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PERTURBATION THEORY FROM STOCHASTIC QUANTIZATION OF SCALAR FIELDS

W. Grimus

CERN - Geneva

and

H. Hüffel *)

Institut für Theoretische Physik
Univ. Wien, A-1090 Wien, Austria

ABSTRACT

By using a diagrammatical technique it is shown that in scalar theories the stochastic quantization method of Parisi and Wu gives the usual perturbation series in Feynman diagrams.

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1. INTRODUCTION

Parisi and Wu ¹⁾ introduced a stochastic quantization method which is based on the Langevin equation ²⁾ of non-equilibrium statistical mechanics. They applied their method to the quantization of scalar and gauge field theories with the main aim of constructing a new perturbation theory for gauge theories without introducing gauge fixing. Therefore for non-Abelian gauge theories their method is free of the Gribov ambiguity ³⁾.

In this paper we deal with scalar theories and show diagrammatically that the perturbation theory of Parisi and Wu gives the usual Feynman diagrams when calculating Green functions. Our paper is organized as follows : In sect. II we recall the basic features of the stochastic quantization method and how to obtain "stochastic diagrams" from an iterative solution of the Langevin equation. In sect. III we show how to perform the time integrations of stochastic diagrams by introducing time ordering of the vertices and in sect. IV we prove the equivalence of sums of stochastic diagrams and Feynman diagrams in the case of self-interacting scalar fields. According to a remark in Ref. 1) this has been shown in Ref. 4) but we find it hard to see what has been proved in this connection.

2. LANGEVIN EQUATION AND STOCHASTIC DIAGRAMS

The starting point of the discussion of stochastic diagrams is the introduction of a fictitious time for the fields whose time evolution is given by the Langevin equation

$$\frac{\partial \phi(x,t)}{\partial t} = - \frac{\delta S}{\delta \phi(x,t)} + \eta(x,t) \quad (2.1a)$$

$$S = \int d^D x \left\{ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi^3 + \frac{1}{4!} \lambda \phi^4 \right\} \quad (2.1b)$$

We have written down (2.1) for a real self-interacting scalar field. S is the Euclidean action in D space-time dimensions. The fictitious time t should not be confused with the physical time contained in x . η is a random source with Gaussian distribution

$$\langle \eta(x,t) \eta(x',t') \rangle_\eta = 2 \delta(x-x') \delta(t-t') \quad (2.2a)$$

$$\langle \eta(x_1, t_1) \dots \eta(x_{2n+1}, t_{2n+1}) \rangle_\eta = 0 \quad (2.2b)$$

$$\begin{aligned} \langle \eta(x_1, t_1) \dots \eta(x_{2n}, t_{2n}) \rangle_\eta &= \\ &= \sum_{\substack{\text{possible} \\ \text{pair comb.}}} \prod_{\text{pairs}} \langle \eta(x_i, t_i) \eta(x_j, t_j) \rangle_\eta \end{aligned} \quad (2.2c)$$

Explicitly we can write the random average over η

$$\langle \dots \rangle_\eta = \frac{\int [d\eta] (\dots) \exp(-\frac{1}{4} \int d^D x dt \eta^2(x, t))}{\int [d\eta] \exp(-\frac{1}{4} \int d^D x dt \eta^2(x, t))} \quad (2.3)$$

Now the crucial point is that in the limit $t \rightarrow \infty$ equilibrium is reached and the random average of correlation functions of $\phi(x, t)$ tends to the corresponding Green functions of the quantum field theory with action (2.1b)

$$\lim_{t \rightarrow \infty} \langle \phi(x_1, t) \dots \phi(x_L, t) \rangle_\eta = \langle \phi(x_1) \dots \phi(x_L) \rangle \quad (2.4)$$

This has been shown in Ref. 1) for scalar theories by using the Fokker-Planck equation 2). For gauge theories the situation is more complicated 5). In contrast to Ref. 1) we prove (2.4) by a diagrammatical technique in the case of scalar fields.

The Langevin equation (2.1a) can be transformed into an integral equation in momentum space

$$\begin{aligned} \phi(k, t) &= \int_0^t d\tau G(k, t-\tau) \left\{ \eta(k, \tau) - \right. \\ &\left. - \frac{1}{2!} g \int \frac{d^D p}{(2\pi)^D} \phi(p, \tau) \phi(k-p, \tau) - \frac{1}{3!} \lambda \int \frac{d^D p}{(2\pi)^D} \int \frac{d^D q}{(2\pi)^D} \phi(p, \tau) \phi(q, \tau) \phi(k-p-q, \tau) \right\} \end{aligned} \quad (2.5a)$$

$$G(k, t-\tau) = e^{-(t-\tau)(k^2+m^2)} \quad (2.5b)$$

where the boundary condition $\phi(x, 0) = 0$ has been used. However, any trace of a specific boundary condition should die out with $t \rightarrow \infty$ due to the equilibrium property of the system. In momentum space (2.2a) reads

$$\langle \eta(k, t) \eta(k', t') \rangle_\eta = 2(2\pi)^D \delta(k+k') \delta(t-t') \quad (2.6)$$

Solving (2.5) by iteration one arrives at a power series expansion of ϕ in the coupling constants, which can be written diagrammatically

$$\phi = \text{---} \times + \text{---} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} + \text{---} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} + \dots \quad (2.7)$$

$\begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array} \begin{array}{c} \times \\ \diagup \quad \diagdown \\ \times \end{array}$

where we denote G by a line and η by a cross; integration over the momenta and the fictitious times at all the vertices and crosses is included.

Let us now consider the L point function $\langle \phi(x_1, t) \dots \phi(x_L, t) \rangle_\eta$ and substitute for ϕ its diagrammatical expansion (2.7). When the random average over the η 's is taken all crosses are joined together in all possible ways due to the Wick-decomposition property (2.2c). In this way diagrams are obtained which we call stochastic diagrams. Each of them has the form of an ordinary Feynman diagram of the theory described by the action S apart from crosses on the lines where two η 's have been joined together (Fig. 1). Conversely, to every Feynman diagram there exists a number of stochastic diagrams with the same topology⁶⁾. Actually we will show in sect. IV that the sum of all stochastic diagrams to a given Feynman diagram yields just this Feynman diagram. Obviously cutting a stochastic diagram of an L point function at all crosses gives L connected trees contained in (2.7).

The momentum δ function in (2.6) guarantees momentum conservation at a cross and eliminates one momentum integration, so momentum integration is reduced just to the usual one of Feynman diagrams. Also the time integrations on both sides of a cross can be performed and lead to the propagator

$$\begin{aligned} D(k, \tau, \tau') &= 2 \int_0^{\min(\tau, \tau')} d\tau'' G(k, \tau - \tau'') G(k, \tau' - \tau'') = \\ &= \frac{1}{k^2 + m^2} \left(e^{-|\tau - \tau'| (k^2 + m^2)} - e^{-(\tau + \tau') (k^2 + m^2)} \right) \end{aligned} \quad (2.8)$$

τ, τ' are the times of the neighbour vertices. This means that in a stochastic diagram each line represents the Green function G and each crossed line the propagator D respectively (Fig. 2).

3. INTEGRATIONS OVER THE FICTITIOUS TIMES

In this section we will show how to perform the time integration of stochastic diagrams by introducing time ordering of the vertices. Let us consider a stochastic diagram belonging to some Feynman diagram with L external lines and N vertices. Each vertex carries a time τ_i which has to be integrated over between 0 and the time of some next vertex according to (2.7). Due to the absolute values of time differences in the D propagators it is however convenient to introduce fixed time orderings of the vertices. Given the Feynman graph we will allow all time orderings of vertices which are compatible with at least

one of the corresponding stochastic diagrams (e.g. in Fig. 2 τ_2 or τ_4 cannot be greater than all the other τ 's, because one could not go then from vertex 2(4) to an external leg via an increasing sequence of times at the vertices for any distribution of the crosses). In general, a given allowed time ordering belongs to several stochastic diagrams. In performing the time integrations we will only keep the largest terms thus leaving out the second term in D and always dropping terms, which come from the lower boundaries of the integrations. We will see that the integration over the largest terms compensates exactly the exponential t dependence of the G 's on the external lines and all smaller terms will go to zero exponentially in t .

Let us concentrate now on a fixed time ordering of the N vertices which we can choose to be

$$0 < \tau_1 < \tau_2 < \dots < \tau_N < t \quad (3.1)$$

We denote the set of momenta of the i -th vertex by V_i and leave out the momentum denominators of the D 's for the moment. For convenience of notation we substitute $p^2 + m^2 \rightarrow p^2$ for all momenta. Thus masses are trivially contained in our discussion. Now we integrate over τ_1 . Because it is the smallest time all τ_1 exponents in G 's and D 's have positive signs (except those we have already dropped from D , which do not concern us anymore) and we obtain

$$\int_0^{\tau_2} d\tau_1 e^{\tau_1 \sum_{V_1} p^2} = \frac{e^{\tau_2 \sum_{V_1} p^2} - 1}{\sum_{V_1} p^2} \quad (3.2)$$

We drop the 1 in (3.2) and go to the τ_2 integration. Now there are two possibilities :

- a) vertex 2 is not a neighbour vertex of the first one, which means that all neighbour vertices of vertex 2 have times larger than τ_2 . In this case again all τ_2 exponents have positive signs and

$$\int_0^{\tau_3} d\tau_2 e^{\tau_2 (\sum_{V_1} p^2 + \sum_{V_2} p^2)} = \frac{e^{\tau_3 \sum_{V_1 \cup V_2} p^2} - 1}{\sum_{V_1 \cup V_2} p^2} \quad (3.3)$$

The 1 will be dropped as before.

- b) vertex 2 is a neighbour of vertex 1. In this case special care has to be taken for the lines which connect vertices 1 and 2 as the corresponding G 's and D 's depend on both τ_1 and τ_2 ; they are of the form $e^{-(\tau_2 - \tau_1)p^2}$ before τ_1 integration. From this one sees immediately that all τ_2 exponents coming from momenta, which connect vertices 1 and 2, are cancelled by τ_1 integration. On the other hand all other neighbour vertices of 2 will give positive τ_2 exponents. Explicitly we have

$$\int_0^{\tau_3} d\tau_2 e^{\tau_2 (\sum_{V_1} p^2 - \sum_{V_1 \cap V_2} p^2 + \sum_{V_2 - V_1 \cap V_2} p^2)} = \frac{e^{\tau_3 \sum_{W_2} p^2} - 1}{\sum_{W_2} p^2} \quad (3.4a)$$

$$W_2 = V_1 \cup V_2 - V_1 \cap V_2 \quad (3.4b)$$

(3.4) also includes (3.3), so it is the general result of the second integration. Using the same arguments as in (3.4) one obtains for the k^{th} integration

$$\int_0^{\tau_{k+1}} d\tau_k e^{\tau_k (\sum_{W_{k-1}} p^2 - \sum_{\substack{U(V_i \cap V_k) \\ 1 \leq i < k}} p^2 + \sum_{V_k - \bigcup_{1 \leq i < k} (V_i \cap V_k)} p^2)} = \frac{e^{\tau_{k+1} \sum_{W_k} p^2} - 1}{\sum_{W_k} p^2} \quad (3.5a)$$

$$W_k = (W_{k-1} - \bigcup_{1 \leq i < k} (V_i \cap V_k)) \cup (V_k - \bigcup_{1 \leq i < k} (V_i \cap V_k)) = \bigcup_{i=1}^k V_k - \bigcup_{1 \leq i < j \leq k} (V_i \cap V_j) \quad (3.5b)$$

$$W_1 \equiv V_1, \tau_{N+1} \equiv t$$

(3.5b) can easily be shown by induction.

Now we would like to make some remarks to (3.5).

- a) The procedure is independent of the number of momenta leading to a vertex.
- b) The contribution from the N^{th} time integration is given by

$$\frac{e^{t \sum_{\text{external lines}} p^2} - 1}{\sum_{\text{external lines}} p^2} \quad (3.6)$$

so that the t exponents of the G 's and D 's corresponding to external lines are exactly cancelled. We see at this point that the neglect of non-leading contributions is justified due to their exponential fall off in t .

- c) From all time integrations we finally get for $t \rightarrow \infty$

$$\prod_{k=1}^N \frac{1}{\sum_{W_k} p^2} \quad (3.7)$$

- d) Note that leaving out the last time integration in (3.7) we get exactly the expression of the time integration of a diagram obtained from the original one by dropping the τ_N vertex with its external line(s). In the example of Fig. 3 for instance we get

$$\frac{1}{k_1^2 + p_1^2 + p_3^2} \frac{1}{k_1^2 + k_2^2 + p_2^2 + p_3^2} \frac{1}{k_1^2 + k_1^2 + k_3^2} \rightarrow \frac{1}{k_1^2 + p_1^2 + p_3^2} \frac{1}{k_1^2 + k_2^2 + p_2^2 + p_3^2} \quad (3.8)$$

e) Given a time ordered stochastic diagram we find therefore

$$\prod_{k=1}^N \frac{1}{\sum_{w_k} p^2} \prod_{crosses} \frac{1}{p^2} \quad (3.9)$$

where the second product of momenta arises simply from the so far neglected denominators of the D 's.

f) To obtain the full contribution of a stochastic diagram one has to sum (3.9) over all allowed time orderings.

4. EQUIVALENCE PROOF

In this section we prove by induction on the number of vertices N of a given Feynman diagram that the sum over all stochastic diagrams with the same topology gives exactly the Feynman diagram in the case of the real scalar field theory (2.1b).

For $N = 1$ we only have the 3 point and the 4 point vertex function in lowest order. According to the simplest application of the time integration rule (3.7) and (3.9) we get

$$-\frac{1}{2!} g \left(\begin{array}{c} k_3 \\ \diagup \quad \diagdown \\ * \quad * \\ \diagdown \quad \diagup \\ k_1 \end{array} \quad * \quad k_2 + \begin{array}{c} k_3 \\ * \quad \diagdown \\ \diagup \quad * \\ \diagdown \quad \diagup \\ k_1 \end{array} \quad * \quad k_2 + \begin{array}{c} k_3 \\ * \quad \diagdown \\ \diagup \quad * \\ * \quad \diagup \\ k_1 \end{array} \quad k_2 \right) \cdot 2! \quad (4.1a)$$

$$= -g \frac{1}{k_1^2 + k_2^2 + k_3^2} \left(\frac{1}{k_1^2 k_2^2} + \frac{1}{k_2^2 k_3^2} + \frac{1}{k_3^2 k_1^2} \right) = \frac{-g}{k_1^2 k_2^2 k_3^2}$$

$$-\frac{1}{3!} \lambda \left(\begin{array}{c} k_4 \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ * \quad * \quad * \quad * \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ k_1 \quad k_2 \end{array} + \begin{array}{c} k_4 \\ * \quad \diagdown \quad \diagup \quad * \\ \diagup \quad * \quad \diagdown \quad * \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ k_1 \quad k_2 \end{array} + \dots \right) \cdot 3! = -\lambda \frac{1}{k_1^2 + k_2^2 + k_3^2 + k_4^2} \left(\frac{1}{k_1^2 k_2^2 k_3^2} + \frac{1}{k_2^2 k_3^2 k_4^2} + \dots \right) = \frac{-\lambda}{k_1^2 k_2^2 k_3^2 k_4^2} \quad (4.1b)$$

$2! (3!)$ is the number of possibilities of joining the η 's together to obtain the desired Feynman diagram and for the crosses we have inserted the propagators stemming from the D 's according to (3.9). We have exactly obtained the corresponding Feynman diagrams. Because of the expansion (2.7) all Green functions we consider are non-truncated.

Now we assume that we have proved the equivalence to Feynman diagrams for all numbers of vertices smaller than N with any number of external lines.

Let us consider a Feynman diagram F with N vertices and L external lines and a stochastic diagram S_F with the same topology. In F and S_F momentum integration is not contained. It is easy to see that S_F has the number of crosses

$$Z = \frac{L + N_3 + 2N_4}{2} \quad (4.2)$$

where $N_{3(4)}$ is the number of 3(4) vertices of F and $N = N_3 + N_4$.

Now we introduce time ordering of the vertices compatible with S_F . The largest time has to be at a vertex with at least one external line (see (2.7) and sect. III). For an external line there are five topological possibilities shown in Fig. 4. All lines leading into the blobs of Fig. 4 are inner lines. We call the largest time τ_N . With k 's we denote external momenta and with p 's inner momenta. Integration over τ_N yields

$$\frac{1}{\sum_{i=1}^L k_i^2} \equiv K \quad (4.3)$$

We discuss now the different cases of Fig. 4 :

Case (a)

Because τ_N is the largest time there cannot be a cross on k_1 . Removing the momentum k_1 and $-1/2! g$ from F we obtain a Feynman diagram F' (Fig. 5a). Doing the same with S_F we get $\tilde{S}_F$. The factor $1/2!$ comes from the first iteration of $\phi(k_1)$ and is exactly needed for a loop with two identical bosons. The change in the number of crosses going from topology F to F' is $\Delta Z = 0$ (4.2). Because there was no cross on k_1 this means that $\tilde{S}_F$ has the right number of crosses to be a stochastic diagram with topology F' and because of remark d) in section III the time integration of S_F with the given time ordering is just (time integration of $\tilde{S}_F$) $\times K$. Now summing over all time ordered stochastic diagrams S_F keeping τ_N as largest time one obtains (sum over all stochastic diagrams of F') $\times K$. According to our induction assumption the sum over all stochastic diagrams of F' gives F' , because F' has one vertex less than F . So we obtain for the sum over all time ordered stochastic diagrams of F with τ_N as largest time

$$-\frac{1}{2!} g K F' = k_1^2 K \left(-\frac{1}{2!} g\right) \frac{1}{k_1^2} F' = \frac{k_1^2}{\sum_{i=1}^L k_i^2} F \quad (4.4)$$

Case (b)

There must be now either a cross on k_1 or k_2 , so we consider the two stochastic diagrams of Fig. 6. Because the iterative expansion (2.7) of $\phi(k_2)$ in Fig. 6a for example contains two identical pieces we have only $-g$ at the vertex τ_N (see Fig. 7) in accordance with the Feynman rules.

Now we remove k_1, k_2 and $-g$ from F and obtain F' (Fig. 5b) and $\tilde{S}_F$. The change in the number of crosses going from F to F' is $\Delta Z = -1$. Again $\tilde{S}_F$ is a correct stochastic diagram with topology F' . Doing the same summation as in case (a) we get

$$-g \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} \right) KF' = \frac{k_1^2 + k_2^2}{\sum_{i=1}^L k_i^2} F \quad (4.5)$$

$1/k_1^2$ and $1/k_2^2$ come from the D propagators at the places of the crosses in Fig. 6.

Analogously case (c) yields the same as (a), case (d) the same as (b) and case (e) gives

$$\frac{k_1^2 + k_2^2 + k_3^2}{\sum_{i=1}^L k_i^2} F \quad (4.6)$$

Thus summing over all possible places for the largest time τ_N we get a contribution $k_i^2 KF$ from each external leg with momentum k_i and all the contributions sum up to give F .

We want to stress that the correct momentum integration of the Feynman diagram has already been obtained by the iterative expansion of ϕ (2.7) and the random average over the η 's (2.6).

We have shown now the equivalence in the case of one real scalar field. It is straightforward to generalize our considerations to many scalar fields and complex scalar fields. Then every field has its own random source, also the charge conjugate of a complex field ¹⁾. The time integrations are the same as in section III and the only point, where one has to be careful in section IV, is to get the right combinatorial factors of the Feynman diagrams, but this can easily be seen from the iterative expansion of the fields.

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Though their method of stochastic quantization is quite different from that of Parisi and Wu, the diagrammatic representation is the same in both methods, but the interpretation is different.

FIGURE CAPTIONS

- Fig. 1 : All stochastic diagrams belonging to a given Feynman diagram. This is an example of what is proved in section IV.
- Fig. 2 : A stochastic diagram and the times and propagators associated to it. In each of the G 's the first time must be greater than the second one because of the time integration in the expansion (2.7). Momenta have been omitted.
- Fig. 3 : This figure illustrates remark d) in section III. The second diagram has been obtained from the first one by removing the vertex with the largest time τ_3 and the external line k_3 .
- Fig. 4 : The topological possibilities for an external line in a self-interacting scalar field theory with action (2.1b).
- Fig. 5 : Removing the vertex with the largest time τ_N and its external lines from F one obtains F' .
- Fig. 6 : The two possibilities for crosses on the external lines of the vertex with τ_N of Fig. 4b.
- Fig. 7 : This figure shows how the $1/2!$ at the vertex with time τ_N in Fig. 6a is cancelled by two identical terms in the expansion of $\phi(k_2)$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5a

Fig. 5b

Fig. 6

Fig. 7