Ref.TH.3433-CERN

PERTURBATION THEORY FOR QED BOUND STATES

E. Remiddi

CERN -- Geneva

and

Istituto di Fisica, Università di Bologna,
INFN, Sezione di Bologna

and

M. Semeria

Istituto di Fisica, Università di Bologna,
INFN, Sezione di Bologna

ABSTRACT

A new method is presented for deriving a systematic perturbative expansion for QED bound states, which does not rely upon solving any new or old equation. The starting point is a given non-perturbative zeroth order Green's function, obtained by a suitable "relativistic dressing" of the non-relativistic Green's function for the Schrödinger equation with Coulomb potential, which embodies the Coulombic bound states and is known. The comparison with the complete Green's function as given by standard perturbative QED gives a perturbative kernel which is then used for the expansion of the QED Green's function in terms of the given non-perturbative zeroth order Green's function.

1. Coulomb bound states can be studied with the Schrödinger equation in non-relativistic approximation, with the Dirac equation to account for relativistic effects; it is a common belief that a proper treatment of their QED radiative corrections requires going through the Bethe-Salpeter equation¹⁾ or some reasonable approximation of it. In this paper, we present a different approach; it is not based on the solution of any new proposed equation, but it is rather a systematic expansion for the energy levels, which can be carried out up to any level of accuracy in closed explicit form. The basic ingredients for writing down the expansion are the non-relativistic Green's function for the Schrödinger equation with Coulomb potential, which is known, plus the Feynman graphs for the related scattering process, which are known from standard renormalized QED perturbation theory. The Bethe-Salpeter formalism (kernels, poles and residues of Green's functions, etc.) is used in the derivation but, as already stated, the Bethe-Salpeter equation never comes in as an equation to be actually solved. In lowest orders the new expansion obviously reproduces the good old results already well established in the literature; in the authors' opinion, the new approach can be relevant for a better understanding of the QED bound state problem and also for providing one more tool for the evaluation of higher order corrections to various bound systems (μ^+e^- , e^+e^- , etc.) in order to match the theoretical predictions with the ever increasing precision of experiments.

The essential lines of the derivation are given in Section 2. In Section 3 they are implemented, as a simple application, to the description of the hydrogen atom, for the sole purpose of showing how they work in practice. Their extension to other systems, including positronium, is then shortly discussed in Section 4, together with the relation to previous work²⁾.

2. Let $G(W)$ be the Green's function for some system of total energy W (spatial momenta and spin indices will not be written for simplicity), expected to develop bound states of Coulomb type. We will consider $G(W)$ as given by perturbative renormalized QED as a (formal) power series in the fine structure constant α

$$G(W) = G_0(W) + \alpha G_1(W) + \alpha^2 G_2(W) + \dots, \quad (1)$$

where $G_0(W)$ is the free propagator of the system. Always in perturbation theory, we can evaluate its inverse $F(W)$ and its kernel $K(W)$ defined as

$$F(W) = G^{-1}(W), \quad (2)$$

$$K(W) = G_o^{-1}(W) - G^{-1}(W); \quad (3)$$

they are both known because $G(W)$ is known. Equation (3) is, of course, equivalent to the Bethe-Salpeter equation

$$G(W) = G_o(W) + G_o(W) K(W) G(W). \quad (4)$$

When studying Eq. (4), one usually intends that $K(W)$ is known perturbatively, while the unknown non-perturbative solution provides with the resummation of the series (1). In the following we will not use Eq. (4) in that sense, but only Eq. (3) which gives an expansion for $K(W)$ in terms of a known expansion for $G(W)$.

$G(W)$ is expected to have bound state poles, which are surely not visible in Eq. (1). The gross features of the QED bound states are, however, well known from non-relativistic quantum mechanics; in particular, the Green's function $\bar{G}(E)$ for the Schrödinger equation with Coulomb potential is known. The starting point of the present approach consists in using that knowledge for constructing a relativistic Green's function $G_b(W)$, a kind of "relativistic dressing" of $\bar{G}(E)$, to be used as a zeroth order approximation to $G(W)$. Explicit forms of $G_b(W)$ will be given or discussed in next sections; here we develop the formalism for its use in the bound state problem.

By assumption, $G_b(W)$ is explicitly known in closed form and embodies bound states to be considered as a reasonable zeroth order approximation to the otherwise unknown bound states of $G(W)$. Further, the asymptotic behaviour of $G_b(W)$ for high momenta is such that standard perturbative renormalization applies. The meaning of those conditions and the way of fulfilling them will become clear in the example given in next section.

Given $G_b(W)$, we define as in Eqs. (2) and (3)

$$F_b(W) = G_b^{-1}(W), \quad (5)$$

$$K_b(W) = G_o^{-1}(W) - G_b^{-1}(W), \quad (6)$$

where $G_o(W)$ is the same as in Eq. (3). Let $W_{b,n}$ be the position of the n^{th} pole of $G_b(W)$; we can write, for any n ,

$$G_b(W) = i \frac{R_{b,n}}{W - W_{b,n}} + \hat{G}_{b,n}(W), \quad (7)$$

where $R_{b,n}$ is the residuum and $\hat{G}_{b,n}(W)$ is regular at $W = W_{b,n}$. It would be customary to write the residuum as a product (or a sum of products, if there is degeneracy) of bound state wave functions, but that factorization is of no use here. From Eq. (5) one has obviously

$$F_b(W) G_b(W) = G_b(W) F_b(W) = 1. \quad (8)$$

By inserting Eq. (7) into Eq. (8) it follows

$$F_b(W_{b,n}) R_{b,n} = R_{b,n} F_b(W_{b,n}) = 0, \quad (9)$$

$$i F'_b(W_{b,n}) R_{b,n} + i F_b(W_{b,n}) \hat{G}_{b,n}(W_{b,n}) = 1, \quad (10)$$

where $F'_b(W)$ is the W derivative of $F_b(W)$. By defining further

$$\mathcal{R}_{b,n} = i F'_b(W_{b,n}) R_{b,n} \quad (11)$$

and multiplying Eq. (10) on the left by $\mathcal{R}_{b,n}$ gives

$$\mathcal{R}_{b,n}^2 = \mathcal{R}_{b,n}. \quad (12)$$

From Eq. (12) we see that the trace $N_{b,n}$ of $\mathcal{R}_{b,n}$

$$N_{b,n} = \text{Tr} \{ \mathcal{R}_{b,n} \} \quad (13)$$

is an integer number, the degeneracy of the level $W_{b,n}$. If $N_{b,n} > 1$, $\mathcal{R}_{b,n}$ can be split as

$$\mathcal{R}_{b,n} = \sum_i \mathcal{R}_{b,n}^{(i)} \quad (14)$$

where the $\mathcal{R}_{b,n}^{(i)}$ satisfy the orthogonality conditions

$$\mathcal{R}_{b,n}^{(i)} \mathcal{R}_{b,n}^{(j)} = \delta_{ij} \mathcal{R}_{b,n}^{(j)} \quad (15)$$

and

$$N_{b,n}^{(i)} = \text{Tr} \{ \mathcal{R}_{b,n}^{(i)} \} \quad (16)$$

is the degeneracy of $\mathcal{R}_{b,n}^{(i)}$. It is convenient in practice to choose a decomposition in which the index (i) refers to some set of good quantum numbers for the bound states of $G(W)$, such as total angular momentum, parity, etc. [It is, of course, expected that $G(W)$ will partly remove the degeneracy.] Note here, in passing, that simple orthogonality conditions for different energy levels do not exist, in general, if $F_b'(W)$ is energy dependent (as it is the case in general), but they were not needed in the previous analysis.

As $G_b(W)$ is known, all the $W_{b,n}$, $\mathcal{R}_{b,n}^{(i)}$ are also known. To set up the desired perturbative expansion of $G(W)$ we define further

$$\delta K(W) = K(W) - K_b(W), \quad (17)$$

which on account of Eqs. (3) and (6) gives

$$G(W) = G_b(W) + G_b(W) \delta K(W) G(W). \quad (18)$$

Equation (18) is a new Bethe-Salpeter equation with "free propagator" $G_b(W)$ and kernel $\delta K(W)$. As implied in the choice of the notation for $\delta K(W)$, we will treat it perturbatively and the sole use of Eq. (18) will be its perturbative iteration, giving the expansion of $G(W)$ in terms of $G_b(W)$ and $\delta K(W)$

$$G(W) = G_b(W) + G_b(W) \delta K(W) G_b(W) + G_b(W) \delta K(W) G_b(W) \delta K(W) G_b(W) + \dots \quad (19)$$

We can now turn our attention to the bound state poles of $G(W)$. By assumption, they lie near the (more degenerate) poles of $G_b(W)$; we will indicate by W_{nj} any of the poles of $G(W)$ near to $W_{b,n}$ whose "good quantum numbers" are described by j . Following a standard procedure, we can now write

$$W_{nj} = \frac{\oint dW W \text{Tr} \{ G(W) \mathcal{R}_{b,n}^{(j)} \}}{\oint dW \text{Tr} \{ G(W) \mathcal{R}_{b,n}^{(j)} \}}, \quad (20)$$

where the integral is performed in the complex W plane along a closed circuit enclosing $W_{b,n}$ and all the corresponding nearby poles W_{ni} of $G(W)$. The factor $\mathcal{R}_{b,n}^{(j)}$ picks up the contribution from the pole at W_{nj} , dropping those with $i \neq j$. By inserting the perturbative expansion Eq. (19) into Eq. (20) we finally obtain

$$\begin{aligned} W_{nj} = W_{b,n} + \langle \delta K(W_{b,n}) \rangle + \langle \delta K(W_{b,n}) \hat{G}_{b,n}(W_{b,n}) \delta K(W_{b,n}) \rangle \\ + \langle \delta K(W_{b,n}) \rangle \langle \frac{\partial}{\partial W} \delta K(W_{b,n}) \rangle + \dots \end{aligned} \quad (21)$$

where $\hat{G}_{b,n}(W)$ is given by Eq. (7) and, for short,

$$\langle \delta K(W) \rangle = \frac{1}{N_{b,n}^{(j)}} \text{Tr} \left\{ \delta K(W) \mathcal{R}_{b,n}^{(j)} \right\}, \text{ etc.} \quad (22)$$

Equation (27) is the desired result for the shift of the energy levels. It is a kind of "quadrature formula" which involves only known quantities, namely the "unperturbed" Green's function $G_0(W)$ and its residua, assumedly known as the starting point of the whole approach, and the perturbative kernel $\delta K(W)$, as given through Eqs. (3) and (17), by standard QED perturbation theory.

From a rigorous point of view, it is of course almost impossible to prove the convergence of the expansion (21), as $\delta K(W)$ itself is given as a (formal) power series, so that one can only check a posteriori if the corrections to the energy levels do become smaller as the order in $\delta K(W)$ increases.

It would be not difficult, but will not be done here, to obtain a perturbative expansion in $\delta K(W)$ also for the residuum of $G(W)$ at the pole W_{nj} .

3. As an application of the formalism of the previous section, we now consider the hydrogen atom; the extension to other systems, including positronium, will be shortly discussed afterwards. In this case, $G_0(W)$ is the Dirac free electron propagator

$$G_0(W) = \frac{i}{W^2 - m^2 - \vec{p}^2 + i\epsilon} \left(m + W \gamma_4 - i \vec{p} \cdot \vec{\gamma} \right); \quad (23)$$

the kernel $K(W)$ corresponding to the complete Green's function as given by QED perturbation theory is shown in Fig. 1 up to first order radiative corrections. It can be written as

$$K = K_c + K_{se} + K_{vp} \quad (24)$$

where K_c is the interaction with the external Coulomb field of a heavy nucleus of charge Z , for momentum transfer $\vec{k}$

$$K_c = -i\gamma_4 V_c(\vec{k}), \quad (25)$$

$$V_c(\vec{k}) = - (Z\alpha) \frac{4\pi}{k^2}; \quad (26)$$

K_{se} is the self-energy kernel, which consists in turn of the series of kernels depicted in Fig. 2: they are all of first order in α but of increasing order in $(Z\alpha)$; K_{vp} is the vacuum polarization kernel: it can also be expanded in the same way in $(Z\alpha)$, but for brevity we will not consider it anymore.

The hydrogen atom can traditionally be discussed in the Furry picture, where the radiative correction kernels are treated as a perturbation with respect to K_c . The unperturbed problem corresponds to the relativistic Dirac equation with Coulomb potential, so that in the perturbative expansion, which can be written in the same form as Eq. (21), one needs the Green's function of the Coulomb-Dirac equation. At variance with wave functions, however, convenient ways of writing down explicitly the Green's function for that problem, as required in an analytical calculation, have not yet been found. Therefore, the Furry picture, although of course correct, is not handy in practice. To go on with our approach, we need a notation for the known solution of the non-relativistic Schrödinger equation for the Coulomb potential. The Schrödinger free propagator for non-relativistic energy E is

$$\bar{G}_0(E) = \frac{2mi}{2mE - \vec{p}^2 + i\epsilon}; \quad (27)$$

the equation for the Green's function $\bar{G}(E)$ in the presence of a Coulomb interaction

$$V = -i V_c, \quad (28)$$

where V_c is the same as in Eqs. (25) and (26), can be written in the equivalent ways

$$\begin{aligned}\bar{G}(E) &= \bar{G}_0(E) + \bar{G}_0(E) V \bar{G}(E), \\ \bar{G}(E) &= \bar{G}_0(E) + \bar{G}(E) V \bar{G}_0(E), \\ \bar{G}(E) &= \bar{G}_0(E) + \bar{G}_0(E) V \bar{G}_0(E) + \bar{G}_0(E) V \bar{G}(E) V \bar{G}_0(E)\end{aligned}\quad (29)$$

(all the dependence and integrations on spatial momenta are omitted for simplicity). The solution to Eq. (29) is known; the perhaps best way of representing it is given in Ref. 3), where use is made of Fock's $O(4)$ formalism⁴⁾. In particular, $\bar{G}(E)$ has bound state poles of the form

$$\bar{G}(E) \simeq i \frac{\bar{R}_n}{E - E_n}, \quad (30)$$

where E_n are the Balmer levels and the residua can be written as

$$\bar{R}_n(\vec{p}, \vec{q}) = \sum_{\ell, m} \varphi_{n\ell m}(\vec{p}) \varphi_{n\ell m}^*(\vec{q}), \quad (31)$$

the $\varphi_{n\ell m}(\vec{p})$ being the familiar linear independent bound state wave functions of the n^2 degenerate n^{th} level. For our purpose, however, it is sufficient to take the residua simply split as

$$\bar{R}_n(\vec{p}, \vec{q}) = \sum_{\ell=0}^{n-1} \bar{R}_{n\ell}(p, q) (2\ell+1) P_{\ell}\left(\frac{\vec{p} \cdot \vec{q}}{pq}\right), \quad (32)$$

where $p = |\vec{p}|$, $q = |\vec{q}|$, $P_{\ell}(z)$ is the Legendre polynomial of order ℓ and $\bar{R}_{n\ell}$ the corresponding "radial part".

After such preliminaries with an eye to the third of Eq. (29), we write the Green's function $G_b(W)$ to be used as the zeroth order approximation to the relativistic hydrogen atom problem as

$$G_b(W) = G_0(W) + G_0(W) \gamma_4 V G_0(W) + G_0(W) \gamma_4 V \bar{G}\left(\frac{W^2 - m^2}{2m}\right) V G_0(W), \quad (33)$$

where spatial momenta are again omitted for simplicity and all the symbols have already been introduced in Eqs. (27) and (29). According to Eq. (6), the kernel of $G_b(W)$ is

$$\begin{aligned}
 K_b(W) &= G_o^{-1}(W) - G_b^{-1}(W) \\
 &= \gamma_4 V + \gamma_4 V (\bar{G}_o(E) - G_o(W) \gamma_4) V \\
 &\quad + \gamma_4 V (\bar{G}_o(E) - G_o(W) \gamma_4) V (\bar{G}_o(E) - G_o(W) \gamma_4) V + \dots
 \end{aligned} \tag{34}$$

where $E = (W^2 - m^2)/2m$. The bound state poles of $G_b(W)$ corresponding to those of $\bar{G}(W^2 - m^2)/2m$ are at

$$W_{b,n} = \sqrt{m^2 + 2m E_n}, \tag{35}$$

with residues, according to Eqs. (32) and (33)

$$\begin{aligned}
 R_{b,n} &= \frac{m}{W_n} \sum_{\ell=0}^{n-1} (-i \vec{\gamma} \vec{p} + W_{b,n} \gamma_4 + m) \bar{R}_{n\ell}(p, q) (2\ell+1) P_\ell\left(\frac{\vec{p} \cdot \vec{q}}{pq}\right) \\
 &\quad \cdot (-i \vec{\gamma} \vec{q} + W_{b,n} \gamma_4 + m).
 \end{aligned} \tag{36}$$

For future use, it is convenient to separate the $j = \ell + 1/2$ and $j = \ell - 1/2$ parts of the ℓ wave. This is achieved by writing, for any two unity vectors $\vec{a}, \vec{b}$

$$(2\ell+1) P_\ell(\vec{a} \cdot \vec{b}) = \sum_{\lambda=\pm 1} D_\ell^{\ell+\frac{\lambda}{2}}(\vec{a}, \vec{b}) \tag{37}$$

with

$$D_\ell^{\ell+\frac{\lambda}{2}}(\vec{a}, \vec{b}) = \left(\ell + \frac{1+\lambda}{2}\right) P_\ell(\vec{a} \cdot \vec{b}) - \lambda (\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) - \vec{a} \cdot \vec{b} P'_\ell(\vec{a} \cdot \vec{b}), \tag{38}$$

where $P'_\ell(z)$ is the derivative of $P_\ell(z)$ and for $\ell = 0$ only $\lambda = +1$ makes sense. The D_ℓ^j defined above correspond to the definite values of parity and total angular momentum $(-1)^\ell, j(j+1)$ and therefore satisfy the usual orthogonality relations

$$D_\ell^j(\vec{a}, \vec{b}) \cdot D_\ell^{j'}(\vec{b}, \vec{c}) = \delta_{\ell\ell'} \delta_{jj'} D_\ell^j(\vec{a}, \vec{c}), \tag{39}$$

where the integration on the unit vector $\vec{b}$ is understood. In the notation of Eqs. (11)-(16) the normalized residuum of $G_b(W)$ at the n^{th} pole can then be written as

$$i F'_b(W_{b,n}) R_n = \sum_{\ell=0}^{n-1} \sum_{\lambda=\pm 1} R_{b,n}^{(\ell\lambda)}, \quad (40)$$

with

$$R_{b,n}^{(\ell\lambda)} = i F'_b(W_{b,n}) \frac{m}{W_{b,n}} \left(-i \vec{p} \cdot \vec{\gamma} + W_{b,n} \gamma_4 + m \right) \bar{R}_{n\ell}(p, q) D_{\ell}^{\ell+\frac{1}{2}} \left(\frac{1}{p} \vec{p}, \frac{1}{q} \vec{q} \right) \cdot \\ \cdot \left(-i \vec{q} \cdot \vec{\gamma} + W_{b,n} \gamma_4 + m \right). \quad (41)$$

By subtracting K_b Eq. (34) from K Eq. (24), one obtains the perturbative kernel to be used, together with $R_{b,n}^{(\ell\lambda)}$ Eq. (41) in the perturbative expansion for the energy levels Eq. (21). We write

$$\delta K = \delta K_c + K_{se} + K_{vp} \quad (42)$$

with

$$\delta K_c = \gamma_4 V \left(\bar{G}_0(E) - G_0(W) \gamma_4 \right) V \\ + \gamma_4 V \left(\bar{G}_0(E) - G_0(W) \gamma_4 \right) V \left(\bar{G}_0(E) - G_0(W) \gamma_4 \right) V + \dots \quad (43)$$

We now look at the contribution from δK_c to the energy shift in first order perturbation theory according to Eqs. (21) and (22). When averaging kernels on bound states, it is customary to scale all spatial momenta by $(Z\alpha)$; in the simplest cases the procedure gives the right order in $(Z\alpha)$ of the contribution. Knowing that the mean value of l is l , the normalization, the mean value of $\vec{p}^2$ is found to be of order $(Z\alpha)^2$ according to the virial theorem, etc. In the case of δK_c , $\bar{G}_0(E)$ and $G_0(W)$ are of order $(Z\alpha)^{-2}$, but the combination $(G_0(E) - G_0(W)\gamma_4)$ is of order $(Z\alpha)^{-1}$, so that, accounting for all the rest which includes also a $(Z\alpha)^3$ factor due to the integration on intermediate momenta, the first term of Eq. (43) turns out to be of order $(Z\alpha)^3$, reducing actually to $(Z\alpha)^4$ for symmetry reasons; together with the next term, which is and remains of order $(Z\alpha)^4$, it gives the expected fine structure corrections to the Balmer energy levels.

Consider then K_{se} . According to Fig. 2 it consists of a series of terms of increasing nominal order in $(Z\alpha)$, actually, however, all of a same (as yet unspecified) order if the above naïve $(Z\alpha)$ counting is performed. To be less sloppy, the first two terms require the renormalization of their ultra-violet (u.v.) divergences, so that the counting does not apply to them; it does apply to the other terms, for which a resummation procedure is therefore needed.

One could try to sum all of them "exactly", identifying the double line on the left of Fig. 2 with the Dirac propagator in the Coulomb field, G_D : that is, however, unpractical because, as already remarked, we do not dispose of an handy expression for G_D . Within our approach, it is natural to approximate G_D in terms of G_b , δK_c

$$G_D = G_b + G_b \delta K_c G_b + \dots ; \quad (44)$$

neglecting higher order terms, we further restrict here to the contribution from the first term only. According to the definition of G_b Eq. (33), it consists of three pieces: free propagator, one Coulomb exchange and a "remainder" with the bound states. After having approximated G_D by G_b in the left-hand side of Fig. 2 we write the self-energy shift as

$$\langle \delta K_{se} \rangle = \langle \delta K_1 \rangle + \langle \delta K_2 \rangle + \langle \delta K_3 \rangle, \quad (45)$$

the three contributions being diagrammatically showed in Fig. 3. It is to be noted that the first two terms, which require renormalization of u.v. divergences, are identical to the first two terms of Fig. 2, which is "exact". In the third graph of Fig. 3 the newly introduced dotted-undotted double line represents the known "remainder" $G_0 V \bar{G} V G_0$ of Eq. (33); its contribution is u.v. finite.

In order to work out the first two terms of Eq. (3), we use Feynman gauge and Pauli-Villars regularization. After mass renormalization, but without wave function renormalization, one finds in leading order in $(Z\alpha)$ and on account of the Schrödinger equation

$$\langle \delta K_1 \rangle \simeq \frac{\alpha}{\pi} \left\langle \left(-\frac{1}{4} \lg \frac{A^2}{m^2} - \frac{1}{8} - \lg \frac{p^2 + m^2}{m^2} \right) V_c \right\rangle, \quad (46)$$

where A is the Pauli-Villars mass and $p^2 = \vec{p}^2 - W_{b,n}^2$. Similarly, without carrying out the vertex renormalization, in leading order the second term gives

$$\begin{aligned} \langle \delta K_2 \rangle \simeq \frac{\alpha}{\pi} \left\langle \left(\frac{1}{4} \lg \frac{A^2}{m^2} + \frac{1}{8} \right. \right. \\ \left. \left. + \frac{1}{p^2 - q^2} \left((p^2 + m^2) \lg \frac{p^2 + m^2}{m^2} - (q^2 + m^2) \lg \frac{q^2 + m^2}{m^2} \right) \right) V_c \right\rangle, \end{aligned} \quad (47)$$

with $q^2 = \vec{q}^2 - W_{b,n}^2$. Again in leading order, the third term is found to be

$$\langle \delta K_3 \rangle \simeq \frac{\alpha}{\pi} \left\langle \left(\lg \frac{p^2 + m^2}{m^2} - \frac{1}{p^2 - q^2} \left((p^2 + m^2) \lg \frac{p^2 + m^2}{m^2} - (q^2 + m^2) \lg \frac{q^2 + m^2}{m^2} \right) \right) V_c \right\rangle. \quad (48)$$

The sum of the three contributions in leading order vanishes

$$\langle \delta K_1 \rangle + \langle \delta K_2 \rangle + \langle \delta K_3 \rangle = 0. \quad (49)$$

As a first comment, the dependence on the Pauli-Villars mass A cancels between (46) and (47), as required by the Ward identity, so that no u.v. divergences remain. One might recall here that infra-red (i.r.) divergences, such those often regularized in scattering processes by means of, say, a fictitious small photon mass, can appear only in processes on the mass shell, not in bound state calculations where the binding itself keeps particles away from the mass shell. Additional i.r. divergences are sometimes artificially introduced also by means of separate renormalization of wave function and vertex on the mass shell, but as remarked above those two renormalizations are not actually needed, so that there is really no place for i.r. divergences to show up.

As a second comment to Eq. (49), the three contributions Eqs. (46) are of leading order $\alpha(Z\alpha)^2$, so that their cancellation is necessary for recovering the expected correct radiative level shift of order $\alpha(Z\alpha)^4$. Equations (46)-(48) were obtained in the Feynman gauge; it is usual in the literature to use some "better" gauge, in which the three contributions (or better the analogue of the three contributions in the various approaches) are separately of order $\alpha(Z\alpha)^4$. As we have shown, the cancellation of $\alpha(Z\alpha)^2$ is easily established even in the Feynman gauge which is perhaps the most convenient for the precise evaluation of contributions like $\langle \delta K_2 \rangle$ Eq. (47).

A last comment concerns the choice Eq. (33) of G_b . Its form is by no means unique; one might, for instance, keep γ_4 in the second term and change γ_4 into $(1+\gamma_4)/2$ in the third one, or use $(1+\gamma_4)/2$ in both (which gives a particularly simple δK_c) etc.; the third term might also be modified by keeping in any case $V\bar{G}V$ but multiplying it by some "convergence factor" of value around 1 for small momenta and vanishing for high momenta, but in the hydrogen case that last change is not needed.

The same G_b has been used both for writing the perturbative expansion of the energy levels and for approximating G_D within δK_{se} ; that is a natural and convenient choice (in particular for the discussion of the gauge invariant

properties of the perturbative expansion), but again not strictly necessary, as any other choice explicitly exhibiting the one-Coulomb interaction and an u.v. finite "remainder" with the bound states would work.

4. The previous discussion brings us to the extension of the approach to more general Coulomb systems, including positronium. As a matter of fact, this work was partly originated as an attempt to generalize the results of Ref. 2). There, a simple but somewhat ad hoc Coulomb kernel K_C for positronium was taken as the starting point and the solution G_C for the corresponding Bethe-Salpeter equation was obtained in closed analytic form; the same perturbative expansion as in Eqs. (17)-(21) is then worked out.

The approach of Ref. 2) lacks in flexibility: practically any change in the initial K_C leads to an equation for G_C which is impossible to solve in closed form. One has therefore to stick to those unique K_C and G_C , which generate in an analytic calculation a host of nasty integrals, clearly an artifact due to the many square roots present in K_C and G_C , whose cancellation is hard to trace back.

In the spirit of this paper, one might start from that G_C obtaining from it K_C and all the rest identically; but one can also change G_C to a more manageable form, provided its crucial features are preserved. In the case of positronium, we recall, one also needs a "convergence factor" for the bound state wave functions, as the simple use of a relativistically dressed Schrödinger wave function leads to artificial u.v. divergences when dealing with one-photon annihilation kernels; in the present approach the explicit form of that convergence factor is not a priori fixed, as it was in Ref. 2) (where it was the main source of the unpleasant square roots).

In conclusion, we have shown how to write a systematic perturbative expansion for QED bound states, without really using (i.e., trying to solve) any equations, if not at most for book-keeping purposes. The ingredients for the expansion are the known Green's function for the non-relativistic Schrödinger equation with the Coulomb potential and the perturbative expansion for the relativistic Green's function as given by standard renormalized QED. A central rôle is played in the formalism by Green's functions, their kernels and their residues, rather than by separate wave functions as in non-relativistic potential problems.

The u.v. divergences are the same as in standard QED perturbation theory, so that renormalization can be carried out in the standard way even in studying bound states, while i.r. divergences are intrinsically absent (i.e., naturally regularized) for bound particles.

ACKNOWLEDGEMENT

One of the authors (E.R.) is pleased to thank V. Glaser for several discussions on the bound state problem.

REFERENCES

- 1) E.E. Salpeter and H.A. Bethe, Phys. Rev. 84 (1951) 1232;
M. Gell-Mann and F. Low, Phys. Rev. 84 (1951) 350.
- 2) R. Barbieri and E. Remiddi, Nucl. Phys. B141 (1978) 413;
see also
E. Remiddi, "The Bethe-Salpeter Equation in QED", Eds. G. Costa and R.R. Gatto,
in Theory of Fundamental Interactions, LXXXI Course, Varenna, Academic
Press (1982);
An equivalent approach is followed in
W.E. Caswell and G.P. Lepage, Phys. Rev. A18 (1978) 810.
- 3) J. Schwinger, J. Math. Phys. (N.Y.) 5 (1964) 1601.
- 4) J. Fock, Zeit. f. Phys. 98 (1935) 145.

FIGURE CAPTIONS

- Fig. 1 : The kernel of the complete Green's function for the hydrogen atom
up to first order radiative corrections.
- Fig. 2 : The self-energy kernel and its expansion in $(Z\alpha)$.
- Fig. 3 : The approximated self-energy kernel.

Fig. 1

Fig. 2

Fig. 3