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THERMODYNAMICS OF HADRONS: DELIMITING THE TEMPERATURE

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ABSTRACT

We assume an exponentially rising hadronic mass spectrum $\sim \exp(m/T_0)$ and consider means for avoiding a limiting temperature at $T = T_0$. It is possible to do so if the single-particle potential rises at least as fast as m , or if the finite size of hadrons is taken into account, or both. Applications to heavy ion collisions, the early Universe and Monte Carlo simulations of the thermodynamics of lattice QCD are mentioned.

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1. INTRODUCTION

Certainly the concepts of negative temperature and maximum temperature are among the most bizarre and counter-intuitive in statistical physics. The former arises when a system has only a finite number of energy levels, whereas the latter arises when a system has an exponentially rising level density. Experimentally negative temperatures have been achieved, for example in nuclear spin systems¹⁾. So far, the maximum temperature concept seems to have relevance only in hadronic physics where a number of models such as the bootstrap²⁻⁵⁾, dual⁶⁾, bag^{7,8)}, etc., all predict an exponentially rising hadronic level density $\sim \exp(m/T_0)$ with $T_0 = 160 \pm 20$ MeV. There is some experimental support coming from the Particle Data Handbook, and from high energy hadron-hadron collisions which exhibit a transverse momentum cut-off $\sim \exp(-p_\perp/T_0)$ and a fragment mass cut-off $\sim \exp(-m_f/T_0)^2$.

It appears that QCD is the fundamental theory of the strong interactions. Various analytic approximations suggest that QCD undergoes a deconfining phase transition, from a gas of hadrons to a gas of quarks and gluons, at a temperature $T_c = 190 \pm 30$ MeV⁹⁾. Indeed Monte Carlo simulations¹⁰⁾ of the thermodynamics of lattice QCD explicitly show such a phase transition. Therefore one might conclude that an exponentially increasing mass spectrum does not exist in nature, or if it does then $T_0 = T_c$ ¹¹⁾.

The phenomenological successes of the above mentioned models compel us, in this paper, to assume an exponential mass spectrum. We envisage a phase diagram as shown in Fig. 1. At temperature T_c there is a phase transition separating the high temperature quark-gluon phase from the low temperature hadron phase. If the phase transition is of first order there is a possibility of superheating. For example, suppose we collide two very large nuclei at high energy. After first contact they will begin to interpenetrate and to be compressed and subsequently heated. If the nuclei were to be heated quickly enough they would pass right through T_c and remain in the metastable, superheated hadron phase. Eventually they would jump over to the stable quark-gluon phase.

The reverse situation might happen if we start at high temperature in the quark-gluon phase. The system might be an expanding nuclear fireball or the expanding early universe. The system could remain in the supercooled quark-gluon phase below T_c before making the transition to the stable hadron phase, depending on the dynamics.

According to the bag model, and according to current theoretical prejudice concerning QCD, the true ground state lies lower in energy than the perturbative ground state. This means that if we "analytically continue" the quark-gluon pressure below T_c it will pass through zero at some point T_0 and intersect the vertical axis at some negative pressure $-B$ at $T = 0$. Here B is the energy density difference between true and perturbative vacua.

Since the pressure of the supercooled phase passes through zero at T_0 we know that the asymptotic form of the hadron mass spectrum will be $\sim \exp(m/T_0)^{7,8}$. The simplest way to see this is to imagine that we have a single, very large hadron filled with quarks, antiquarks, and gluons. We calculate its properties by using the grand partition function. Since $P_{q-g}(T_0) = 0$ this hadronic bag or bubble is mechanically stable. Its total entropy S_{tot} is related to its total energy E_{tot} via $S_{tot} = E_{tot}/T_0$. Then the level density is $\sim \exp(S_{tot}) \sim \exp(E_{tot}/T_0)$. The energy of the bag or bubble increases by increasing its volume, not its internal temperature.

This then poses a problem. If we consider a gas of such hadrons and ignore any repulsive interactions between them, then T_0 will be an absolute limiting temperature for the hadronic phase. Furthermore $P_{had}(T_0) > 0$ whereas $P_{q-g}(T_0) = 0$. Since the pressure must be a monotonically increasing, continuous function of T , how is it possible to connect the two phases? We shall devote the remainder of this paper to a search for an answer to this question⁸⁾.

In Section 2 we will review the properties of a gas of hadrons with an exponential mass spectrum with particular emphasis on the structure of the singularity. Then we will consider various types of repulsive potentials between the hadrons, and treat the thermodynamics in the framework of an effective potential (mean field) model¹²⁾. If a common Yukawa-type potential is used between all pairs

of hadrons then, no matter how strong the repulsion, it is not possible to avoid the original singularity.

In Section 3 we consider single-particle potentials which can depend on the mass of the hadron. If the potential is proportional to the mass then the singularity may or may not be avoided depending on the strength of the coefficient. If the potential is proportional to the mass-squared then no singularity can occur and generally all high mass hadronic states are irrelevant. Such is the case in the bag model, for instance, where the volume of a bag is proportional to its mass.

In Section 4 we consider a technically different but philosophically related approach to overcome the singularity. This is the excluded volume approximation for deformable finite size hadrons¹³⁾. In general the singularity is shifted to higher temperature or is eliminated altogether. However a limiting energy density or entropy density is seen to occur even if there is no longer a limiting temperature.

In Section 5 we present our conclusions and suggest possibilities for further study.

Before proceeding it is worth while to note that all of our numerical results will be given for the entropy density as a function of temperature. One reason is that a closed form expression for any of the other thermodynamic quantities does not exist for the types of models studied in Section 3. Another reason is that the entropy tells us approximately how many degrees of freedom are being excited at a given temperature. Finally the entropy may be a convenient way of thinking about heavy ion collisions in order to obtain information on the hadronic matter equation of state¹⁴⁾.

2. EXPONENTIAL MASS SPECTRUM AND THE EFFECTIVE INTERACTION MODEL

The hadrons we will include in this paper may be delineated as follows. We include as discrete states the 0^{-+} mesons: $\pi(137)$, $K(495)$, $\eta(549)$, $\eta'(958)$; the 1^{--} mesons: $\rho(770)$, $\omega(783)$, $K^*(892)$, $\phi(1020)$; and the baryon octet: $N(939)$, $\Lambda(1115)$, $\Sigma(1192)$, $\Xi(1318)$. We include all other hadrons in the form of a continuous level density,

$$\rho(m) = \Theta(m-m_0) c T_0^2 m^{-3} \exp(m/T_0) \quad (2.1)$$

$$m_0 = 1200 \text{ MeV},$$

$$T_0 = 160 \text{ MeV}.$$

This functional form has been obtained in the statistical bootstrap model^{3,4)} with approximately $c = 4$, and in the bag model⁸⁾ with approximately $c = 40$. Both values of c will be used, along with $c = 0$, to test the sensitivity of the results. Other functional forms for the pre-exponential factor are also possible¹⁵⁾, and would not qualitatively affect the results of our study.

Repulsive interactions among the hadrons are treated in an effective interaction, or mean field, approach¹²⁾. The single particle energies are

$$\mathcal{E}(q, m, n) = (q^2 + m^2)^{1/2} + U(n) \quad (2.2)$$

where n is the total hadron number density. At this stage, we will assume that the potential energy U represents all repulsive interactions, taken to be the same for all particles. Furthermore four different interaction strengths will be tested. The first is

$$U_1(n) = 0 \quad (2.3)$$

which reproduces all of the old bootstrap results. The second is obtained by considering the soft core potential between nucleons in the bag model. When a pair of nucleons sit on top of each other there is a cost in energy of about 500 MeV¹⁶⁾. We then somewhat arbitrarily parametrize the inter-hadron potential as $V(r) = 500 \text{ MeV} \cdot \exp(-\mu r)$ with μ being the ω meson mass. Integrating over all space one obtains

$$U_2(n) = 200 n \quad (2.4)$$

with n in units of fm^{-3} and U in units of MeV. Similarly one may take Weinberg's effective Lagrangian for π - π interactions¹⁷⁾ to obtain a potential energy

$$U_3(n) = 663 n \quad (2.5)$$

which is thus generalized to encompass all hadron-hadron interactions. Finally an extreme case is obtained if we use a repulsive ω -exchange potential $V(r) = g^2 \exp(-\mu r)/r$ with $g^2/4\pi = 9$,

$$U_4(n) = 1.77 \times 10^4 n \quad (2.6)$$

[A much smaller coefficient of 680 would occur if the full Reid potential was used instead¹²⁾.] Notice that $U(n)$ always has the form Kn , and the four potentials differ only in that they have different values of K . We consider U_2 and U_3 to be the most physically reasonable choices.

The number density for a hadron of type i is

$$n_i = \frac{g_i}{2\pi^2} \int_0^\infty dq \, q^2 \exp[-\mathcal{E}_i(q)/T] \quad (2.7)$$

where g_i is the degeneracy. The total density is

$$n = \sum_i n_i \quad (2.8)$$

We will only consider temperatures up to 250 MeV, which ought to be sufficient to reach beyond the quark-gluon phase transition. It is then a fairly good approximation to employ Maxwell-Boltzmann statistics, which has been done in eq. (2.7). Furthermore we make a non-relativistic approximation in eq. (2.2) for all particles except the pion. The pion number density reduces to

$$n_\pi = \frac{3}{2\pi^2} m_\pi^2 T K_2(m_\pi/T) \exp(-U(n)/T) \quad (2.9)$$

the number density for all other discrete states is

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp(-m_i/T) \exp(-U(n)/T) \quad (2.10)$$

and the number density of the continuum is

$$n_c = \frac{c T_0^2 T}{(2\pi)^{3/2}} I(T) \exp(-U(n)/T)$$

$$\begin{aligned}
 I(T) &= \int_{m_0/T}^{\infty} dy \, y^{-3/2} \exp[(T/T_0 - 1)y] \\
 &= 2 \left(\frac{T}{m_0} \right)^{1/2} \left[\exp(-\frac{1}{2}x^2) - \left(\frac{\pi}{2} \right)^{1/2} x Q(x^2|1) \right] \\
 x^2 &= 2m_0(T_0 - T)/T_0 T
 \end{aligned} \tag{2.11}$$

Here $Q(x^2|1)$ is the probability function¹⁸⁾. A self-consistent equation must now be solved since the right hand side of

$$n = n_{\pi} + \sum_{i=k}^{\infty} n_i + n_{>} \tag{2.12}$$

depends on n through $U(n)$. In fact for fixed T the form of the self-consistency equation is

$$n = \text{constant} \cdot \exp(-Kn/T) \tag{2.13}$$

Once eq. (2.13) has been solved (numerically) one may proceed to calculate the entropy density. With Boltzmann statistics it becomes¹²⁾

$$S = \sum_i g_i \int \frac{d^3q}{(2\pi)^3} [\mathcal{E}_i(q)/T + 1] \exp[-\mathcal{E}_i(q)/T] \tag{2.14}$$

After some calculus this expression reduces to

$$S = S_{\pi} + \sum_{i=k}^{\infty} S_i + S_{>} \tag{2.15}$$

where

$$S_{\pi} = (4 + U(n)/T) n_{\pi} + \frac{3}{2\pi^2} m_{\pi}^3 K_1(m_{\pi}/T) \exp(-U(n)/T) \tag{2.16}$$

$$S_i = \left(\frac{5}{2} + \frac{m_i}{T} + \frac{U(n)}{T} \right) n_i \tag{2.17}$$

$$S_{>} = \left(\frac{5}{2} + U(n)/T \right) n_{>} + \frac{c}{2\pi} T_0^2 (m_0 T)^{1/2} \frac{Q(x^2|1)}{x} \exp(-U(n)/T) \tag{2.18}$$

Numerical results are presented in Figs. 2-4. In Fig. 2 we have taken $c = 0$ and test the sensitivity of the entropy density to the strength of the potential. At fixed temperature the entropy decreases as the strength of the potential is

increased. Physically the reason is that with a strong potential a smaller fraction of the total energy per particle is in the form of random thermal motion, hence the entropy is less. Of course with $c = 0$ there can be no limiting temperature.

In Fig. 3 we have taken $c = 40$ and test the sensitivity of the entropy density to the strength of the potential. The case of $U_1 = 0$ reproduces the old bootstrap results. The ordering of the curves remains the same as in Fig. 2 and for the same reason. The striking feature is that the limiting temperature of $T_0 = 160$ MeV remains no matter how strong the potential is! The reason is the following. The discrete number densities eqs. (2.9) and (2.10) are of course finite as $T \rightarrow T_0$, and so is the continuum number density eq. (2.11) since $Q(0|\nu) = 1$. Therefore $U(n) = Kn$ is finite at T_0 no matter how big K is. The continuum contribution to the entropy density eq. (2.18) is the piece which diverges, in fact as $\sim x^{-1} \sim (T_0 - T)^{-1/2}$. The only effect of the potential is to multiply this divergent piece by the finite number $\exp [-U(n)/T_0]$.

In Fig. 4 we fix the potential to be either U_2 or U_4 and test the sensitivity to the three values of c . The effect of $c \neq 0$, i.e. an exponential mass spectrum, only begins to manifest itself when $T \geq 130$ MeV. After that the results are dramatic: if $c \neq 0$ there is a limiting temperature, if $c = 0$ there is not.

At this point it may be worth while mentioning the connection between the effective interaction model¹²⁾ used here and the relativistic Lagrangian mean field models of Walecka¹⁹⁾. Although there are other differences, the essential difference is that in the latter approach a mean field arises only when a net baryon number is present, whereas in the former approach a mean field interaction is operating among all hadrons even at zero baryon density. Unless quantum corrections are included Walecka's approach would correspond to $U_1 = 0$ when the baryon density is zero²⁰⁾.

3. MASS DEPENDENT POTENTIALS

Let us recall the contribution to the entropy from the mass continuum. From eq. (2.14),

$$S_> = \frac{5}{2} n_> + \frac{c T_0^2 T^{1/2}}{(2\pi)^{3/2}} \int_{m_0}^{\infty} dm m^{-3/2} (m + U(n)) \times \exp\left[\frac{m(T-T_0)}{T T_0} - \frac{U(n)}{T}\right] \quad (3.1)$$

Since $U(n)$ is independent of m the exponential of $-U/T$ factors out of the integral and the same divergence appears as when $U = 0$, namely $\int_{m_0}^{\infty} dm m^{-1/2}$ at $T = T_0$.

Within the framework of the effective interaction model the only possibility for avoiding this divergence is to allow the single particle potential U to depend on the mass m . This would allow for a damping of the integrand with the possibility of going to temperatures greater than T_0 .

Are there any models which predict such an effect, and if so what is the mass dependence of U ? Consider the expression

$$U(n) = n \int d^3r V(r) = n \bar{V} v \quad (3.2)$$

where $\bar{V}$ is an average potential and v is the volume over which this potential extends. In the bag model the volume occupied by a hadron is proportional to its mass, volume = $m/4B$ where B is the bag constant. Therefore large hadrons will not have an interaction radius equal to the inverse pion mass or the inverse ω mass, rather their interaction volume will be proportional to their own mass, $v \propto m$. Furthermore the bag model predicts¹⁶⁾ that the average strength of the potential $\bar{V}$ is also proportional to mass, $\bar{V} \propto m$. For example, consider a pair of identical hadrons which are very massive in the sense that $m \gg 4B/T_0^3$. Then they may be treated approximately as a gas of quarks and gluons. Each hadron will have some total entropy S_0 . Now put one hadron on top of the other and ask for the total energy of the system. We want to do this without changing the original configuration of quarks and gluons in each hadron, i.e. we bring the hadrons together adiabatically (total entropy $2S_0$) so as to determine the static potential energy. The answer is $E_{\text{tot}} = 2m + 0.14m$.

This leads us to consider the following two variants of previously considered single particle potentials. Suppose that either the interaction volume v or the

strength of the average potential $\bar{V}$ increases with mass. Then we take

$$U'(n, m) = \begin{cases} Kn & \text{for the discrete states,} \\ Kn(m/m_N) & \text{for the continuum.} \end{cases} \quad (3.3)$$

Suppose that both the interaction volume and the strength of the average potential increase with mass. Then we take

$$U''(n, m) = \begin{cases} Kn & \text{for the discrete states,} \\ Kn(m/m_N)^2 & \text{for the continuum.} \end{cases} \quad (3.4)$$

Also U' and U'' are to be labelled with sub-indices according to the value of K chosen, see eqs. (2.3) to (2.6).

Of course we realize that a description in terms of mass dependent single particle potentials is only a rather crude approximation. Nevertheless we believe that at least qualitatively, and maybe even semi-quantitatively, the physical effects are correctly being taken into account.

Now the calculations performed in Section 2 may be repeated with these mass dependent potentials. Only the formulae for $n_>$ and $S_>$ will be different because of the forms of U' and U'' . For U' ,

$$\begin{aligned} n'_> &= \frac{c T_0^2 T}{(2\pi)^{3/2}} I'(T, n) \\ I'(T, n) &= \int_{m_0/T}^{\infty} dy y^{-3/2} \exp\left(-\frac{1}{2} \frac{T}{m_0} x^2 y\right) \\ &= 2 \left(\frac{T}{m_0}\right)^{1/2} \left[\exp\left(-\frac{1}{2} x^2\right) - \left(\frac{\pi}{2}\right)^{1/2} x Q(x^2|1) \right] \\ x^2 &= 2 m_0 \left(\frac{1}{T} - \frac{1}{T_0} + \frac{Kn}{m_N T} \right) \end{aligned} \quad (3.5)$$

and

$$S'_> = \frac{5}{2} n'_> + \frac{c}{2\pi} T_0^2 (m_0 T)^{1/2} \frac{Q(x^2|1)}{x} \left(1 + \frac{Kn}{m_N} \right) \quad (3.6)$$

For U'' we cannot write the results in terms of tabulated functions. They must be done numerically.

$$n''_> = c T_0^2 \left(\frac{T}{2\pi} \right)^{3/2} \int_{m_0}^{\infty} dm m^{-3/2} \exp \left[m \frac{(T-T_0)}{TT_0} - m^2 (Kn/m_N^2 T) \right] \quad (3.7)$$

$$S''_> = \frac{5}{2} n''_> + \frac{c T_0^2 T^{1/2}}{(2\pi)^{3/2}} \int_{m_0}^{\infty} dm m^{-1/2} \left(1 + \frac{mKn}{m_N^2} \right) \times \exp \left[m \frac{(T-T_0)}{TT_0} - m^2 (Kn/m_N^2 T) \right] \quad (3.8)$$

In Fig. 5 we plot the entropy density as a function of temperature for the potentials U''_2 and U''_4 . There is no limiting temperature since the Gaussian cut-off in eq. (3.8) always controls the exponential growth of the mass spectrum. The curves are drawn for the two values of $c = 4, 40$. The effect of finite size and increased potential strength is so great that the continuum part of the mass spectrum contributes a negligible amount. Therefore, if this was the state of affairs in the real world, we could forget about all high mass hadrons and concentrate on obtaining a more accurate description of the forces between the low mass hadrons.

Whether the exponential cut-off provided by U' is sufficient to damp the exponential growth of the continuum mass spectrum is not obvious. It depends on how fast the density n grows with T as can be seen from eqs. (3.5) and (3.6). If $x^2 \rightarrow 0$ as $T \rightarrow T'_0$ for some $T'_0 > T_0$, T'_0 becomes a new limiting temperature. Some numerical results for U'_2 and U'_4 are plotted in Fig. 6. For U'_4 , $K = 1.77 \times 10^4$ is so big that the product $Kn/m_N T$ is always sufficient to obtain convergence. As a function of T , x^2 begins at infinity at $T = 0$, decreases and reaches a minimum at about $T \approx 225$ MeV for $c = 4$ and at $T \approx 175$ MeV for $c = 40$, and then increases monotonically as $T \rightarrow \infty$. In contrast, for U'_2 , K is too small. When $c = 4$ a limiting temperature of 170 MeV follows, and when $c = 40$ a limiting value of 180 MeV follows.

So the results can be summarized like this. For a given value of c there exists a critical K_{crit} such that for $K < K_{\text{crit}}$ the cut-off is insufficient and a new limiting temperature $> T_0$ arises, whereas for $K > K_{\text{crit}}$ the cut-off is sufficient and there is no limiting temperature. For fixed $K < K_{\text{crit}}$ an increase in c results in an increase in the limiting temperature since then $n(T)$ is larger and so is the cut-off. Clearly K_{crit} decreases with increasing c . We have not attempted to compute the function $K_{\text{crit}}(c)$.

Finally for the sake of completeness we plot U'_2 and U''_2 together and U'_4 and U''_4 together in Fig. 7, all with the same value of $c = 40$.

4. EXCLUDED VOLUME APPROXIMATION

In this section we consider a different approach to bypass the limiting temperature T_0 . This approach is based on a Van der Waals treatment, or excluded volume approximation, of deformable hadron bags^{21,13,8,22,23}). Here we wish to quickly review this model, to then obtain some new analytic formulae, and finally to compare numerically with the effective interaction model.

The partition function for a gas of non-interacting point particles, with mass spectrum $\rho(m)$, which are treated relativistically but with Boltzmann statistics, can be written as

$$Z_{pe}(\beta, V) = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \int \frac{d^3p}{(2\pi)^3} dm_i \rho(m_i) V^N \exp(-\beta \epsilon_i) \quad (4.1)$$

where $\beta = 1/T$ and $\epsilon_i = (p^2 + m_i^2)^{1/2}$. Note that in accordance with special relativity, particle production is allowed so that the number of hadrons N in the box of volume V is not fixed. Unlike Refs. 13 and 23 we are concerned only with the usual thermodynamic limit, i.e. total energy $E \rightarrow \infty$, volume $V \rightarrow \infty$, with energy density E/V fixed at a finite value.

Since the product in eq. (4.1) factorizes into separate mass and momentum integrations the thermodynamic quantities of interest may easily be obtained:

$$P_{pe}(\beta) = \ln Z_{pe}(\beta, V) / \beta V = \int \frac{d^3p}{(2\pi)^3} dm \rho(m) \frac{p^2}{3E} \exp(-\beta E) \quad (4.2)$$

$$\rho_{p_t}(\beta) = \int \frac{d^3p}{(2\pi)^3} dm \rho(m) \exp(-\beta \varepsilon) \quad (4.3)$$

$$S_{p_t}(\beta) = \frac{\partial}{\partial T} P_{p_t}(\beta) = \beta [P_{p_t}(\beta) + \rho_{p_t}(\beta)] \quad (4.4)$$

$$n_{p_t}(\beta) = \int \frac{d^3p}{(2\pi)^3} dm \rho(m) \exp(-\beta \varepsilon) = \beta P_{p_t}(\beta) \quad (4.5)$$

The pressure, energy density, entropy density, and average hadron number density are denoted by P , ρ , S , and n respectively. These expressions reduce to the ones given in Section 2 for the potential $U_1(n) = 0$.

According to the bag model hadrons occupy a finite volume proportional to their own mass, $m/4B$. The excluded volume approximation for deformable hadrons says that it costs an infinite amount of energy for two hadrons to overlap, i.e. they are impenetrable. In this respect the effective interaction model is more realistic in that the cost in energy is finite. However, the excluded volume approximation has the minor computational advantage of avoiding a self-consistency condition, as we shall see.

The approximation consists of the replacement in eq. (4.1) of

$$V \rightarrow V - \sum_{j=1}^N \varepsilon_j / 4B \quad (4.6)$$

which means that the volume available for hadrons to move in is reduced by the total volume occupied by the hadrons themselves. The energy rather than the mass appears in eq. (4.6) because of relativity, although we are only treating the pion with relativistic kinematics in this paper. Then

$$\begin{aligned} Z_{xv}(\beta, V) = & \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \int \frac{d^3p}{(2\pi)^3} dm_i \rho(m_i) \exp(-\beta \varepsilon_i) \\ & \times \left(V - \sum_{j=1}^N \varepsilon_j / 4B \right)^N \end{aligned} \quad (4.7)$$

Now, unfortunately, the product in eq. (4.7) does not factorize into independent integrations, so it is not possible to simplify this expression further.

Progress is made by employing the pressure ensemble instead^{22,23}). It is obtained by taking the Laplace transform,

$$\hat{Z}(\beta, \xi) = \int_0^\infty dV Z(\beta, V) \exp(-\xi V) \quad (4.8)$$

Thus

$$\begin{aligned} \hat{Z}_{xv}(\beta, \xi) &= \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^N \int \frac{d^3 p}{(2\pi)^3} dm_i \rho(m_i) \exp(-\beta \varepsilon_i) \\ &\quad \times \int_{V_0}^{\infty} dV (V - V_0)^N \exp(-\xi V) \\ &= \frac{1}{\xi} \sum_{N=0}^{\infty} \left[\frac{1}{\xi} \int \frac{d^3 p}{(2\pi)^3} dm \rho(m) \exp\left\{-\left(\beta + \frac{\xi}{4B}\right) \varepsilon\right\} \right]^N \\ &= \left[\xi - \int \frac{d^3 p}{(2\pi)^3} dm \rho(m) \exp(-\beta^* \varepsilon) \right]^{-1} \end{aligned} \quad (4.9)$$

where

$$V_0 = \sum_{j=1}^N \varepsilon_j / 4B \quad (4.10)$$

and

$$\beta^* = \beta + \xi / 4B \quad (4.11)$$

The pressure in the thermodynamic limit is obtained by locating the pole ξ_0 of $\hat{Z}(\beta, \xi)$ with the largest real part in the complex ξ -plane, as $\xi_0 = \beta P_{xv}(\beta)$. From eqs. (4.9) and (4.5),

$$\xi_0 = \beta P_{xv}(\beta) = \beta^* P_{pt}(\beta^*) \quad (4.12)$$

Then from eqs. (4.11) and (4.12) we obtain the parametric relation

$$P_{xv}(T) = \frac{P_{pt}(T^*)}{1 - P_{pt}(T^*)/4B} \quad (4.13)$$

$$T = \frac{T^*}{1 - P_{pt}(T^*)/4B} \quad (4.14)$$

that is, we choose a numerical value for T^* and then compute the real temperature and pressure in excluded volume approximation by making use of the point particle pressure at T^* . Using thermodynamic identities we readily obtain the entropy and energy densities as

$$S_{xv}(T) = \frac{S_{pt}(T^*)}{1 + P_{pt}(T^*)/4B} \quad (4.15)$$

$$\rho_{xv}(T) = \frac{\rho_{pt}(T^*)}{1 + P_{pt}(T^*)/4B} \quad (4.16)$$

From eq. (4.6) note that the point particle limit is recovered by letting $B \rightarrow \infty$, in which case eqs. (4.13) to (4.16) become identities. Note also that these relations are somewhat similar to those given in Refs. 13 and 23 but different because we employ the usual thermodynamic limit.

The form of the mass spectrum we chose to consider in this paper, eq. (2.1), implies that P_{pt} is finite at T_0 but that ρ_{pt} and S_{pt} diverge. This has important consequences for the excluded volume approximation. There are two cases. Suppose that $P_{pt}(T_0) < 4B$. Then, since T is a monotonically increasing function of T^* [see eq. (4.14)], the excluded volume model also has a limiting temperature given by $T_0^{xv} = T_0/[1 - P_{pt}(T_0)/4B] > T_0$. In this case the entropy density and energy density are also bounded: $S_{xv}(T_0^{xv}) = 4B/T_0$, $\rho_{xv}(T_0^{xv}) = 4B$.

Suppose that $P_{pt}(T_0) \geq 4B$. Then $T \rightarrow \infty$ for some value of T^* less than or equal to T_0 , as can be seen from eq. (4.14). There is no limiting temperature in this case. Furthermore $P_{xv}(T) \rightarrow \infty$ as $T \rightarrow \infty$, whereas $S_{xv}(T) \leq 4B/T_0$ and $\rho_{xv}(T) \leq 4B$ for all T . Thus, in the excluded volume model, the existence of a limiting temperature depends only on $P_{pt}(T_0)/4B$.

Numerical results for the entropy density are displayed in Fig. 8. For the bag constant we take $B^{1/4} = 145$ MeV, which is determined by a fit to the low-lying hadron masses²⁴⁾. For the case $c = 0$, $P_{pt}(T_0) < 4B$, which results in a new limiting temperature of $T_0^{xv} = 400$ MeV. For the cases $c = 4$ and $c = 40$, $P_{pt}(T_0) > 4B$, so there is no limiting temperature. Notice how flat the entropy density is for $T > 400$ MeV. For comparison the results for a gas of point particles with $c = 0$ and $c = 40$ are also shown. The curves coincide for $T < 110$ MeV, but are vastly different at higher temperatures.

5. CONCLUSION

We have assumed an exponentially rising hadronic mass spectrum and have considered means for avoiding a limiting temperature. This is not just of academic interest, for if the pressure of the deconfined quark-gluon phase passes through zero, as it does in the bag model, then it follows that the mass spectrum does increase exponentially. Treating these hadrons in ideal gas approximation results in a limiting temperature for the hadronic phase which is too small to allow for a consistent description of a phase transition to a gas of deconfined quarks and gluons.

The effective interaction model can be employed as a semi-quantitative guide to study the resolution of this problem. If the average potential energy of a hadron is independent of mass, or if it increases more slowly than the mass to the first power, then the limiting temperature remains. If the average potential energy increases more rapidly than the mass to the first power then no limiting temperature can arise. If the average potential energy is proportional to the mass, then the limiting temperature either will be shifted to a higher numerical value or it will be eliminated altogether, depending on the parameters in the problem. Unfortunately, for mass dependent potentials a closed form expression exists only for the entropy density. This is due to a double counting of potential energies¹²⁾. Other thermodynamic quantities must be obtained numerically via thermodynamic identities, i.e. $P(T) = \int_0^T S(T') dT'$, etc.

A second model can also be employed to either postpone or cancel a limiting temperature. This is the excluded volume model for deformable hadrons. A curious feature of this model is that the entropy and energy densities are bounded. Since the hadrons are treated as impenetrable by each other we find this model less physically appealing than the effective interaction model.

Numerical differences among the models and among different, reasonable, parametrizations of the same model arise only when the temperature is greater than about 120 MeV. Experimentally this is quite significant. It probably would take beam energies greater than 2 GeV/nucleon to see the differences in nucleus-nucleus collisions.

Due to finite size and potential energy effects the heavier hadrons are probably less significant than previously thought. This means that more effort should be spent on solving the many-body problem with realistic interactions among the lower mass hadrons. Although we have not discussed the deconfined quark-gluon phase very much, there are indications that the first few terms of a weak coupling expansion may be sufficient to describe this phase. Consider Fig. 1. The thermodynamic potential through order g^3 has been calculated¹¹⁾, and it has been shown²⁵⁾ that the order g^4 term can be fit to parametrize rather well the lattice Monte Carlo calculations¹⁰⁾. This would take us down to temperatures on the order of T_c . Furthermore, interactions calculated through order g^2 are quite sufficient to give a good fit to the light hadron spectroscopy in the bag model^{24,16)}. Since hadrons live essentially at the point corresponding to T_0 in Fig. 1, it may be that one only needs to know the interactions calculated through order g^2 or g^4 to describe well the whole curve for $T \geq T_0$.

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Figure captions

- Fig. 1 : Qualitative plot of the pressure versus temperature. There is a phase transition at T_c , assumed here to be first order. There is an energy density difference between true and perturbative vacua of amount B . There is an exponential mass spectrum with inverse slope T_0 determined by the intersection of the quark-gluon phase pressure with the $P = 0$ axis.
- Fig. 2 : Entropy density as a function of temperature. Only the discrete hadron states are included ($c = 0$). Notice the dependence on the strength of the potential used, eqs. (2.3) to (2.6).
- Fig. 3 : Entropy density as a function of temperature. An exponential mass spectrum ($c = 40$) causes a limiting temperature $T_0 = 160$ MeV no matter how strong the potential is.
- Fig. 4 : Entropy density as a function of temperature. The exponential mass spectrum is not important until $T \gtrsim 120$ MeV when it suddenly hits like a brick wall.
- Fig. 5 : Entropy density for the single particle potentials which yield a Gaussian damping. There is no limiting temperature, and in fact the contribution from high mass states is rather small.
- Fig. 6 : Entropy density for the single particle potentials which yield an exponential damping. For weak enough potentials such as U'_2 this damping is insufficient and a new limiting temperature ensues.
- Fig. 7 : Comparison of the entropy densities for exponential (U') versus Gaussian (U'') dampings.
- Fig. 8 : Entropy density in excluded volume (xv) approximation. For the case $c = 0$ there is a new limiting temperature $T_0^{xv} = 400$ MeV denoted by the arrow; for the cases $c = 4$ and $c = 40$ there is no limiting temperature, but the entropy density is limited, $S_{xv} < 4B/T_0$. For comparison, the results for a gas of point particles with potential U_2 and with $c = 0$ and $c = 40$ are also shown.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8