU(N)$, that is $(n_0 = 0, n_i \geq 0)$. However, this observation, while useful in computing explicitly the integral for small N , is not helpful in finding the large N limit (we report in the Appendix on results for $N = 1, 2$).

When $N \rightarrow \infty$ anyway we can use the following properties of the characters of $U(N)$:

$$\frac{\chi_{\{2\}}^*(V)}{d_{\{2\}}} \xrightarrow{N \rightarrow \infty} \left(\frac{T_2 \bar{V}}{N} \right)^{-\sum_{n_i < 0} n_i} \left(\frac{T_2 V^+}{N} \right)^{\sum_{n_i > 0} n_i} + O(1/N^2) \quad (14)$$

Therefore, each term is $O(1)$ and can be computed by a saddle point approximation and by using the factorization property

$$Z_2(\beta) = \sum_{\{2\}} \left\langle \frac{T_2 V}{N} \right\rangle^{-\sum_{n_i < 0} n_i} \left\langle \frac{T_2 V^+}{N} \right\rangle^{\sum_{n_i > 0} n_i} \int dV e^{N\beta T_2(V+V^*)} = \quad (15)$$

$$\xrightarrow{\beta \rightarrow 0} \sum_{\{2\}} \beta^{-\sum_{n_i < 0} n_i + \sum_{n_i > 0} n_i} \exp N^2 \beta^2$$

Since we are in the strong coupling regime, the relevant representations ($|n_i| \ll N$) are those such that $n_i < 0$ when $i \ll N$, $n_i > 0$ when $i \approx N$ and $n_i = 0$ otherwise. (N is assumed to be larger than any other number in the problem.) Therefore

$$\begin{aligned} \sum_{\{2\}} \beta^{-\sum_{n_i < 0} n_i + \sum_{n_i > 0} n_i} &= \\ &= \left(\sum_{\{2\}_-} \beta^{-\sum n_i} \right) \left(\sum_{\{2\}_+} \beta^{\sum n_i} \right) = \left(\sum_{\{2\}_+} \beta^{\sum n_i} \right)^2 \end{aligned} \quad (16)$$

where $\{r\}_-$ are the representations with all $n_i \leq 0$ and $\{r\}_+$ are the representations with all $n_i \geq 0$. This is equivalent to stating that each representation with $|n_i| \ll N$ can be generated by considering all combinations of purely polynomial representations with their conjugates. The sum of purely polynomial representations is exactly the problem we already faced in discussing the previous integral. Therefore we finally obtain

$$Z_2(\beta) \xrightarrow[N \rightarrow \infty]{\beta \rightarrow 0} \left(\prod_{p=1}^N \frac{1}{(1-\beta^p)} \right)^2 \exp N^2 \beta^2 \quad (17)$$

The series expansion of $Z_2(\beta)$ in powers of β is

$$Z_2(\beta) = 1 + 2\beta + 5\beta^2 + N^2\beta^2 + 10\beta^3 + 2N^2\beta^3 + \dots \quad (18)$$

and matches with the large N limit of the exact strong coupling expansion^{6),8)}

$$Z(\beta) = 1 + 2\beta + 4 \frac{N^2}{N^2-1} \beta^2 + \frac{N^2}{N^2-1} (N^2 \beta^2) + \frac{N^4}{(N^2-1)(N^2-4)} 2N^2 \beta^3 \quad (19)$$

Therefore, the EK model reproduces in an extremely accurate way the strong coupling expansion of the Wilson model. This is an interesting result per se in view of the fact that the limits $N \rightarrow \infty$ and $\beta \rightarrow 0$ cannot be exchanged for smoother lattice actions [see Ref. 9) for details]. However, we are not able to determine the convergence radius of the strong coupling series for $Z_2(\beta)$; we find only that it is certainly smaller or equal to $\beta_0 = 1/2$, where we know that the strong coupling expansion for

$$Z(\beta) = \int \mathcal{D}\vec{V} e^{+p N \beta [T_2(\vec{V} + \vec{V}^*)]} \quad (20)$$

breaks down⁶⁾.

However, while the series breaks down, the analyticity domain of its formal summation is the whole complex plane. In turn, the formal summation of the strong coupling series for $Z_2(\beta)$ has a much smaller analyticity domain since it shows a ring of essential singularities on the unit circle, corresponding to the n^{th} roots of the unity $\beta^n = 1$.

3. - WEAK COUPLING

We apply a straightforward saddle point method which generally gives reliable results near $\beta = \infty$ for the large N limits. The hope is that it would give information about the eventual discrepancies with the well-known results for the standard model⁶⁾. To simplify matters, we evaluate first the integral of Eq. (3) using the result

$$\mathcal{W}(\beta) = \frac{1}{N!} \frac{1}{(N\beta)^{N(N-1)/2}} \prod_{p=1}^{N-1} p! \int \prod_i \frac{d\Phi_i}{2\pi} \det \left[\exp N\beta (e^{i\frac{\Phi_k}{N}} - e^{i\frac{\Phi_l}{N}}) \right] \quad (21)$$

It is not so difficult to see that the saddle corresponds to $\Phi_k = \Phi_l$ in the determinant. Therefore,

$$\int \prod_i \frac{d\Phi_i}{2\pi} \det \left[\exp N\beta (e^{i\frac{\Phi_k}{N}} - e^{i\frac{\Phi_l}{N}}) \right] \xrightarrow{\beta \rightarrow \infty} e^{N^2 \beta} \quad (22)$$

giving the leading contribution,

$$\prod_{p=1}^{N-1} \frac{p!}{N!} \frac{\exp(N^2 \beta)}{(N\beta)^{N(N-1)/2}} \quad (23)$$

By the use of the Stirling approximation for the product of factorials, one arrives at the asymptotic limit ($N \rightarrow \infty$) of $W(\beta)$.

For evaluating the leading term of $Z_2(\beta)$ one proceeds as above, finding only a factor of 2 in the exponent of Eq. (22). The final result for the free energy is

$$F_2(\beta) \equiv \frac{1}{N^2} \ln Z_2(\beta) = 2\beta - \frac{1-1/N}{2} \ln 2\beta - \frac{1}{2} \ln N - \frac{3}{4} = F(\beta) - \frac{1}{2N} \ln 2\beta - \frac{1}{2N} \ln N \quad (24)$$

i.e., the correction terms to the Gross-Witten result $(\beta)^{6)}$ are down by a factor $1/N$, as already noted in Ref. 2) for $d > 2$, using a different method. It is interesting to trace back the origin of such a term. Looking at Eq. (23), one realizes that it comes from the factor $(N\beta)^{N(N-1)/2}$ in the denominator; in turn, such a factor is the product of the integration over the off-diagonal elements of the unitary matrix U_i , $i = 1, 2$ (i.e., the degrees of freedom not "quenched"). Moreover, the explicit evaluation shows that such a correction term originates from the subleading term in the usual large N expansion, i.e., $O(1/N^2)$ which gains a weight N in front. We shall not dwell here on the unfruitful attempts to find a systematic $1/N$ expansion by the use of the technique of orthogonal polynomials⁸⁾.

4. - CONCLUSIONS

We have shown that the partition function for the EK model at $d = 2$, reproduces exactly the strong coupling expansion of the usual model, but its formal series show a ring of essential singularities on the unit circle of the complex β plane. Therefore, it is very likely that the quenched reduced model enjoys a less smooth crossover from strong to weak couplings for any $d \geq 2$. In other words, one would expect singularities other than the usual ones found for $U(N)$ gauge models for large N ^{6), 11)}, more precisely for bigger values of β . This could be a motivation for analyzing in more detail the intermediate coupling region with Monte Carlo methods so as to make sure that no pathology is present.

On the other hand, the result of Eq. (24) for the weak coupling limit of the free energy, together with the failure in applying standard methods for the evaluation of the next-to-leading terms in the large N expansion, could mean that

the model has an intrinsic weakness which has to be solved in some way. Of course, the restriction of the analyticity domain in the β plane could also be a peculiarity of $d = 2$.

We hope to be able to report on eventual progress in this direction.

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NOTE ADDED

While this paper was being completed, we became aware of some recent work by Bhanot et al., Phys. Lett. 115B (1982) 237 and M. Okawa, BNL preprint (1982). They report on Monte Carlo simulations showing a first order phase transition at $\beta_c = 0.32$ for the quenched model at $d = 4$. Since they do not analyze values of β in the intermediate region between weak and strong couplings, no conclusions can be drawn concerning the arguments given here.

APPENDIX

Some result of explicit evaluations of the integrals $W_2(\beta)$, $Z_2(\beta)$ for low N are reported here. We use the formula

$$\int \mathcal{D}u_2 \exp[N\beta \text{Tr}(u_1 u_2 u_1^+ u_2^+)] = (N\beta)^{-\frac{N(N-1)}{2}} \prod_{p=1}^{N-1} p! \frac{\det[\exp N\beta e^{i(\Phi_i - \Phi_j)}]}{\Delta(e^{i\Phi_i}) \Delta(e^{i\Phi_j})} \quad (\text{A.1})$$

and reduce the integration to an integral over the eigenvalues

$$\begin{aligned} W(\beta) &= \int \prod_i \frac{d\Phi_i}{2\pi} |\Delta(e^{i\Phi_i})|^2 (N\beta)^{\frac{N(N-1)}{2}} \prod_{p=1}^{N-1} p! \frac{\det[\exp N\beta e^{i(\Phi_i - \Phi_j)}]}{|\Delta(e^{i\Phi_i})|^2} \\ &= \frac{1}{(N\beta)^{\frac{N(N-1)}{2}}} \prod_{p=1}^{N-1} \frac{p!}{N!} \int \prod_i \frac{d\Phi_i}{2\pi} \det[\exp N\beta e^{i(\Phi_i - \Phi_j)}] \end{aligned} \quad (\text{A.2})$$

One can use any of the two methods (or even direct integration) to check that

$$\begin{aligned} W_{N=1}(\beta) &= e^\beta \\ W_{N=2}(\beta) &= \frac{1}{4\beta} [e^{4\beta} - I_0(4\beta)] \end{aligned} \quad (\text{A.3})$$

and the result for finite N , $N \geq 3$ cannot be expressed in terms of elementary functions. Again by explicit computation we can evaluate exactly the $U(1)$ and $U(2)$ integrals:

$$\begin{aligned} Z_2(\beta)_{N=1} &= e^{2\beta} \\ Z_2(\beta)_{N=2} &= \frac{1}{8\beta} [e^{8\beta} - I_0(8\beta)] \end{aligned} \quad (\text{A.4})$$

In both cases $Z_2(\beta) = W(2\beta)$ but it is immediate to check that no such relation exists for $N \geq 3$. For the sake of comparison, let us recall that the $Y.M._2$ partition functions are

$$\begin{aligned} Z_{YM}(\beta)_{N=1} &= I_0(2\beta) \\ Z_{YM}(\beta)_{N=2} &= I_0^2(4\beta) - I_1^2(4\beta) \end{aligned} \quad (\text{A.5})$$

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