Ref.TH.3436-CERN

IS THE WESS-ZUMINO MODEL INCONSISTENT ?

N.V. Krasnikov^{*)} and H. Nicolai
CERN - Geneva

A B S T R A C T

We present evidence that the Wess-Zumino model cannot be defined as a mathematically consistent quantum field theory. Our arguments are based on the use of Schwinger's equations, the Källen-Lehmann inequality and the supersymmetry Ward identities. Possible consequences of our results are briefly discussed.

^{*)} On leave from the Institute for Nuclear Research, Moscow 117312.

At present, it is widely believed that supersymmetry^{1),2)} may help to solve the gauge hierarchy problem^{3),4)}. These hopes are closely related to the remarkable fact that a wave function renormalization is all that is needed to remove the ultra-violet divergences and to render supersymmetric models finite in each order of perturbation theory⁵⁾. On the other hand, most of the (semi) realistic supersymmetric models that have been proposed so far are not asymptotically free. The investigation of the ultra-violet behaviour of these theories therefore constitutes an important problem. Unfortunately, not much is presently known about this question; the main obstacle is the growth of the effective charge in the ultra-violet region for theories which are not asymptotically free so perturbation theory cannot be applied. Moreover, there is convincing evidence⁶⁾ that the simplest non-asymptotically free model -- the ϕ_4^4 model -- does not exist and therefore does not correspond to a mathematically consistent quantum field theory. The question then arises whether this is a feature common to all non-asymptotically free theories. The simplest candidate to study this question is the Wess-Zumino (WZ) model²⁾. The constructive aspects of this model were first studied in Ref. 7) but in that work no final conclusion was reached as to existence or non-existence of this model. In Ref. 8), an investigation of the massless WZ model was performed which was based on the use of Schwinger's equations⁹⁾ and the Källen-Lehmann inequality^{10),11)}. It was found there that the invariant charge of this model exhibits a rather strange behaviour: for large momenta it has to decrease faster than $(\log p^2)^{-1}$.

In this paper, we further investigate the massive WZ model and present evidence that this model also cannot lead to a mathematically consistent quantum field theory. We believe that our arguments are sufficiently general to be extendable to more general WZ type models involving several chiral supermultiplets and, perhaps, to any globally supersymmetric model which is not asymptotically free. We interpret this as yet another indication that a consistent unification scheme based on supersymmetry must include gravity since the inclusion of gravitational interactions can drastically alter the picture, and that consistency requirements inevitably lead us to local supersymmetry and supergravity [for a review, see Ref. 12)].

Our analysis again relies on the use of Schwinger's equations and the Källen-Lehmann inequality. The latter states that^{10),11)}

$$0 \leq Z(\Lambda = \infty, g_R, \frac{m_R}{\mu}) \leq 1 + \frac{m_R^2}{\mu^2} * \quad (1)$$

*) For the regularized theory ($\Lambda < \infty$), this inequality is only valid if the regularization respects positivity. Such a regularization may be obtained by smearing all fields in space but not in time. Note that the upper bound in (1) depends on the normalization convention.

for the wave function renormalization $Z(\dots)$. Here Λ denotes the UV-cut-off parameter, μ the normalization point and m_R and g_R the renormalized mass and coupling constant, respectively; these are the only parameters that appear in the WZ model. In fact, we only use the finiteness of $Z(\infty, \dots)$ and this is enough to derive a contradiction. Note that the inequality (1) is essentially a consequence of the existence of a positive definite metric in Hilbert space^{10),11)}. [This postulate is violated in the case of Yang-Mills theories and this is the main reason why in these models one can have asymptotic freedom; there is no inequality that prevents $Z(\Lambda, \dots)$ from tending to infinity as $\Lambda \rightarrow \infty$, and no inconsistency arises.] We separately consider the cases $Z(\infty, \dots) = 0$ (infinite charge renormalization) and $Z(\infty, \dots) \neq 0$ (finite renormalization). In the first case, we find that the usual perturbation expansion breaks down in that it does not properly describe the theory which, in this limit, corresponds to an ultra-local theory [for the WZ model, this fact was first noted in Ref. 7)]. Such a theory is of no interest since its spectrum is empty. If, however, $Z(\infty, \dots) \neq 0$ we can derive a contradiction. So we conclude that no choice of the wave function renormalization $Z(\dots)$ which satisfies the inequality (1) can lead to a consistent theory.

Let us now briefly recall the essential features of the WZ model; we follow the notation and conventions of Ref. 13). The model describes the interactions of a chiral multiplet

$$\phi(x, \theta) = A(x) + \sqrt{2} \theta \psi(x) + \theta \theta F(x) \quad (2)$$

and its Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \quad (3)$$

with^{2),13)}

$$\mathcal{L}_0 = A^* \square A + i \partial_\mu \bar{\psi} \bar{\sigma}^\mu \psi + FF^* \quad (4)$$

$$\begin{aligned} \mathcal{L}_I = & m (FA - \frac{1}{2} \psi\psi) + \\ & + g (FA^2 - A\psi\psi) + \\ & + \text{h.c.} \end{aligned} \quad (5)$$

The Lagrangian is invariant under the supersymmetry transformations

$$\begin{aligned}\delta A &= \sqrt{2} \xi \psi \\ \delta \psi &= i\sqrt{2} \sigma^\mu \bar{\xi} \partial_\mu A + \sqrt{2} \xi F \\ \delta F &= i\sqrt{2} \bar{\xi} \sigma^\mu \partial_\mu \psi\end{aligned}\tag{6}$$

It has been shown that only one wave function renormalization $Z(\Lambda/\mu, m_R/\mu, g_R)$ is required to remove all perturbative divergences in this model⁵⁾. The renormalized Lagrangian is therefore

$$\mathcal{L}^R = Z\left(\frac{\Lambda}{\mu}, \frac{m_R}{\mu}, g_R\right) \mathcal{L}_0^R + \mathcal{L}_I^R\tag{7}$$

where $\mathcal{L}_0^R$ and $\mathcal{L}_I^R$ are simply obtained from (4) and (5) by attaching the subscript R to all fields and to the parameters m and g . Thus m_R and g_R are the renormalized mass and coupling constants. The counterterm Z depends on these parameters as well as on the UV-cut-off Λ and the normalization parameter μ ; in perturbation theory, one obtains⁵⁾

$$Z\left(\frac{\Lambda}{\mu}, \frac{m_R}{\mu}, g_R\right) = 1 - \alpha g_R^2 \log \frac{\Lambda}{\mu} + O(g_R^4)\tag{8}$$

($\alpha > 0$)

We are assuming here that the theory has been regularized supersymmetrically such as by introduction of higher derivatives⁵⁾ or by smearing the fields⁷⁾. We emphasize that the remarkable renormalization properties of the WZ model have so far only been established in perturbation theory. We shall assume that these properties also hold beyond perturbation theory. According to general arguments based on the renormalization group¹⁴⁾, one obtains finite expressions for the Green's functions in each order of perturbation theory in the limit $\Lambda \rightarrow \infty$ if $Z(\Lambda, \dots)$ is appropriately varied, and this should be true for the full theory. Furthermore, we shall make the assumption that the Green's functions which are obtained in this limit obey the usual axioms of quantum field theory¹⁴⁾; this assumption is needed for the derivation of the Källen-Lehmann inequality^{10), 11)}. Just as in perturbation theory, the relation between bare and renormalized parameters of the theory is given by

$$m = Z^{-1}\left(\frac{\Lambda}{\mu}, \frac{m_R}{\mu}, g_R\right) m_R\tag{9}$$

$$g = Z^{-3/2}\left(\frac{\Lambda}{\mu}, \frac{m_R}{\mu}, g_R\right) g_R\tag{10}$$

Observe that, on account of inequality (1), the bare charge (10) either stays bounded (if $Z(\infty, \dots) \neq 0$) or is infinitely renormalized (if $Z(\infty, \dots) = 0$) but in no case can tend to zero as would be the case for an asymptotically free theory. Note also that, in Ref. 15), it has been pointed out that no fixed point exists which can be continuously reached from the origin. The Källén-Lehmann inequality (1) contains powerful and non-trivial information about properties of the full theory^{*)}. The second ingredient of our analysis is the use of Schwinger's equations⁹⁾. These are most conveniently obtained in the functional integral formalism by means of the identity

$$\int [d\phi] \prod_i \frac{\delta}{\delta \phi_i(x_i)} \{ R[\phi] \exp i S[\phi] \} = 0 \quad (11)$$

Here, ϕ denotes the multiplet (A, F, ψ) , $S[\phi]$ the action $\int d^4x \mathcal{L}^R$ and $R[\phi]$ an arbitrary function; ϕ_i may be taken to be any of the fields A, F or ψ [for convenience one may also choose to work with an Euclidean signature⁷⁾]. The crucial point is now that we can derive a set of equations from (11) for any $\Lambda < \infty$; we then remove the cut-off and obtain the corresponding equations for the renormalized theory. This step relies on our knowledge of the behaviour (9) and (10) of the bare parameters: if there were genuine mass and charge renormalizations, it would be very difficult to derive these equations for the renormalized propagators and vertices.

We first consider the case $Z(\infty, \dots) = 0$ (infinite renormalization). For this purpose we take the differential operator in (11) to be $(\delta/\delta F_R(x))(\delta/\delta F_R^*(y))$ and $R = 1$. The resulting equations are

$$\begin{aligned} & i Z\left(\frac{\Lambda}{\mu}, \dots\right) \delta(x-y) = \\ & = \left\langle \left\{ Z\left(\frac{\Lambda}{\mu}, \dots\right) F_R^*(x) + (m_R A_R(x) + g_R A_R^2(x)) \right\} \cdot \right. \\ & \quad \left. \cdot \left\{ Z\left(\frac{\Lambda}{\mu}, \dots\right) F_R(y) + (m_R A_R^*(y) + g_R A_R^{*2}(y)) \right\} \right\rangle \end{aligned} \quad (12)$$

Taking the limit $\Lambda \rightarrow \infty$ in this equation and assuming $Z(\infty, \dots) = 0$, one gets

$$\begin{aligned} & \left\langle (m_R A_R(x) + g_R A_R^2(x)) (m_R A_R^*(y) + g_R A_R^{*2}(y)) \right\rangle = \\ & = 0 \end{aligned} \quad (13)$$

^{*)} For example, it is not valid order by order in perturbation theory as is immediately obvious from (8).

It is important here that the composite operator A_R does not require any additional renormalization because $(A^2)_R = (A_R)^2$ ^{*} and therefore all terms in (13) are well defined separately. The result (13) is in contradiction with the standard lore of renormalized perturbation theory which should be valid at least for sufficiently small values of the renormalized coupling g_R . For fixed $|x - y| \neq 0$ and small g_R , (13) becomes

$$m_R^2 \langle A_R(x) A_R^*(y) \rangle_{g_R} + O(g_R) = 0 \quad (14)$$

and therefore is arbitrarily close to the free field propagator which certainly does not vanish ! This contradiction can be more easily understood if one realizes that, in the limit $Z \rightarrow 0$, the WZ model degenerates into an ultra-local field theory: there are no kinetic terms, and therefore the functional integration measure becomes a product of ordinary integration measures over all space time points. In addition, integration over the fermions and the auxiliary fields shows that these measures are just δ function measures [there is a remarkable cancellation between the determinant resulting from the δ function and the fermionic determinant¹⁶⁾] and all Green's functions can be evaluated explicitly; such a theory is clearly without physical interest.

Turning now to the case $Z(\infty, \dots) \neq 0$, we will argue that, for sufficiently large momenta, the Green's functions of the massive WZ model coincide with those of the massless one; that is, mass terms are not relevant for the ultra-violet behaviour of the theory. To see this it is advantageous to formulate the renormalization group equations à la 't Hooft-Weinberg¹⁷⁾, such that the wave function renormalization parameter Z no longer depends on the renormalized mass m_R and therefore is the same as in the massless theory. The renormalization group equations for any n point Green's function $G_n(p_1, \dots, p_n)$ tell us that

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g_R) \frac{\partial}{\partial g_R} + m_R \gamma_m(g_R) \frac{\partial}{\partial m_R} + n \gamma(g_R) \right\} G_n(p_1, \dots, p_n; \mu, m_R, g_R) = 0 \quad (15)$$

where, from (9) and (10), we have

$$\beta(g_R) = 3 \gamma(g_R) g_R; \quad \gamma_m(g_R) = 2 \gamma(g_R) \quad (16)$$

^{*}) We are indebted to K. Sibold for pointing out to us that, in fact, $(\phi^n)_R = (\phi_R)^n$ for a chiral superfield ϕ and any n .

The solution of (15) reads as follows

$$\begin{aligned} G_n(\lambda p_i; \mu, m_R, g_R) &= \\ &= \lambda^{d(G_n)} G_n(p_i; \mu, \bar{m}(\lambda, g_R), \bar{g}(\lambda, g_R)) \end{aligned} \quad (17)$$

where $d(G_n)$ denotes the naïve dimension of G_n and

$$\lambda \frac{d}{d\lambda} \bar{g}(\lambda, g_R) = \beta(\bar{g}) ; \bar{g}(1, g_R) = g_R \quad (18)$$

$$\bar{m}(\lambda, g_R) = m_R \left(\frac{\bar{g}(\lambda, g_R)}{g_R} \right)^{2/3} \quad (19)$$

From $Z(\infty, \dots) \neq 0$, we infer that $\bar{g}(\lambda, g_R)$ stays bounded as $\lambda \rightarrow \infty$, and therefore $\bar{m}(\lambda, g_R)$ stays also bounded in this limit. Thus,

$$\lim_{\lambda \rightarrow \infty} \frac{\bar{m}(\lambda, g_R)}{\lambda p} = 0 \quad (20)$$

if $Z(\infty, \dots) \neq 0$. Next, we again use Eq. (11), this time taking the differential operator to be $\delta/\delta F_R(x)$ and $R[\phi] = F_R(0)$. In this way, one arrives at the following equation

$$\begin{aligned} i \delta(x) - Z(\infty, g_R) \langle F_R^*(x) F_R(0) \rangle &= \\ &= m_R \langle F_R(0) A_R(x) \rangle + \\ &+ g_R \langle F_R(0) A_R^2(x) \rangle \end{aligned} \quad (21)$$

or, in momentum space,

$$\begin{aligned} 1 - Z(\infty, g_R) D_R^{FF*}(p^2; \mu, m_R, g_R) &= \\ &= m_R D_R^{FA}(p^2; \mu, m_R, g_R) + \\ &+ g_R \int G_R^{FAA}(p^2, (p-k)^2, k^2; \mu, m_R, g_R) \frac{d^4 k}{(2\pi)^4} \end{aligned} \quad (22)$$

where we have defined

$$D_R^{FF*}(p) = i \int \langle T(F_R^*(x) F_R(0)) \rangle e^{ipx} d^4x \quad (23)$$

etc.

From the supersymmetry Ward identities⁵⁾, we learn that

$$D_R^{FF*}(p^2, \dots) = p^2 D_R^{AA*}(p^2, \dots) \quad (24)$$

Inserting this into (22), differentiating with respect to p^2 and putting $p^2 = 0$, we get

$$\begin{aligned} & -Z(\infty, g_R) D_R^{AA*}(p^2 = 0, \dots) - \\ & - m_R \frac{\partial D_R^{FA}(p^2, \dots)}{\partial p^2} \Big|_{p^2=0} = \\ & = g_R \int \frac{\partial G_R^{FAA}(p^2, \dots)}{\partial p^2} \Big|_{p^2=0} \frac{d^4k}{(2\pi)^4} \end{aligned} \quad (25)$$

We have previously argued that in analyzing the ultra-violet behaviour, one may neglect the mass terms, see Eq. (20). In that limit, we can therefore write

$$\begin{aligned} & G_R^{FAA}(p^2, (p-k)^2, k^2, \dots) = \\ & \approx D_R^{FF*}(p^2) D_R^{AA*}((p-k)^2) D_R^{AA*}(k^2) \cdot \\ & \cdot \Gamma_R^{FAA}(p^2, (p-k)^2, k^2, \dots) \end{aligned} \quad (26)$$

where Γ_R^{FAA} is the one-particle irreducible three-point vertex function. For the massless theory, the value of Γ_R^{FAA} at $p^2 = 0$ is equal to $2g_R$ due to non-renormalization theorems⁵⁾. Using this knowledge and the Ward identity (24) once more, we find that the right-hand side of (25) is dominated by a term proportional to

$$D_R^{AA*}(p^2 = 0, \dots) \int \left[D_R^{AA*}(k^2, \dots) \right]^2 \frac{d^4k}{(2\pi)^4} \quad (27)$$

From the Källen-Lehmann representation^{10),11)}

$$D_R^{AA*}(p^2, \dots) = \int_0^\infty \frac{\rho(t, \dots)}{p^2 + t - i\varepsilon} dt \quad (28)$$

where $\rho(t, \dots) \geq 0$ is the associated spectral density we deduce that $D_R^{AA*}(p^2, \dots)$ decreases as $1/p^2$ in the ultra-violet region. Therefore, the integral (27) and, thus, the right-hand side of (25) are ultra-violet divergent. But this contradicts the presumed finiteness of the left-hand side (there are no infra-red divergences for the massive theory). Thus, we obtain a contradiction for any value of the wave function renormalization $Z(\infty, g_R)$ compatible with the inequality (1), and we are led to the conclusion that the WZ model cannot be defined in the limit where the ultra-violet cut-off is removed. (Observe that our argument does not apply to WZ-type models in less than four dimensions.)

It appears that our main result also holds in the case of more general WZ type models with several chiral multiplets, since the main steps in our argument (no asymptotic freedom, Ward identities, etc.) can still be performed. It will be more difficult to see whether our methods also apply to supersymmetric gauge theories; however, the lack of asymptotic freedom in the spin-0 and spin-1/2 sector of such models may cause problems there as well.

ACKNOWLEDGEMENTS

One of us (N.V.K.) is indebted to the CERN TH Division for its kind hospitality. We are grateful to K. Symanzik and K. Sibold for useful discussions.

REFERENCES

- 1) D. Volkov and V.P. Akulov, Phys. Lett. 46B (1973) 109.
- 2) J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39;
see also:
P. Fayet and S. Ferrara, Phys. Rep. 32 (1977) 251;
A. Salam and J. Strathdee, Fortschritte der Physik 26 (1978) 57.
- 3) E. Witten, Nucl. Phys. B188 (1981) 513.
- 4) S. Dimopoulos and H. Georgi, Nucl. Phys. B193 (1981) 150;
N. Sakai, Zeit. für Physik C11 (1982) 153.
- 5) J. Wess and B. Zumino, Phys. Lett. 49 (1974) 52;
J. Iliopoulos and B. Zumino, Nucl. Phys. B76 (1974) 310.
- 6) J. Glimm and A. Jaffe, Phys. Rev. Lett. 33 (1974) 440;
J. Kogut and K.G. Wilson, Phys. Rep. 12 (1975) 75;
R. Schrader, Comm. Math. Phys. 49 (1976) 131; 50 (1976) 97.
- 7) H. Nicolai, Nucl. Phys. B140 (1978) 294; B156 (1979) 157, 177.
- 8) N.V. Krasnikov, CERN preprint TH.3407 (1982).
- 9) J. Schwinger, Proc. Nat. Ac. Sci. 37 (1951) 452.
- 10) G. Källen, Helv. Phys. Acta 23 (1952) 416; Nuovo Cim. 12 (1954) 217.
- 11) H. Lehmann, Nuov. Cim. 11 (1954) 342.
- 12) P. van Nieuwenhuizen, Phys. Rep. 68C (1981) 189.
- 13) J. Wess, Lectures given at Princeton University (1981).
- 14) N.N. Bogoliubov and D.V. Shirkov, "Introduction to the Theory of Quantized Fields", Interscience, New York (1959), Chapter VIII.
- 15) S. Ferrara, J. Iliopoulos and B. Zumino, Nucl. Phys. B77 (1974) 413.
- 16) H. Nicolai, Phys. Lett. 89B (1980) 341.
- 17) G. 't Hooft, Nucl. Phys. B61 (1973) 455;
S. Weinberg, Phys. Rev. D8 (1973) 3497.