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REDUCED CHIRAL $U(N) \times U(N)$ MODEL

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A B S T R A C T

We investigate analytically the reduced chiral model for any N and any dimension d . The relevant group integral for computing the free energy and correlation functions is performed in the semi-classical approximation which becomes exact for $d = 1$ any N , and some special values of N for any d . The quenching procedure is described and the appropriate expressions, which should be investigated in the large N limit are written down. For the one-dimensional case the free energy is computed and it agrees with the exact result. The relevance of this reduced model for the underlying continuum chiral theory is discussed and the structure of the vacuum is analyzed.

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1. - INTRODUCTION

The $SU(N) \otimes SU(N)$ matrix chiral models are of great interest because of the resemblance they share with the $SU(N)$ gauge theories. Some relationships between chiral and gauge theories were noted in the past. The similarities between the chiral models in two dimensions and the gauge theory in four dimensions were emphasized by Migdal and Polyakov^{1),2)}. In particular, both theories are asymptotically free (in two and four dimensions, respectively), and share similar global topological structure. Since the chiral models are much easier to handle, they have been extensively used as a laboratory to check ideas and methods which eventually were applied to gauge theories³⁾.

Recently, following a modification⁴⁾ of the Eguchi-Kawai⁵⁾ approach, a general $N \rightarrow \infty$ reduction prescription was suggested^{6),7)} for treating theories with global or local $U(N)$ symmetry both on the lattice and in the continuum⁶⁾⁻¹¹⁾. The reduced model was shown to give the large N limit of the original model in the sense that the reduced model reproduces all the leading planar diagrams^{6),7)} (in the large N limit). The reduced model, thus, defines the full sum of the planar graphs and includes non-perturbative contributions. This important development raises the hope that one would be able to eventually solve the continuum reduced gauge theory^{7),8)} in the large N limit. It is a common belief that the physics in this limit is similar to the physics of QCD ($N = 3$). Hence any progress towards solving the large N limit is very welcome.

We must add a warning that while it is easy to see the equivalence of the reduced model to the underlying model for less than N loops, the equivalence is blurred for more than N planar loops. This is because the reduced model has N momenta and thus could not describe more than N perturbative loops (but it does describe other non-perturbative effects beyond N loops). It is possible that the solution to the question we raise is simply that since every term for more than N loops is multiplied by $(g^2)^N$, they may disappear for $g^2 < 1$ $N \rightarrow \infty$, by vanishing exponentially as $e^{N \ln g^2} \rightarrow 0$, provided there are no divergences in the sum. We will assume that this is the case in reduced models, and in the chiral model in particular. This assumption is reasonable in view of 't Hooft's recent claim of Borel summability of planar diagrams.

The reduced model is equivalent (certainly for symmetric quantities) to the original one only in the large N limit. Yet, it would be extremely interesting to investigate it for finite N as well. We might be lucky enough that the convergence to $N \rightarrow \infty$ is approached very fast so that the finite N results may already provide a meaningful approximation (specially because of the non-perturbative effects). The $d = 1$ chiral model provides a laboratory to

investigate such questions since it is completely solvable. If convergence is fast, it would be very interesting from the point of view of comparing QCD with $N = 3$ to $N \rightarrow \infty$. Indeed, the $N = 3$ reduced QCD theory ^{7),8)} should be compared to the real world in any case, as a phenomenological model.

The present paper is devoted to the investigation of the reduced chiral model. In the presence of a cut-off, we find a semiclassical approximation to the (continuum) model for any N in any number of dimensions d . For $d=1$ the semiclassical approximation is exact. Here we will mainly concentrate on the free energy and the two-point correlation function. We derive, for finite N , a closed expression for these quantities as a function of the fixed N momenta which then should be integrated. For large N the integration over momenta can be replaced by the quenching procedure which we describe in detail. Thus, we replace the momenta by a fixed flat distribution and take the large N limit. For $d = 1$ or $d = 2$ dimensions, the renormalization limit corresponds to $\beta \rightarrow \infty$ where $\beta = (\Lambda/2\pi)^{-d+2}/g^2(\Lambda)$, Λ is the cut-off and $g^2(\Lambda)$ the (bare) coupling. In contrast, for $d > 2$, $\beta \rightarrow 0$. We shall be interested mainly in the $d = 1$ and $d = 2$ cases. The exact solution of the $d = 1$ chiral model is included in the Appendix, hence it can serve us as a check on our results and the quenching prescription for large as well as for finite N . As far as the analogy with the gauge theory in four dimensions goes, it is the two-dimensional renormalizable case which is of interest.

The action of the reduced model, unlike the original one, is not invariant under $U(N)_L \times U(N)_R / U(1)$. This should not come as a surprise since the reduced model includes the momenta as constants in $N \times N$ space, and reproduces the integrand rather than the integral of planar graphs. However, for any value of the momenta, the reduced model does possess a global $[U(1)_L]^N \times [U(1)_R]^N / U(1)$ symmetry which is a remnant of the original invariance. Naïvely, one would expect this remaining symmetry to force the vacuum to be in the unbroken phase of the chiral symmetry; is this correct¹¹⁾? Indeed, in two dimensions, Coleman's theorem requires the absence of Goldstone bosons and hence the vacuum of the underlying field theory should be in the unbroken symmetry phase. Thus, for the reduced model to be equivalent to the underlying model, it is necessary to show that the vacuum of the reduced model is indeed in the unbroken phase. In general, having a one-to-one correspondence between the perturbative planar diagrams is insufficient without ensuring that the vacuum is the correct one. We will argue in Section 4 that our reduced model has the correct vacuum, in disagreement with Ref. 11). The diagonal elements of the $N \times N$ matrix play an important rôle in enforcing the symmetry and they should not be constrained naïvely.

The plan of the paper is as follows. In Section 2 we review three approaches to the quantization of the chiral model : path integral, Hamiltonian, reduced model. We include some new comments on quantization in the Hamiltonian approach. For $d = 2$ we introduce a set of variables that show further relations to gauge theories. The reduced model and the calculational procedures with it are described in detail in this Section. In Section 3, we obtain a semiclassical approximation to the $U(N)$ group integral for the reduced model which may be of interest in its own right. The result is exact for $d = 1$ but not for $d \geq 2$. Based on this result, a closed formula for the free energy and the two-point function is provided. After quenching the momenta, further algebra is performed to completely evaluate the $d = 1$ free energy for any value of the cut-off Λ . Based on the leading semiclassical approximation, the free energy for $d = 2$ is also evaluated in the limit $\Lambda \rightarrow \infty$. In section 4 we discuss the vacuum and order parameter of the reduced model and argue that the chiral symmetry is unbroken. Here, special attention is paid to the diagonal matrix elements of the $N \times N$ matrix which turn out to be the zero modes of the chiral symmetry. An Appendix contains the exact solution of the $d = 1$ chiral model in both the Hamiltonian and path integral approaches. Since the $d = 1$ chiral model is the only exactly solvable $N \times N$ matrix field theory, the results of this Appendix provide important test cases for the new reduced model approach.

2. - $U(N)_L \times U(N)_R$ CHIRAL MODEL IN d DIMENSIONS

In this Section we review three approaches to the quantization of the model which will be useful in our analysis.

A. - Field theoretical path integral

The model we wish to study is defined by the following action, written in Euclidean space

$$S = \frac{N}{2g^2} \int d^d x \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) \quad (2.1)$$

where a sum over $\mu = 1, 2, \dots, d$ is implied. The $U(N)$ unitary matrix $U_1^j(x)$, $i, j = 1, 2, \dots, N$ satisfies the constraint

$$(U^\dagger(x) U(x))_i^j = \delta_i^j \quad (2.2)$$

at every x . It would be convenient to define the action on a lattice by replacing every derivative by a difference

$$S = \frac{Na^{d-2}}{2g^2} \sum_{\vec{n}, \hat{\mu}} \text{Tr} \left(2 - U_{\vec{n}+\hat{\mu}} U_{\vec{n}}^\dagger - U_{\vec{n}} U_{\vec{n}+\hat{\mu}}^\dagger \right) \quad (2.3)$$

where the lattice distance (a) plays the rôle of a cut-off.

The quantum mechanical path integral provides any n point Green function in the form of "averages"

$$G(x_1, x_2, \dots, x_n)_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n} = \frac{1}{Z} \int \prod_x (dU) e^{-S(U)} U_{i_1}(x_1)^{j_1} U_{i_2}(x_2)^{j_2} \dots U_{i_n}(x_n)^{j_n} \quad (2.4a)$$

where the "partition function" Z is given by

$$Z = \int \prod_x (dU) e^{-S(U)} \quad (2.4b)$$

We will use the notation of averages

$$G(x_1, x_2, \dots, x_n)_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n} = \langle U_{i_1}(x_1)^{j_1} U_{i_2}(x_2)^{j_2} \dots U_{i_n}(x_n)^{j_n} \rangle \quad (2.5)$$

The integrals are performed with the Haar measure for the U(N) unitary matrix U(x) at every $\vec{x}$, or every lattice site. When the symmetry is unbroken, the non-vanishing Green functions involve an equal number of U's and U[†]'s in Eq. (2.5). This is certainly the case for $d \leq 2$ (Coleman's theorem).

From the invariance of the action and the Haar measure under both left (V_L) and right (V_R) unitary transformations on U,

$$\begin{aligned} U &\longrightarrow V_L U \\ U &\longrightarrow U V_R^\dagger \end{aligned} \quad (2.6)$$

we can deduce many relations among the non-vanishing components of the Green functions. For example,

$$\langle U_i^j(x) U_k^l(y) \rangle = \frac{1}{N} \delta_i^l \delta_k^j \langle \text{Tr}(U(x) U^\dagger(y)) \rangle \quad (2.7)$$

Therefore, we can concentrate, in the unbroken case, on particular components i, j, k, l, or on invariant combinations such as $\text{Tr}(U(x) U^\dagger(y))$ in order to get the full answer.

The path integral is normally performed by using the lattice form of the action, and at the end remove the cut-off ($a \rightarrow 0$) by performing renormalization, if necessary. In one dimension the theory is finite (for the exact solution see the Appendix). In two dimensions it requires renormalization.

B. - Canonical formalism

The quantum theory can also be treated in the canonical Hamiltonian formalism. This is done through the Minkowski version of the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{N}{2g^2} \text{Tr} (\partial_\mu U \partial^\mu U^\dagger) \\ &= - \frac{N}{2g^2} \text{Tr} ((U^\dagger \partial_\mu U)(U^\dagger \partial^\mu U)) \end{aligned} \quad (2.8)$$

and defining the canonical conjugate as

$$\Lambda_i^j = \frac{\delta \mathcal{L}}{\delta (\partial_0 U_j^i)} = - \frac{N}{g^2} (U^\dagger \partial_0 U U^\dagger)_i^j \quad (2.9)$$

The naive commutation rules are inconsistent with the constraint $U^\dagger U = 1$. After taking the constraint into account via the Dirac formalism, the correct canonical commutation rules can be obtained¹²⁾. First we define the following combinations of canonical variables

$$\begin{aligned} \left(\frac{\lambda^a}{2}\right)_i^j G_L^a &= (G_L)_i^j \equiv \frac{i}{2} (\Lambda U)_i^j = \frac{N}{2g^2} [U^\dagger (-i \partial_0 U)]_i^j \\ \left(\frac{\lambda^a}{2}\right)_i^j G_R^a &= (G_R)_i^j \equiv \frac{-i}{2} (U \Lambda)_i^j = \frac{N}{2g^2} (i \partial_0 U U^\dagger)_i^j \end{aligned} \quad (2.10)$$

These are proportional to the zeroth component of the conserved currents associated with the invariances V_L and V_R of Eq. (2.6). They turn out to satisfy the equal time commutation rules given below, which are obtained from the Dirac brackets between U and Λ ¹²⁾.

$$\begin{aligned} [G_R^a(\bar{z}), G_R^b(\bar{y})] &= i f^{abc} G_R^c(\bar{z}) \delta^{(d-1)}(\bar{z} - \bar{y}) \\ [G_L^a(\bar{z}), G_L^b(\bar{y})] &= i f^{abc} G_L^c(\bar{z}) \delta^{(d-1)}(\bar{z} - \bar{y}) \\ [G_L^a(\bar{z}), U_i^j(\bar{y})] &= - \left(\frac{\lambda^a}{2} U(\bar{y})\right)_i^j \delta^{(d-1)}(\bar{z} - \bar{y}) \\ [G_R^a(\bar{z}), U_i^j(\bar{y})] &= (U \frac{\lambda^a}{2})_i^j \delta^{(d-1)}(\bar{z} - \bar{y}) \\ [G_L^a(\bar{z}), G_R^b(\bar{y})] &= 0 \end{aligned} \quad (2.11)$$

Thus, the $G_L^a(\vec{x})$, $G_R^a(\vec{x})$ act like the generators of a local $U(N) \times U(N)$ algebra. From Eq. (2.10) it is also seen that the left and right quadratic Casimir operators are equal to each other

$$G_L^a(x) G_L^a(x) = G_R^a(x) G_R^a(x) \quad (2.12)$$

The Hamiltonian density can now be written as

$$\mathcal{H} = \frac{g^2}{N} G_R^a(x) G_R^a(x) + \frac{N}{2g^2} \text{Tr} (U^\dagger \partial U)^2 \quad (2.13)$$

so that the first term is the local Casimir operator. This Hamiltonian can be put into special forms in one and two dimensions:

In one dimension (only time) there is no $\vec{x}$ dependence and the Hamiltonian is just equal to the quadratic Casimir operator constructed from the $U(N) U(N)$ charges

$$H = \frac{g^2}{N} G_R^2 \quad (2.14)$$

Furthermore, in the commutation rules, Eq. (2.11), there are no delta functions. In this form all eigenstates and eigenvalues can immediately be obtained, for any N , as in the Appendix.

In two dimensions, it is convenient to define new canonical variables analogous to a gauge potential A^a and its canonical conjugate E^a (like an electric field). Defining A and E through relations to U and G as

$$\begin{aligned} \frac{i\sqrt{N}}{g} (U^\dagger \frac{\partial U}{\partial x})_i^j &\equiv (\lambda^a)_i^j A^a \\ \frac{g}{\sqrt{N}} G_L^a(x) &= \frac{\partial E^a}{\partial x} + \frac{g}{\sqrt{N}} f^{abc} A^b E^c \equiv (DE)^a \end{aligned} \quad (2.15)$$

where D is a covariant derivative, we find that the commutation relations between U and G become equivalent to the canonical commutation rules between A^a and E^a

$$[A^a(x), E^b(y)] = i \delta^{ab} \delta(x-y) \quad (2.16)$$

The Hamiltonian density takes then the form

$$\mathcal{H} = (\mathcal{D}E^a)^2 + (A^a)^2 \quad (2.17)$$

This form is convenient for perturbative expansions in powers of g and shows many analogies to gauge theories.

C. - The reduced model for large N

In Refs. 6) and 7), a prescription is given for building a reduced model which is equivalent to the large N limit of any $N \times N$ matrix model. The method could be useful in gauge theories as well as chiral models. We apply this recipe to the chiral model. First $U(x_\mu)$ is translated to the origin as

$$U(x_\mu) = e^{-i P_\mu \cdot x} U e^{i P_\mu \cdot x} \quad (2.18)$$

where U at the origin is independent of x_μ . Then, instead of the translation operator one uses an $N \times N$ diagonal matrix P_μ

$$P_\mu = \begin{pmatrix} P_\mu^1 & 0 & 0 & \dots \\ 0 & P_\mu^2 & 0 & \dots \\ 0 & 0 & P_\mu^3 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ & & & P_\mu^N \end{pmatrix} \quad (2.19a)$$

With this prescription we obtain

$$i \partial_\mu U(x) \longrightarrow [P_\mu, U] \quad (2.19b)$$

In addition, we replace the volume $\int d^d x$ by $(2\pi/\Lambda)^d$. Thus, the Euclidean action of Eq. (2.1) is reduced to an action containing a single matrix U and $N \times d$ constant momenta

$$\begin{aligned} S(U, P) &= \left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{g^2(\Lambda)} \text{Tr} \left\{ -P_\mu U P_\mu U^\dagger + P^2 \right\} \\ &= \left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{g^2(\Lambda)} \sum_{i,j}^N (\bar{P}_i - \bar{P}_j)^2 |U_{ij}|^2 \end{aligned} \quad (2.20)$$

Note that since P_μ is diagonal, this action is invariant under $(U(1)_L)^N \times [U(1)_R]^N / U(1)$ which is a remnant of the original $U(N)_L \times U(N)_R / U(1)$. We point out that the diagonal elements of U_{ij} are absent in the action automatically. Thus they do not contribute to propagators. They are in fact zero modes related to the symmetries of the theory. We will deviate from Refs. 7) and 9) by including

them in the measure of integration below, and as we will discuss in Section 4, they will be responsible for the correct vacuum structure ^{*}).

Using the reduced action, the Green functions of Eq. (2.5) can be calculated in the $N \rightarrow \infty$ limit exactly as follows. First, define a "partition function"

$$Z_N(P) = \int dU e^{-S(U, P)} \quad (2.21)$$

where the integral is done with the $U(N)$ Haar measure over the single $U(N)$ matrix U for any fixed momenta. The free energy per unit volume follows from

$$\left(\frac{2\pi}{\Lambda}\right)^d \mathcal{E}(\Lambda, g(\Lambda), d) = \lim_{N \rightarrow \infty} \left[\frac{1}{N^2} \ln Z_N(P) \right]_{\text{quenched}} \quad (2.22)$$

where the momenta P_i are frozen (quenched) in a way specified below. The quenching corresponds to doing the integrals (with cut-off Λ) over the momenta in planar diagrams order by order in perturbation theory as we shall argue below. Then define "averages" with the reduced action for any fixed momenta,

$$\langle U_{i_1}^{j_1} U_{i_2}^{j_2} \dots U_{i_n}^{j_n} \rangle_P = \frac{1}{Z_N(P)} \int dU e^{-S(U, P)} U_{i_1}^{j_1} U_{i_2}^{j_2} \dots U_{i_n}^{j_n} \quad (2.23)$$

Again, provided the symmetry is unbroken (see Section 4) a non-zero average is obtained only if there are an equal number of U 's and $U^\dagger$'s and the indices are correlated as, e.g., Eq. (2.7). (There is index or charge conservation because of the $(U(1)_L)^N \times [U(1)_R]^N / U(1)$.) The x dependence in the Green functions of the original theory is recovered by substituting the reduced form

$$U_i^j(x) \rightarrow \left(e^{-iP \cdot x} U e^{iP \cdot x} \right)_i^j = e^{-i(\vec{P}_i - \vec{P}_j) \cdot \vec{x}} U_i^j \quad (2.24)$$

For example, in order to obtain the correlation function $\langle U_1^j(x) U_j^{\dagger 1}(0) \rangle$ of the original theory, Eq. (2.7), we construct the Green function $G(x)$ as

^{*}) In Ref. 7), it was shown that diagonal constraints on a gauge potential are necessary. This is because the gauge potential is gauge variant. If instead, gauge invariant variables are used, one can follow the naïve reduction procedure without diagonal constraints on gauge invariant variables. The resulting model reproduces automatically the diagonal constraints on the gauge potential as shown in Refs. 7,8).

$$\sum_{i \neq j} e^{-i(P_i - P_j) \cdot x} \langle U_i^i U_j^i \rangle_P = G(x, P) \quad (2.25)$$

in the reduced theory. Next, one must perform the following steps :

- quench all momenta (see below)

$$G(x, \Lambda) = [G(x, P)]_{\text{quenched}} \quad (2.26)$$

- let $N \rightarrow \infty$,

- renormalize and remove the cut-off Λ .

Fortunately, it is possible to avoid doing the momentum integrals, provided we consider only singlet Green functions [as justified by Eq. (2.7)]. Then in the underlying infinite volume theory, one notices that a contribution from some index loop "i" in a planar graph such as Fig. 1 can be put in the form

$$\sum_i \int dp_i F(p_i, \dots) \quad (2.27a)$$

where for every momentum integral over p_i there is also a sum over i . The dots in $F(p_i, \dots)$ represent the momenta of other loops. Since the momentum p_i is a dummy variable, we could replace it by p and then do the sum over i to recover the usual factor of N and the usual momentum dependence of the graph

$$N \int_{\Lambda} d^d p F(p, \dots) \quad (2.27b)$$

However, by using the form of (2.27a), we can justify the quenching of momenta as follows.

For simplicity of the discussion, let us specialize to one dimension. Then the integral in eq. (2.27b) can be approximated by discretizing it into N pieces

$$dp = \frac{\Lambda}{2\pi N} ; \quad p \rightarrow \frac{i\Lambda}{2\pi N} ; \quad -\frac{N}{2} \leq i \leq \frac{N}{2} \quad (2.28)$$

$$N \int_{\Lambda} dp F(p, \dots) \rightarrow \frac{\Lambda}{2\pi} \sum_i F(i\Lambda/2\pi N, \dots)$$

However, comparing eq. (2.28) to (2.27a) we see that the net result is equivalent to omitting the momentum integration $\int dp_i$ in (2.27a) provided the momenta are quenched in a uniform and dense way. It was shown in refs. (6,7), that the momentum dependence of the planar diagrams $F(p_1, \dots)$ was correctly reproduced by the reduced model. The model also supplies automatically all summations as well as all factors of $\Lambda/2\pi$ for correct dimensionality. Therefore, the reduced model automatically performs the momentum integrations provided the momenta are quenched. While our description of quenching is in agreement with the order by order integration of the momenta in planar graphs, as it should, it is not obvious how it could be made to agree with the integration prescription of ref. 7. It is, however, in agreement with their quenching prescription. We will thus always use quenching to define the Green functions of the physical theory. This justification of quenching is, of course, more general than the chiral model and applies generally to all large N reduced models.

We argued that the net result for $d = 1$ is that the N momenta p_i ($i = 1, 2, \dots, N$) can be frozen (quenched) to the values $p_i = i\Lambda/2\pi N$ and the integration completely omitted. Since all expressions depend only on the differences $(p_i - p_j)$ the origin in momentum space can be chosen anywhere. Then, in one dimension, we can simply take

$$P_i = \frac{i}{N} \frac{\Lambda}{2\pi} \quad ; \quad 1 \leq i \leq N \quad (2.29)$$

To do the quenching in d dimensions we label N cells $i = 1, 2, \dots, N$ by specifying a set of d integers $\vec{i}_\mu$, for each cell i

$$\vec{i} = (i_1, i_2, \dots, i_d) \quad (2.30)$$

The range of each integer is taken as

$$1 \leq i_\mu \leq N^{1/d} \quad (2.31)$$

so that N cells cover uniformly the space. We then freeze the momenta P_i by taking their value to be equal to the cell label up to an over-all factor,

$$\vec{P}_i = \frac{\Lambda}{2\pi N^{1/d}} (\vec{i}_1, \vec{i}_2, \dots, \vec{i}_d) \quad (2.32)$$

In two dimensions, we have the grid (e.g., $N = 16$) of Fig. 2 where $\vec{p}_1 = \Lambda/2\pi N^{1/2} \vec{q}_1$ and

$$\begin{aligned} \vec{q}_1 &= (1, 1) \quad , \quad \vec{q}_2 = (1, 2) \quad , \quad \vec{q}_3 = (1, 3) \quad , \quad \vec{q}_4 = (1, 4) \\ \vec{q}_5 &= (2, 1) \quad , \quad \vec{q}_6 = (2, 2) \quad , \quad \dots \\ &\dots \quad , \quad \vec{q}_{16} = (4, 4) . \end{aligned} \quad (2.33)$$

Now, we return to the Green function, Eq. (2.26). We consider summing over the indices i, j in order to obtain a singlet to which we apply the above quenching procedure. There is, however, a delicate point concerning the case $i = j$. For these special values of the indices one cannot translate the diagonal elements $U_i^i(x)$ to the point x in the reduced model, since the exponential vanishes. Therefore, the x dependence of the Green function in the momentum integral in Eq. (2.26) could not be obtained from the diagonal elements of $U(x)$. Thus, we must leave out the diagonal elements when we sum over i, j . Since, as in Eq. (2.7), we expect the same contribution for every i, j the omission of the $i = j$ elements is a correction of order $1/N$ to the leading term, which is negligible as $N \rightarrow \infty$.

We will demonstrate some explicit calculations in Section 3, using the procedure outlined above.

3. - ANALYTIC COMPUTATIONS WITH THE REDUCED MODEL IN d DIMENSIONS

The prescription for calculating physical quantities with the reduced model is given in Section 2. Now we perform some of these steps analytically. First, we need to do the group integral for the partition function in Eq. (2.21). and then perform the computations in Eqs. (2.22) and (2.35) with the momenta (2.32).

A. - Group integral

We will consider a more general integrand which will be useful for calculating Green functions as well. Thus we start from

$$Z_{N,d}(\vec{a}, \vec{b}) = \int dU \, e^{\text{Tr}(\alpha_\mu U b_\mu U^\dagger - \frac{1}{2} \alpha_\mu^2 - \frac{1}{2} b_\mu^2)} \quad (3.1)$$

where both α_μ and b_μ are diagonal matrices

$$\alpha_\mu = \begin{pmatrix} \alpha_{1\mu} & & 0 \\ & \alpha_{2\mu} & \\ 0 & & \ddots \\ & & & \alpha_{N\mu} \end{pmatrix} ; \quad b_\mu = \begin{pmatrix} b_{1\mu} & & 0 \\ & b_{2\mu} & \\ 0 & & \ddots \\ & & & b_{N\mu} \end{pmatrix} \quad (3.2)$$

$Z_{N,d}(\vec{a}, \vec{b})$ may also be written as

$$Z_{N,d}(\vec{a}, \vec{b}) = \int dU \, e^{-\frac{1}{2} \sum_{i,j} (\vec{a}_i - \vec{b}_j)^2 |U_{ij}|^2} \quad (3.3)$$

For

$$a_{i\mu} = b_{i\mu} = \left[\frac{N (2\pi)^d}{g^2(\lambda) \lambda^d} \right]^{1/2} P_{i\mu} \quad (3.4)$$

we recover the "partition function" $Z_N(p)$ of Eq. (2.21).

The integral in the case of one dimension, $d = 1$, also occurs in the two-matrix problem, and has already been evaluated¹³⁾. The result is

$$Z_{N,1}(a, b) = \prod_{n=1}^{N-1} (n!) \frac{\text{Det} \left[e^{-\frac{1}{2} (a_i - b_j)^2} \right]}{\Delta(a) \Delta(b)} \quad (3.5)$$

where $\Delta(a)$ and $\Delta(b)$ are the Vandermonde determinants

$$\Delta(a) = \prod_{i>j} (a_i - a_j) \quad (3.6)$$

For more than one dimension, $d \geq 2$, the result should depend on Lorentz invariants formed from $a_{\mu i}$ and $b_{\mu i}$, as is evident from Eq. (3.3), but it does not immediately follow from Eq. (3.5). We first establish some general properties.

- i) We apply a unitary transformation on the matrix a_{μ} , that permutes any two of its diagonal elements $a_{i\mu} \leftrightarrow a_{j\mu}$. This unitary matrix V_L can be absorbed in a redefinition of U

$$V_L U \rightarrow U' \quad ; \quad U^\dagger V_L^\dagger \rightarrow U'^\dagger \quad (3.7)$$

since the Haar measure is invariant under such transformations. We thus conclude that $Z_{N,D}(\vec{a}, \vec{b})$ remains invariant under the interchange of any two $a_{i\mu} \leftrightarrow a_{j\mu}$. Similar statements apply also to independent permutations of $b_{i\mu} \leftrightarrow b_{j\mu}$.

- ii) We now apply a transformation

$$U \rightarrow V U \quad (3.8)$$

where $V = U^\dagger U^\dagger$, so that this transformation becomes

$$U \rightarrow U^\dagger \quad (3.9)$$

Since the Haar measure is invariant under any such V , we see that this transformation is actually equivalent to interchanging

$$a_{\mu i} \longleftrightarrow b_{\mu i} \quad (3.10)$$

while keeping the integral invariant. We thus conclude that $Z_{N,d}(\vec{a}, \vec{b})$ is symmetric under the interchange of the matrices a_{μ} and b_{μ} .

$$Z_{N,d}(\vec{a}, \vec{b}) = Z_{N,d}(\vec{b}, \vec{a}) \quad (3.11)$$

- iii) We take the derivative of Eq. (3.3) with respect to $a_{\mu i}$ and find, after using the unitarity of U ,

$$\left(a_{i\mu} + \frac{\partial}{\partial a_{i\mu}} \right) Z_{N,d}(a, b) = Z_{N,d}(a, b) \sum_j b_{j\mu} \langle |U_{ij}|^2 \rangle_{(a,b)} \quad (3.12)$$

where we have defined

$$\langle |U_{ij}|^2 \rangle_{(a,b)} = \frac{1}{Z_{N,d}(a,b)} \int dU |U_{ij}|^2 e^{-\frac{1}{2} \sum_{k,e} (\vec{a}_k - \vec{b}_e)^2 |U_{ke}|^2} \quad (3.13)$$

Thus, if we have an expression for $Z_{N,d}(a,b)$ and take a derivative with respect to $a_{\mu i}$ to define the sum

$$(a_{i\mu} + \frac{\partial}{\partial a_{i\mu}}) Z_{N,d}(a,b) = \sum_j b_{j\mu} \alpha_{ij}(a,b) Z_{N,d}(a,b) \quad (3.14a)$$

then, the coefficient $\alpha_{ij}(a,b)$ can be identified with $\langle |U_{ij}|^2 \rangle_{a,b}$. We emphasize that this statement is true only in $d \geq 2$ because of the μ index. That is, the relation

$$\sum_j b_{j\mu} \alpha_{ij}(a,b) = \sum_j b_{j\mu} \langle |U_{ij}|^2 \rangle_{(a,b)} \quad (3.14b)$$

has a unique solution

$$\alpha_{ij}(a,b) = \langle |U_{ij}|^2 \rangle_{(a,b)} \quad (3.15a)$$

because Eq. (3.14) must be true for every μ and since we can take $\mu = 1, 2, \dots, d$ with any arbitrarily large d . We warn that this procedure of calculating $\langle |U_{ij}|^2 \rangle_{(a,b)}$ could not be used in one dimension with the expression in Eq. (3.5), since the μ index is missing, and in that case, Eq. (3.14) reduces to a single constraint on the sum rather than on each coefficient. Thus, through the identity of Eq. (3.15) we learn that the coefficients $\alpha_{ij}(a,b)$ obtained by differentiating $Z_{N,d}(\vec{a}, \vec{b})$ must satisfy the unitarity constraints

$$\begin{aligned} \sum_{j=1}^N \alpha_{ij}(a,b) &= 1 && \text{for any } i \\ \sum_{i=1}^N \alpha_{ij}(a,b) &= 1 && \text{for any } j \end{aligned} \quad (3.15b)$$

These constraints amount to second-order differential conditions on $Z_{N,d}(\vec{a}, \vec{b})$. Since we could also write

$$\frac{\partial^2 Z}{\partial a_{i\mu} \partial b_{j\nu}} = g_{\mu\nu} \alpha_{ij} Z + \dots \quad (3.16)$$

where the dots ... represent terms not proportional to $g_{\mu\nu}$.

Note that the constraints i), ii), iii) above use essential properties of the group integration, namely the invariance of the Haar measure and the unitarity of U. We simply write down an expression for $Z_{N,d}(a,b)$ that satisfies these conditions (as we shall see this will correspond to a semiclassical approximation for $d \geq 2$).

$$Z_{N,d}(a,b) = \sum_{j_1 \dots j_N} \delta_{j_1 j_2 \dots j_N}^{12 \dots N} \mathcal{Z}_{N,d}(j_1, j_2, \dots, j_N) \quad (3.17a)$$

where

$$\mathcal{Z}_{N,d}(j_1, \dots, j_N) = \left(\prod_{n=1}^{N-1} n! \right) \frac{e^{-\frac{1}{2}(\vec{a}_1 - \vec{b}_{j_1})^2} e^{-\frac{1}{2}(\vec{a}_2 - \vec{b}_{j_2})^2} \dots e^{-\frac{1}{2}(\vec{a}_N - \vec{b}_{j_N})^2}}{\prod_{1 \leq k < \ell \leq N} (\vec{a}_k - \vec{a}_\ell) \cdot (\vec{b}_{j_k} - \vec{b}_{j_\ell})} \quad (3.17b)$$

and the sum in (3.17a) is over all permutations taken with positive signs. This expression is easily understood by writing the first term with $j_1 = 1, j_2 = 2, \dots, j_N = N$, and then permuting all $\vec{b}_i$ with fixed $\vec{a}_i$. To be more explicit, we write out the $N = 2$ and $N = 3$ cases

$$Z_{2,d}(\vec{a}, \vec{b}) = \frac{e^{-\frac{1}{2}(\vec{a}_1 - \vec{b}_1)^2} e^{-\frac{1}{2}(\vec{a}_2 - \vec{b}_2)^2}}{(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2)} - \frac{e^{-\frac{1}{2}(\vec{a}_1 - \vec{b}_2)^2} e^{-\frac{1}{2}(\vec{a}_2 - \vec{b}_1)^2}}{(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2)} \quad (3.18a)$$

$$Z_{3,d}(\vec{a}, \vec{b}) = 2! \left\{ \frac{e^{-\frac{1}{2}(\vec{a}_1 - \vec{b}_1)^2} e^{-\frac{1}{2}(\vec{a}_2 - \vec{b}_2)^2} e^{-\frac{1}{2}(\vec{a}_3 - \vec{b}_3)^2}}{[(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2)][(\vec{a}_1 - \vec{a}_3) \cdot (\vec{b}_1 - \vec{b}_3)][(\vec{a}_2 - \vec{a}_3) \cdot (\vec{b}_2 - \vec{b}_3)]} \right. \\ \left. + \text{permutation of } \vec{b}_1, \vec{b}_2, \vec{b}_3, \text{ with } \vec{a}_1 \text{ fixed} \right\} \quad (3.18b)$$

One can easily verify that :

- a) the $d = 1$ limit of Eq. (3.17) agrees with Eq. (3.5) ;
- b) the $N = 2, d \geq 2$ group integral is trivial to perform explicitly and agrees with (3.18) ;
- c) the $\vec{a}_1 = \vec{a}_2$ limit, with $\vec{a}_3, \vec{b}_1, \vec{b}_2, \vec{b}_3$ arbitrary for $N = 3$ also is easy to integrate explicitly and the result is in agreement with the same limit of (3.18) ; note that the apparent poles in the denominator of (3.17) are cancelled by zeros in the numerator, as we should expect, on the basis that our compact group integral could not be singular ;

- d) the expression (3.17) is completely symmetric under the permutations $a_{i\mu} \leftrightarrow a_{j\mu}$ or $b_{i\mu} \leftrightarrow b_{j\mu}$ as well as under $a_{i\mu} \leftrightarrow b_{j\mu}$ as required by the Haar measure ;
 e) the derivatives of (3.17) provide the expression for $\langle |u_{ij}|^2 \rangle_{(a,b)}$ through (3.12) or (3.16) ; the result is given below and it can easily be checked that it satisfies the unitarity (or differential) constraints of (3.15)

$$\langle |u_{mn}|^2 \rangle_{(a,b)} = \sum_{\{j\}} \delta_{i_1 i_2 \dots i_N}^{j_1 j_2 \dots j_N} \frac{\delta_{N,d}^{(i_1, i_2, \dots, i_N)}}{Z_{N,d}^{(a,b)}} \left\{ \delta_{n_j} - \sum_{k \neq m} \frac{(\delta_{n_j} - \delta_{n_k})}{(\bar{a}_m - \bar{a}_k) \cdot (\bar{b}_m - \bar{b}_k)} \right\} \quad (3.19)$$

where the sum is over all permutations, as in Eq. (3.17).

Although these formulas (first arrived at by guesswork) satisfy nontrivial constraints, we could not establish that they are exact. To verify their validity it is useful to perform the integral in a semi-classical approximation (as pointed out to us by D. Gross) and compare to our form. We look for the saddle points in the action S of eq. (3.1)

$$-S = \text{Tr} (a_\mu U b_\mu U^\dagger - \frac{1}{2} a_\mu^2 - \frac{1}{2} b_\mu^2) \quad (3.20a)$$

$$-ds = \text{Tr} [b_\mu, U^\dagger a_\mu U] U^\dagger dU$$

Thus, $ds = 0$ for any U such that $U^\dagger a_\mu U$ is diagonal. The general solution is of the form

$$U_{\min} = U_{\text{diag}} P \quad (3.20b)$$

where U_{diag} is any diagonal unitary matrix and P is any $N \times N$ permutation matrix applied on N objects. Note that the appearance of an arbitrary diagonal matrix is intimately connected to the axial symmetry $[U(1)]^N$ in eq. (2.20). Indeed, U_{diag} corresponds to the zero modes of this symmetry as will be discussed in more detail in section (4). The zero modes are treated exactly. Expanding the action S and the Haar measure around the saddle points, $U_{\min}$, doing the saddle integration for the small oscillations, integrating exactly over all diagonal matrices U_{diag} and summing over all permutations, we obtain the answer given in eq. (3.17). An alternative method of dealing with the diagonal modes is to introduce an appropriate $U(1)^N$ "gauge" fixing together with the corresponding Fadeev-Popov determinant.

As a guide to this calculation, we make some remarks. First, there are $N!$ saddle points corresponding to $N!$ permutations. Second, each saddle point contributes an action $\sum_i (a_i - b_{P(i)})^2$, where $P(i)$ is the permutation of the index i . This explains the exponentials in eq. (3.17). Third, the expansion around the saddle gives a gaussian integral with an exponent $\sum_{i \neq j} (a_i - b_{P(j)})^2 \frac{1}{h_{ij}^2}$ which reproduces the denominator in eq. (3.17).

The exact calculations indicated following eq. (3.18) imply that the semiclassical approximation is actually exact for

- (a) $d = 1$ any N
- (b) $N = 2$ any d
- (c) $N = 3, a_1 = a_2$, any d

Could the answer be exact for all cases? The answer is, unfortunately, no, as seen by an explicit calculation of the next order in the semiclassical expansion. The second order correction to eq. (3.17) turns out to modify eq. (3.17b) by the factor

$$1 + \frac{1}{6} \sum_{k \neq l \neq m} \frac{(\vec{a}_k - \vec{a}_l) \cdot (\vec{b}_k - \vec{b}_l) (\vec{a}_l - \vec{a}_m) \cdot (\vec{b}_l - \vec{b}_m) - (\vec{a}_k - \vec{a}_l) \cdot (\vec{b}_l - \vec{b}_m) (\vec{a}_l - \vec{a}_m) \cdot (\vec{b}_k - \vec{b}_l)}{(\vec{a}_k - \vec{a}_l) \cdot (\vec{b}_k - \vec{b}_l) (\vec{a}_l - \vec{a}_m) \cdot (\vec{b}_l - \vec{b}_m) (\vec{a}_m - \vec{a}_k) \cdot (\vec{b}_m - \vec{b}_k)} \quad (3.21)$$

which was first derived by D. Gross. This expression reduces to 1 for the special cases above. Thus, we must understand eq. (3.17) as an approximation for $d > 2$ except in special cases such as (a), (b), (c) above. We do not know yet the general circumstances under which the expansion around the saddle point may stop after a few orders to yield exact results as in the special cases above which stopped at the first order. There may be cases for which (3.17) combined with (3.21) yields an exact answer. In the rest of this section we will use eq. (3.17) as the approximate result of our integral (but exact for $d = 1$).

B. - Free energy and correlation function

We can now substitute the momenta $p_{i\mu}$ instead of $a_{i\mu}, b_{i\mu}$, as in Eq. (3.4). For convenience, it becomes useful to define dimensionless momenta, which we denote by $q_{i\mu}$ and relate to $p_{i\mu}$ with the over-all factor of the cut-off $\Lambda/2\pi$. Thus, we can write, from Eq. (3.4),

$$a_{i\mu} = b_{i\mu} = \sqrt{\beta N} q_{i\mu}$$

where the dimensionless parameter β in d dimensions is

$$\beta = \frac{(\hbar^2/2\pi)^{-d+2}}{q^2(\hbar)} \quad (3.22)$$

With these substitutions we finally have the semiclassical answer:

$$\bar{\delta}_{N,d}^{(j_1, j_2, \dots, j_N)} = \frac{\prod_{n=1}^{N-1} n!}{(\beta N)^{N(N-1)/2}} \frac{e^{-\frac{1}{2}\beta N(\bar{q}_1 - \bar{q}_{j_1})^2} e^{-\frac{1}{2}\beta N(\bar{q}_2 - \bar{q}_{j_2})^2} \dots e^{-\frac{1}{2}\beta N(\bar{q}_N - \bar{q}_{j_N})^2}}{\prod_{1 \leq k < \ell \leq N} (\bar{q}_k - \bar{q}_\ell) \cdot (\bar{q}_{j_k} - \bar{q}_{j_\ell})} \quad (3.23a)$$

$$Z_{N,d}(q) = \sum_{\{j\}} \delta_{j_1, j_2, \dots, j_N}^{12 \dots N} \bar{\delta}_{N,d}^{(j_1, j_2, \dots, j_N)} \quad (3.23b)$$

$$\langle |U_{mn}|^2 \rangle_q = \sum_{\{j\}} \delta_{j_1, j_2, \dots, j_N}^{12 \dots N} \frac{\bar{\delta}_{N,d}^{(j_1, j_2, \dots, j_N)}}{Z_{N,d}(q)} \left\{ \delta_{j_m} - \sum_{k \neq m} \frac{(\delta_{j_m} - \delta_{j_k})}{\beta N (\bar{q}_m - \bar{q}_k) \cdot (\bar{q}_{j_m} - \bar{q}_{j_k})} \right\} \quad (3.24)$$

In one dimension ($d = 1$) these expressions simplify considerably because the denominators in (3.23)-(3.24) are identical for every permutation except for minus signs. This gives a definite form to Eq. (3.23) as in Eq. (3.5). Also, the sums over the j_i in Eq. (3.24) can be performed by noting that the inverse of an $N \times N$ matrix M_{ij} satisfies

$$(\text{Det } M) (M_{j_m m}^{-1} M_{j_k k}^{-1} - M_{j_m k}^{-1} M_{j_k m}^{-1}) = \epsilon_{j_1 j_2 \dots j_m \dots j_k \dots j_N} M_{j_1 j_1} M_{j_2 j_2} \dots M_{j_N j_N} \quad (3.25a)$$

where all j_i except j_m, j_k are summed over. Then Eq. (3.24) simplifies to

$$\langle |U_{mn}|^2 \rangle_{q; d=1} = M_{mn} M_{nm}^{-1} - \frac{1}{\beta N} \sum_{\substack{k \neq m \\ \ell \neq n}} \frac{(M_{mn} M_{k\ell} - M_{kn} M_{m\ell}) (M_{nm}^{-1} M_{\ell k}^{-1} - M_{nk}^{-1} M_{\ell m}^{-1})}{(q_m - q_k)(q_n - q_\ell)} \quad (3.25b)$$

where the matrix M_{ij} is given by

$$M_{ij} = e^{-\frac{1}{2}\beta N (q_i - q_j)^2} \quad (3.25c)$$

This expression is valid only in the semiclassical approximation (i.e. the correction to this expression follows from (3.21) via (3.14a) by first performing derivatives in higher dimensions).

The cut-off dependent free energy and correlation functions of Eqs. (2.22) and (2.25) are now given in any dimension as

$$\mathcal{E}(\Lambda, g(\Lambda), d) = \lim_{N \rightarrow \infty} \left(\frac{\Lambda}{2\pi} \right)^d \left\{ \frac{1}{N^2} \ln Z_{N,d}(q) \right\} \quad (3.26)$$

$$G(\vec{x}-\vec{y}, \Lambda, g(\Lambda), d) = \lim_{N \rightarrow \infty} \left\{ \sum_{m \neq n} e^{-i \frac{\Lambda}{2\pi} (\vec{q}_m - \vec{q}_n) \cdot (\vec{x} - \vec{y})} \langle |u_{mn}|^2 \rangle_q \right\} \quad (3.27)$$

Where the $q_{1\mu}$ are quenched.

In one and two dimensions we must evaluate these expressions as $\beta \rightarrow \infty$ since as we remove the cut-off ($\Lambda \rightarrow \infty$) we expect that

$$\begin{aligned} d=1, \quad g^2(\Lambda) \rightarrow \text{finite} & \quad ; \quad \beta = \frac{\Lambda/2\pi}{g^2(\Lambda)} \rightarrow \infty \\ d=2, \quad g^2(\Lambda) \rightarrow 0 & \quad ; \quad \beta = \frac{1}{g^2(\Lambda)} \rightarrow \infty \end{aligned} \quad (3.28)$$

The $d=2$ behaviour of $g^2(\Lambda)$ follows from the known asymptotic freedom of the theory. In higher dimensions, the behaviour is

$$d > 2 + \epsilon, \quad \beta \rightarrow 0 \quad (3.29)$$

In the next subsection, we will discuss the $\beta \rightarrow \infty$ behaviour for $d=1,2$. This simplifies some of our analytic expressions and will allow us to calculate some quantities completely.

C. - The free energy

In one dimension, instead of Eqs. (3.23) and (3.24), we can directly use the determinat form of Eq. (3.5) with $a_i = b_i = \sqrt{\beta N} q_i$ and with the momenta given as $q_i = i/N$, $i = 1, 2, \dots, N$. Then it is seen that, as $N \rightarrow \infty$

$$\left(\prod_{n=1}^{N-1} n! \right) \rightarrow e^{\frac{N^2}{2} \ln N - \frac{3}{4} N^2} \quad (3.30a)$$

$$\Delta(a)\Delta(b) = \left(\frac{\beta}{N} \right)^{N(N-1)/2} \prod_{i>j}^N (i-j)^2 \xrightarrow{N \rightarrow \infty} e^{\frac{N^2}{2} \ln \beta + \frac{N^2}{2} \ln N - \frac{3}{2} N^2} \quad (3.30b)$$

$$\text{Det} \left(e^{-\frac{1}{2} (a_i - b_j)^2} \right) = \text{Det} \left(e^{-\beta (i-j)^2 / 2N} \right) = \prod_{k=1}^N \left(1 - e^{-\beta k / N} \right)^{N-k} \\ \xrightarrow{N \rightarrow \infty} \exp \left\{ N^2 \int_0^1 dx (1-x) \ln(1 - e^{-\beta x}) \right\} \quad (3.30c)$$

where we have ignored terms of order N in the exponent. We can now write the $N \rightarrow \infty$ free energy per unit volume ϵ , defined through Eqs. (2.21), (2.22), (2.29), (3.22).

$$d=1 :$$

$$\frac{2\pi}{\Lambda} \epsilon(\Lambda, g(\Lambda)) = \frac{1}{2} \ln \left(\frac{\Lambda}{2\pi g^2(\Lambda)} \right) - \frac{3}{4} - \int_0^1 dx (1-x) \ln(1 - e^{-\Lambda x / 2\pi g^2(\Lambda)}) \quad (3.31a)$$

This exact answer is valid for any value of Λ or $g^2(\Lambda)$. For finite Λ , the weak coupling expansion to all orders is

$$\frac{2\pi}{\Lambda} \epsilon(\Lambda, g(\Lambda)) = \frac{1}{2} \ln \left(\frac{\Lambda}{2\pi g^2(\Lambda)} \right) - \frac{3}{4} + \zeta(2) \frac{2\pi g^2(\Lambda)}{\Lambda} - \zeta(3) \left(\frac{2\pi g^2(\Lambda)}{\Lambda} \right)^2 \\ + \left(\frac{2\pi g^2(\Lambda)}{\Lambda} \right)^2 \sum_{n=1}^{\infty} \frac{e^{-\Lambda n / 2\pi g^2(\Lambda)}}{n^3} \quad (3.31b)$$

Note the non-perturbative terms in the last sum. The sum is related to the zeta function and its properties can be found, e.g., in Ref. 14). Furthermore, in (3.31b) $\zeta(z)$ is the Riemann zeta function evaluated at $z = 2, 3$. A strong coupling expansion for finite Λ may also be given by using an analytic continuation of Eq. (3.31b) obtained via Ref. 14)

$$\frac{2\pi}{\Lambda} \epsilon(\Lambda, g(\Lambda)) = - \sum_{n=1}^{\infty} \frac{B_n}{(n+2)!} \frac{1}{n} \left(\frac{\Lambda}{2\pi g^2(\Lambda)} \right)^n \quad (3.32)$$

where B_n are the Bernoulli numbers¹⁴⁾.

This exact result of the reduced model may be compared to the exact $N \rightarrow \infty$, small g result of the path integral in the Appendix, Eq. (A.29). The leading logarithmic terms are seen to agree and a disagreement begins with the non-leading constant terms. The sign difference in $\pm \frac{3}{4}$ is not a typographical

error. Of course, the lattice cut-off (a) and the momentum cut-off (Λ) of the reduced model are different and Eqs. (3.31) and (A.29) may be used to relate them, as well as relating $g(a)$ to $g(\Lambda)$. As the cut-off is removed, (i.e., $\Lambda \rightarrow \infty$, or $a \rightarrow 0$) in one dimension, $g(\Lambda)$ or $g(a)$ tend to a constant g . Then the two expressions agree, provided that we relate the two cut-offs :
 $a = e^{-3} 2\pi/\Lambda + \text{non-leading terms.}$

In two dimensions, the quenched momenta q_i are given by the grid of Fig. 2. We replace them in Eq. (3.23) and note that most terms in $z_{N,d}(j_1, j_2, \dots, j_N)$ have an exponentially decreasing behaviour as $\beta \rightarrow \infty$ (weak coupling)

$$d=2 : \quad z_{N,d}(j_1, j_2, \dots, j_N) \sim e^{-\beta \times (\text{integer})} \quad (3.33)$$

The further away we get from the identity permutation, the faster the exponential decreases. It is therefore tempting to neglect the permutations which are far away from the identity. Note, however, that in this procedure we follow renormalized perturbation theory by first taking $\beta \rightarrow \infty$ ($\Lambda \rightarrow \infty$) and only then $N \rightarrow \infty$. The usual large N saddle point lore corresponds, however, to the reverse order of limits. For the case of one dimension presented above, either order gives the same result. Indeed, as seen from Eq. (3.31b), the surviving terms for $\Lambda \rightarrow \infty$ correspond to neglecting the determinant and hence to keeping only the identity permutation in Eq. (3.23). We hope that the same is true in two dimensions since (superficially) it looks similar to the one-dimensional case. We have not yet carried out the reverse order.

The leading term as $\beta \rightarrow \infty$ corresponds to the trivial permutation in Eq. (3.23)

$$j_1 = 1, \quad j_2 = 2, \quad j_3 = 3, \quad \dots, \quad j_N = N \quad (3.34)$$

for which $z_{N,d}(1, 2, \dots, N)$ has no exponential decrease. The next-to-leading term corresponds to permutations of two momenta p_i, p_j which are next to nearest neighbours on the grid, while all the others remain fixed in the original positions. Let us denote such a term as $z_N(i, j)$.

If we write, for the trivial permutation

$$z_N(1, 2, \dots, N) = \frac{\left(\prod_{n=1}^{N-1} n! \right)}{(\beta N)^{N(N-1)/2}} \frac{1}{\prod_{k>\ell} (\bar{q}_k - \bar{q}_\ell)^2} \quad (3.35)$$

then $z_N(i,j)$ can be put into the form

$$\bar{z}_N(i,j) = \bar{z}_N(1,2,\dots,N) e^{-\beta N (\bar{q}_i - \bar{q}_j)^2} \prod_{n \neq i,j} \frac{(\bar{q}_i - \bar{q}_n)^2 (\bar{q}_j - \bar{q}_n)^2}{\{(\bar{q}_i - \bar{q}_n) \cdot (\bar{q}_j - \bar{q}_n)\}^2} \quad (3.36)$$

in which we are supposed to replace the quenched nearest neighbour momenta

$$q_i = \frac{1}{\sqrt{N}} (i_1, i_2)$$

$$q_j = \frac{1}{\sqrt{N}} (j_1, j_2)$$

such that

$$(i_1 - j_1)^2 + (i_2 - j_2)^2 = 1 \quad (3.38)$$

The last product in Eq. (3.36) is simply written as

$$\prod_{n \neq i,j}^N \frac{1}{\cos^2 \theta_{ij}(n)} \quad (3.39)$$

where $\theta_{ij}(n)$ is the angle defined by the vector $(\vec{q}_i - \vec{q}_n)$ and $(\vec{q}_j - \vec{q}_n)$ shown in Fig. 3. Taking into account all such terms, the main contributions to the sum in (3.23) can now be written as

$$Z_{N,2}(q) = \bar{z}_N(1,2,\dots,N) \left\{ 1 - C_N e^{-\beta} + o(e^{-2\beta}) \right\} \quad (3.40)$$

where C_N is a constant which can be computed by summing Eq. (3.23) for each allowed pair (i,j) of type (3.36).

The free energy is now evaluated from

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \ln Z_{N,2}(q) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln \bar{z}_N(1,2,\dots,N) + \lim_{N \rightarrow \infty} \frac{1}{N^2} \ln(1 - C e^{-\beta} + \dots) \quad (3.41)$$

we see that as $\beta \rightarrow \infty$ and then $N \rightarrow \infty$ only the trivial permutation $z_N(1,2,\dots,N)$ is contributing. We now evaluate this quantity by noting that for quenched momenta as in Eq. (3.37),

$$\begin{aligned}
 \ln \left\{ \prod_{k \neq l}^N (\bar{q}_k - \bar{q}_l)^2 \right\} &= \frac{1}{2} \ln \prod_{k \neq l}^N (\bar{q}_k - \bar{q}_l)^2 \\
 &= \frac{1}{2} \sum_{k \neq l}^N \ln (\bar{q}_k - \bar{q}_l)^2 \\
 &\simeq \frac{1}{2} \int_0^{\sqrt{N}} d\vec{k} d\vec{l} \ln \left[\frac{(\vec{k} - \vec{l})^2}{N} \right] \\
 &= \frac{1}{2} N^2 \int_0^1 d\vec{x} d\vec{y} \ln (\vec{x} - \vec{y})^2
 \end{aligned}$$

Then, taking into account Eqs. (3.30) and (3.35), and the definition of free energy per unit volume, Eqs (2.26) and (2.33), we obtain the answer

$d=2$:

$$\left(\frac{2\pi}{\Lambda} \right)^2 \mathcal{E}(\Lambda, g(\Lambda)) = \frac{1}{2} \ln \left(\frac{1}{g^2(\Lambda)} \right) + \frac{3}{4} + \frac{1}{2} \int_0^1 \int_0^1 d\vec{x} d\vec{y} \ln (\vec{x} - \vec{y})^2 + O(g^2(\Lambda))$$

According to the known asymptotic freedom behaviour of the theory, $g^2(\Lambda)$ is given as

$$\frac{1}{g^2(\Lambda)} = \frac{1}{g^2(\mu)} + C_1 \ln \Lambda/\mu + C_2 \ln \ln \Lambda/\mu + \dots$$

where C_1 and C_2 are constants and μ is the renormalization point. Then, we see that as $\Lambda \rightarrow \infty$ we have obtained the leading terms in the free energy, and the remaining terms $O(g^2(\Lambda))$ simply vanish as $\Lambda \rightarrow \infty$, as $(\ln \Lambda)^{-1}$.

4. - THE VACUUM STRUCTURE

The perturbative expansion of the reduced model around $U = 1$ is easily seen to be formally equivalent to the planar perturbation expansion of the underlying model around $U(x) = 1$ ^{6),7)} (note the comments about more than N loops in the Introduction). However, this is not sufficient ¹¹⁾ to establish the equivalence non-perturbatively. We must check that the two models have the same vacuum state in d dimensions. This is easily demonstrated by comparing the Schwinger Dyson equations of the underlying and the reduced theory. The computation is straightforward and is carried along the lines described in Refs. 5), 11). The outcome is that there is a one-to-one correspondence between the two models provided the vacuum structure is the same. In particular, for $d \leq 2$ Coleman's theorem about the absence of Goldstone bosons imposes an exact $U(N)_L \times U(N)_R / U(1)$ symmetry on the vacuum of the underlying field theory. This is expressed as the vanishing of the order parameter

$$\langle \text{Tr } U(x) \rangle = 0 \quad (4.1)$$

Does the reduced model satisfy the same equation and thus describe the same non-perturbative vacuum? Since the action of the reduced model is invariant under the remnant $[U(1)_L]^N \times [U(1)_R]^N / U(1)$ symmetry, it is clear that if this symmetry is unbroken, Eq. (4.1) is satisfied. We would like therefore to investigate the behaviour of this order parameter.

A standard approach is to introduce a source term J that breaks the symmetry and investigate the vacuum as $J \rightarrow 0$. Thus we take the modified non-symmetric action

$$-S_N(U, q, J) = \beta N \text{Tr} (q_\mu U q_\mu U^\dagger - q^2) + J \text{Tr} (U + U^\dagger) \quad (4.2)$$

where q_μ are the rescaled momenta $\vec{p}_\mu = \Lambda/2\pi \vec{q}_\mu$ and β is given in Eq. (3.22) for any d . We define

$$Z_N(J, q) = \int dU e^{-S_N(U, q, J)} \quad (4.3)$$

and note that

$$\frac{1}{N} \frac{\partial}{\partial J} \ln Z_N(J, q) = \frac{1}{N} \langle \text{Tr} (U + U^\dagger) \rangle_{q, J} \quad (4.4)$$

For $J = 0$, and any momenta q_μ , this expression will determine whether the $[U(1)_L]^N \times [U(1)_R]^N / U(1)$ symmetry remains unbroken in the vacuum for any N , including $N \rightarrow \infty$.

Now we note the following properties.

- i) $Z_N(J, q)$ is an even function of J for any N , including $N \rightarrow \infty$

$$Z_N(J, q) = Z_N(-J, q) \quad (4.5)$$

This is seen by noticing that $J \rightarrow -J$ is equivalent to $U \rightarrow -U$ which is a unitary transformation on U . This implies that $(\partial/\partial J) \ln Z_N(J, q)$ is an odd function of J that vanishes at $J = 0$ unless it is singular at $J = 0$ (which is the case if a phase transition occurs).

- ii) The integrand in Eq. (4.3) is non-singular. Therefore, at least for any finite N , no matter how large it is, $Z_N(J, q)$ is an analytic function of J^2 with a Taylor expansion of the form

$$Z_N(J, q) = Z_N(0, q) + J^2 Z_N''(0, q) + \dots \quad (4.6)$$

Therefore, we obtain $(\partial/\partial J) \ln Z_N(J, q) \rightarrow 0$, indicating that the symmetry is exact in the vacuum for any finite N , any dimension d and any momentum q_μ . Thus the issue reduces to whether the $N \rightarrow \infty$ limit is smooth for any value of β .

- iii) For small β and $N \rightarrow \infty$, Eq. (4.3) reduces to a known integral with the answer ¹⁵⁾

$$\lim_{\substack{\beta \rightarrow 0 \\ N \rightarrow \infty}} Z_N(J, q) \rightarrow e^{J^2} \quad \text{for } J < N/2 \quad (4.7)$$

From this we see $\frac{1}{N}(\partial/\partial J) \ln Z_N \rightarrow 0$, as $J \rightarrow 0$, for any d . The small β expansion can be worked out easily via a generating function given in Ref. 16) and the result remains the same including the perturbations with small β . For $d > 2$ it seems that the continuum theory is driven to $\beta \rightarrow 0$ after renormalization, Eq. (3.29). For $d \leq 2$ it is the weak coupling ($\beta \rightarrow \infty$) behaviour which we have to explore, Eq. (3.28).

- iv) For $\beta \rightarrow \infty$ and any N the integral can be analyzed in more detail by parametrizing the unitary matrix as

$$U = (1 - iH)^{-1} (1 + iH) \quad (4.8)$$

where H is an Hermitian matrix. The Haar measure is then given by ¹⁷⁾

$$dU = C_N \frac{dH}{[\text{Det}(1+H^2)]^N} \quad (4.9)$$

where (dH) is the naïve measure over the real N^2 parameters in H all of which range from $-\infty$ to $+\infty$, and C_N is a numerical normalization constant. The effective action, including the contribution from the measure takes the form

$$\begin{aligned} -S_{\text{eff}}(H, q, J) = & 2\beta N \text{Tr} \left\{ (1+H^2)^{-1} [q_\mu, H] (1+H^2)^{-1} [q_\mu, H] \right\} \\ & - N \text{Tr} \ln(1+H^2) + 2J \text{Tr} (1-H^2)(1+H^2)^{-1} \end{aligned} \quad (4.10)$$

The $\beta \rightarrow \infty$ saddle point is determined by minimizing the first term which gives any diagonal H (plus permutations as in eq. (3.21b)).

$$H_{\text{min}} = \lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_N \end{pmatrix} \quad (4.11)$$

The eigenvalues λ_i are zero modes that correspond to the $(U(1))^N$ axial symmetries, and they should be treated exactly as in any standard saddle point problem with zero modes. These play a crucial rôle in establishing the symmetries of the vacuum and are clearly important in our analysis. We thus differ from Refs. 7), 9), 11) in our treatment of the diagonal of the $N \times N$ reduced matrix.

We now parametrize

$$H = \lambda + \frac{1}{\sqrt{\beta}} h \quad (4.12)$$

where h is off diagonal, and represents the small oscillations around the solution. The expansion in powers of $\frac{1}{\beta}$ of the effective action generates the perturbation series around each saddle parametrized by λ (we ignore the permutations which are not crucial in this discussion). Thus,

$$S_{\text{eff}} = S_0(\lambda, h) + \frac{1}{\sqrt{\beta}} S_{1/2}(\lambda, h) + \frac{1}{\beta} S_1(\lambda, h) + \dots \quad (4.13)$$

where

$$-S_0(\lambda, h) = 2N \text{Tr} \left\{ \frac{1}{1+\lambda^2} [q_\mu, h] \frac{1}{1+\lambda^2} [q_\mu, h] \right\} \\ - N \text{Tr} \ln(1+\lambda^2) + 2J \text{Tr} (1-\lambda^2)(1+\lambda^2)^{-1} \quad (4.14)$$

The perturbation expansion of $Z(J)$ is now written as

$$Z(J) = C_N \int_{-\infty}^{\infty} (d\lambda)(dh) e^{-S_0(\lambda, h)} \left[1 - \frac{1}{\beta} (S_1(\lambda, h) - \frac{1}{2} S_{1/2}^2(\lambda, h)) + \dots \right] \quad (4.15)$$

where

$$(d\lambda) = \prod_{i=1}^N d\lambda_i \quad ; \quad (dh) = \prod_{i>j} d x_{ij} d y_{ij} \quad (4.16)$$

and λ_i as well as x_{ij}, y_{ij} ($h_{ij} = x_{ij} + i y_{ij}$) range from $-\infty$ to $+\infty$. Thus, the zero modes λ_i are treated exactly.

We have obtained the zeroth order contribution in $\frac{1}{\beta}$ as

$$Z(J) = C_0(q) [I_0(J)]^N + O(1/\beta) \quad (4.17a)$$

where $I_0(J)$ is the modified Bessel function and $C_0(q)$ depends on the momenta. From this expression and Eq. (4.4), we calculate

$$\frac{1}{N} \langle \text{Tr}(U + U^\dagger) \rangle_{q, J} = \frac{I_1(J)}{I_0(J)} + O(1/\beta) \quad (4.17b)$$

where $I_1(J) = I'_0(J)$ is odd in $J \rightarrow -J$. In this expression we can take $N \rightarrow \infty$ with J fixed. We note that the order parameter does not vanish as long as $J \neq 0$ even if $N = \infty$. Thus, as expected from the general argument above, the explicit calculation shows that as $J \rightarrow 0$ the order parameter vanishes, for $N \rightarrow \infty$.

The $O(\frac{1}{\beta})$ correction indicated in Eq. (4.13) and (4.15) involves cubic (h^3) vertices in $S_{\frac{1}{2}}$ and quadratic (h^2) and quartic (h^4) vertices in S_1 .

When the 1st-order corrections are included, we obtain the form

$$Z(J) = (C_0 + \frac{1}{\beta} C_1) [I_0(J)]^N + \frac{C_2}{\beta} J I_1(J) [I_0(J)]^{N-1} + O(\beta^{-2})$$

$$\langle \frac{1}{N} \text{Tr}(U+U^\dagger) \rangle = \frac{I_1(J)}{I_0(J)} + \frac{1}{N} \frac{C_3}{\beta} J \left(1 - \left[\frac{I_1(J)}{I_0(J)} \right]^2 \right) + O(\beta^{-2}) \quad (4.18)$$

where $C_i(q)$ are momentum dependent coefficients. These results show that the conclusions drawn from eq. (4.17) remain unchanged. That is, the $[U(1)_L]^N \times [U(1)_R]^N / U(1)$ symmetry is exact in the vacuum, for any values of the momenta, quenched or not, in any number of dimensions.

We need to discuss further why we took $J = \text{fixed}$, as opposed to $J = Nh$ and $h = \text{fixed}$, as $N \rightarrow \infty$. We must keep in mind that a "small" J was introduced in order to break the symmetry and that we really are interested in exactly the $J = 0$ theory. The procedure with $J = Nh$ first sends J to infinity and then sets $h = 0$ at the end. A standard practice in $N \rightarrow \infty$ problems is to take an action whose terms are all of order N . The reasoning in such cases is that unless a term is of order N it is negligible. However, this would not be true if a small perturbation gives a non-zero contribution to a quantity which otherwise would have vanished. In fact, this is precisely the case in our problem. The $\beta \rightarrow \infty$ limit corresponds to solutions (zero modes) with zero action [Eq. (4.11)]. The presence of J is a sufficiently large contribution even for $N \rightarrow \infty$ to contribute to the order parameter [Eq. (4.1)]. Thus, we obtain for $J = \text{fixed}$ and $N \rightarrow \infty$ a non-zero result for the expectation value of $\text{Tr } U$ in the presence of the symmetry breaking source. This allows us to study the theory as $J \rightarrow 0$, and arrive at the conclusion that the vacuum is symmetric.

Our procedure therefore differs from another attempt in the literature that discusses the vacuum structure of the reduced chiral model ¹¹⁾. In the Monte Carlo calculation of Ref. 11), they took $J = Nh$. For such a large J our $\beta \rightarrow \infty$ analytic calculation of Eq. (4.18) gives

$$\frac{I_1(Nh)}{I_0(Nh)} \xrightarrow{N \rightarrow \infty} \varepsilon(h) = \frac{h}{|h|} \quad (4.19)$$

which is a step function around the origin. The Monte Carlo calculation of Ref. 11) overshoot and gave a hysteresis curve for large β . We believe that our procedure which indicates no phase transition is the correct one.

It may be useful to note that, in the infinite volume theory, the signal for Coleman's theorem is the infrared divergent behaviour of the perturbative Feynman diagrams when we calculate chiral non-singlet quantities. The same phenomenon would, of course, occur in the reduced model as well. However, it is known that for chirally invariant Green functions the divergences disappear and so do the Goldstone bosons as well. We expect the same phenomenon in the perturbative reduced model. However, this argument remains incomplete for nonperturbative treatments, in particular, for establishing the equivalence of the reduced model to the infinite volume model on the basis of the Schwinger-Dyson equations as outlined in the beginning of this section. The problem we have addressed in this section is a legitimate question, in any case, for establishing the properties of the vacuum of the reduced model, on its own. Our answer is that the vacuum is chirally invariant.

For fixed distribution of momenta, our analysis does not depend on the number of dimensions. Hence we conclude that the reduced model is always in the symmetric phase, and it describes the theory in its disordered phase for all d . In fact in one dimension where we have an explicit expression for the free energy as a function of β , there is no sign for a phase transition. Moreover in our semiclassical expression for the Green function [Eqs. (3.24), (3.27)] there is no sign for a Goldstone pole in the momentum integral for any d . [The numerator, after the sum over permutations, has zeros which kill the ones in the denominator as argued in item (c) following Eq. (3.19).] We should note that mean field theory on the infinite lattice gives a conflicting result for $d \geq 3$ (19). Thus, either the reduced model or the mean field approximation fails to reproduce the correct theory for $d \geq 3$.

5. - SUMMARY

In the present paper we have investigated to some extent the reduced chiral model. Since the proposed reduced model has some left-over global symmetry $[(U_L^N(1) \times U_R^N(1))/U(1)]$, it was a priori unclear whether it has the correct vacuum structure. The analysis demanded a careful treatment of the diagonal matrix elements which are the zero modes associated with the symmetry. We have discussed this question in detail and have argued that this is indeed the adequate reduced model to describe the original system in its disordered (unbroken) phase.

As for the investigation of the reduced model itself the emphasis was on a semiclassical approximation to the relevant group integral for calculating the free energy and correlation function. It is interesting that our result is actually exact in some special cases, including $d = 1$ any N . This is certainly only the first step in the program. The group integration should be followed by the quenching procedure which we have described and started to carry and then renormalization. For one dimension, where the theory is finite, we could calculate the free energy which agrees with the exact result as $\Lambda \rightarrow \infty$ (Appendix). The most interesting case is the two-dimensional renormalizable case which is expected to show similar behaviour to gauge theories in four dimensions. In particular, we expect to see a non-perturbative mass gap which behaves as e^{-1/g^2} . To establish the behaviour of the mass gap, we should investigate the behaviour of the correlation function at large distances. This is under current study.

For one dimension, the underlying chiral model is exactly solvable. We have presented the solution in the Appendix. This case should be used as a test case for the reduced model and the quenching procedure. In particular, the mass gap should be computed and compared with the exact result. Also, this can give us some feeling on how rapid is the N convergence. It would be extremely interesting to compare the results of the reduced model for various finite N . We hope to come to these questions in the future.

Finally, let us mention that an open and extremely important problem is how to include systematically $\frac{1}{N}$ corrections to the leading large N behaviour which is described by the reduced model.

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APPENDIX - EXACT SOLUTIONS IN ONE DIMENSION FOR ANY N

The one-dimensional chiral model is analogous to the quantum mechanical symmetric top which is exactly solvable. The exact solution can be used as a test case to measure the validity of the reduced model and of the quenching procedure. We include it here for completeness. We will approach these exact solutions from two directions. The first is the Hamiltonian approach which is directly done in the continuum with no cut-offs. The second is the path integral on the lattice, with the lattice cut-off. In the presence of the cut-off, the answers, of course, look different but they agree when the cut-off is removed.

A. - Hamiltonian solution

In Section 2, we saw that in one dimension the Hamiltonian is proportional to the quadratic Casimir operator

$$H = \frac{q^2}{N} G_R^2 = \frac{q^2}{N} G_L^2 \quad (A.1)$$

where G_L^a, G_R^a are the $U(N)_L \times U(N)_R$ charges. The vacuum is chosen with zero charge

$$G_L^a |0\rangle = 0 = G_R^a |0\rangle \quad (A.2)$$

so that

$$H |0\rangle = 0$$

The next state is obtained by applying the operator U_i^j on the vacuum. The indices i, j belong to the fundamental representations of $U(N)_L \times U(N)_R$, and may be denoted by the Young tableau $(\square, \square)$

$$U_i^j |0\rangle \longrightarrow |(\square, \square)\rangle \quad (A.3)$$

Then, using the commutation rules and the neutrality of the vacuum, we see

$$\begin{aligned} G_R^a |(\square, \square)\rangle &\longrightarrow (U \frac{\lambda^a}{2})_i^j |0\rangle \\ G_R^2 |(\square, \square)\rangle &\longrightarrow (U \frac{\lambda^a \lambda^a}{4})_i^j |0\rangle \end{aligned} \quad (A.4)$$

However, since $(\lambda^a/2 \lambda^a/2)_i^j$ is proportional to δ_i^j by virtue of being the $U(N)$ Casimir operator [not $SU(N)$],

$$\sum_{a=1}^{N^2} \left(\frac{\lambda^a}{2} \frac{\lambda^a}{2} \right)_i^j = \frac{N}{2} \delta_i^j \quad (A.5)$$

we see that this state is an eigenstate of the Hamiltonian with the eigenvalue given as [note it is N independent for $U(N)$, not $SU(N)$]

$$H |(\square, \square)\rangle = \frac{1}{2} g^2 |(\square, \square)\rangle \quad (A.6)$$

Another eigenstate with the same eigenvalue is obtained by applying $U_i^{\dagger j}$ on the vacuum

$$|(\square, \square)\rangle \rightarrow U_i^{\dagger j} |0\rangle$$

Similarly, all eigenstates are obtained by taking higher products of the operators U_i^j and $U_i^{\dagger j}$ on the vacuum. These must be symmetrized according to the rules of Young tableaux so that they form irreducible representations. The left-handed $U(N)_L$ Young tableau will always be equal to the right-handed one for $U(N)_R$. Therefore, it is sufficient to specify only one tableau. The most general one contains both dotted and undotted boxes which correspond to upper and lower indices within the same $U(N)_L$ or $U(N)_R$

(A.7a)

Traces must be taken away between upper and lower indices within the same $U(N)$. The number of dotted plus undotted rows should not exceed N for the tableau to be legal

$$0 \leq r + \dot{r} \leq N \quad (A.7b)$$

We have derived the $U(N)$ quadratic Casimir operator corresponding to the labelling with such a tableau

$$C_2(m_i, n_i) = \frac{1}{2} \sum_{i=1}^N \left\{ n_i^2 + m_i^2 + (n_i + m_i)(N+1-2i) \right\} \quad (A.8)$$

To obtain the $SU(N)$ quadratic Casimir operator, we subtract from (A.8) the $U(1)$ piece, which is $1/2N \left[\sum_1^N (n_i - m_i) \right]^2$. Some simple examples are given for any N

$$\begin{aligned} | \square \square > & \sim \frac{1}{2} (U_{i_1}^{i_1} U_{i_2}^{i_2} + U_{i_2}^{i_1} U_{i_1}^{i_2}) | 0 > \\ | \square \square > & \sim \frac{1}{2} (U_{i_1}^{i_1} U_{i_2}^{i_2} - U_{i_2}^{i_1} U_{i_1}^{i_2}) | 0 > \\ | \square \square > & \sim (U_{i_1}^{i_1} U_{i_2}^{i_2} - \frac{1}{N} \delta_{i_1}^{i_2} \delta_{i_2}^{i_1}) | 0 > \end{aligned} \quad (A.9)$$

The last one corresponds to the adjoint representation. The eigenvalues are easily obtained from the eigenvalues of the Casimir operators for $U(N)$ [not $SU(N)$] in Eq. (A.8). For the above examples, we have

$$\begin{aligned} H | \square \square > & = \frac{N+1}{2} g^2 | \square \square > \\ H | \square \square > & = \frac{N-1}{2} g^2 | \square \square > \\ H | \square \square > & = g^2 | \square \square > \end{aligned} \quad (A.10)$$

Any Green function can now be calculated since we have all eigenstates and eigenvalues. The simplest Green function is

$$\begin{aligned} G(t) & = \langle 0 | \text{Tr} (U(t) U^\dagger(0)) | 0 \rangle \\ & = \langle 0 | \text{Tr} \left\{ e^{-iHt} U(0) e^{iHt} U^\dagger(0) \right\} | 0 \rangle \end{aligned} \quad (A.11)$$

Since $U^\dagger|0\rangle$ is an eigenstate of H we can simply write

$$G(t) = \langle 0 | \text{Tr} (U(0) U^\dagger(0)) | 0 \rangle e^{-i \frac{g^2}{2} t} \quad (\text{A.12})$$

finally, using that U is unitary, we arrive at

$$G(t) = N e^{-ig^2 t/2} \quad (\text{A.13})$$

For Euclidean times, $t = -iT$, this becomes

$$G(\tau) = N e^{-g^2 \tau/2} \quad (\text{A.14})$$

We give one other simple example of such computations

$$\begin{aligned} G(t_1, t_2, t_3) &= \langle 0 | \text{Tr} (U(t_1) U^\dagger(t_2) U(t_3) U^\dagger(0)) | 0 \rangle \\ &= \langle 0 | \text{Tr} (U e^{i(t_1-t_2)H} U^\dagger e^{i(t_2-t_3)H} U e^{i t_3 H} U^\dagger) | 0 \rangle \end{aligned} \quad (\text{A.15})$$

The first and last exponentials can easily be computed since $U^\dagger|0\rangle$ and $\langle 0|U$ are eigenstates.

$$G(t_1, t_2, t_3) = e^{i \frac{g^2}{2} (t_1 - t_2 + t_3)} \langle 0 | \text{Tr} (U U^\dagger e^{i(t_2-t_3)H} U U^\dagger) | 0 \rangle \quad (\text{A.16})$$

using the unitarity of U and neutrality of the vacuum, we arrive at

$$G(t_1, t_2, t_3) = N e^{\frac{i}{2} g^2 (t_1 - t_2 + t_3)} \quad (\text{A.17})$$

B. - Path integral solution

We now perform the path integrals of Section 2 in the case of the one-dimensional chiral model. The presence of the cut-off will be useful for comparison to the cut-off dependence of the reduced model.

The partition function Z is given by the lattice integrals

$$Z = \int \prod_n dU_n e^{\beta N \sum_n \text{Tr} [U_{n+1}^\dagger U_n + U_n^\dagger U_{n+1} - 2]} \quad (\text{A.18})$$

where we have defined $\tilde{\beta}$ [as opposed to β in Eq. (3.22)]

$$\tilde{\beta} = \frac{1}{2q^2a} \quad (\text{A.19})$$

The Green function $G(T)$ will be given by

$$\begin{aligned} G(T) &= \langle \text{Tr}(U(T) U^\dagger(0)) \rangle \\ &= \frac{1}{Z} \int \prod_n dU_n e^{\tilde{\beta} N \sum_n (U_{n+1}^\dagger U_n + U_n^\dagger U_{n+1} - 2)} \text{Tr}(U(T) U^\dagger(0)) \end{aligned} \quad (\text{A.20})$$

The calculation below was briefly outlined in an added note in Ref. 16), and carried out in Appendix B of Ref. 18).

These integrals are performed by writing the action in the form

$$e^{-S} = \prod_n e^{\beta N \text{Tr}(U_{n+1}^\dagger U_n + U_n^\dagger U_{n+1} - 2)} \quad (\text{A.21})$$

and using the character expansion method

$$e^{\tilde{\beta} N (U + U^\dagger - 2)} = \sum_r d_r Z_r(\beta N) \chi_r(U) \quad (\text{A.22})$$

where the sum is over all representations of $U(N)$, d_r is the dimension of the representation, $Z_r(\beta N)$ are coefficients that were computed¹⁶⁾. By using the orthogonality of characters in the form

$$\int dU_n \chi_r(U_{n+1}^\dagger U_n) \chi_{r'}(U_n^\dagger U_{n-1}) = \frac{\chi_r(U_{n+1}^\dagger U_{n-1})}{d_r} \delta_{rr'} \quad (\text{A.23})$$

one can perform all the integrals at each site. As seen from this integral the new $\chi_r(U_{n+1}^\dagger U_{n-1})$ depends on the product of further neighbours. After each integration, we accumulate one power of $(Z_r(\beta N))$. After doing all integrals (including those at open boundaries) except for those at sites T and 0 , we obtain the following expression for the Green function

$$G(T) = \frac{\int dU(T) dU(0) \text{Tr}[U(T) U^\dagger(0)] \sum_r d_r (Z_r)^{T/a} \chi_r(U(0) U^\dagger(T))}{\int dU(T) dU(0) \sum_r d_r (Z_r)^{T/a} \chi_r(U(0) U^\dagger(T))} \quad (\text{A.24})$$

The remaining integrals in the numerator and denominator are performed by using the orthogonality of characters, Eq. (A.23), once again, so that we obtain

$$G(\tau) = N \left(\frac{Z_1}{Z_0} \right)^{\tau/\alpha} = N e^{-(\tau/\alpha) \ln(Z_0/Z_1)} \quad (\text{A.25})$$

This form is in agreement with the Hamiltonian result, except that we need to verify (below) that the exponent becomes $(-g^2 T/2)$ after the cut-off is removed ($\alpha \rightarrow 0$). Here, Z_0 and Z_1 are given by the integrals

$$\begin{aligned} Z_0(\tilde{\beta} N) &= \int dU e^{\tilde{\beta} N \text{Tr}(U+U^\dagger-2)} \\ Z_1(\tilde{\beta} N) &= \frac{1}{N} \int dU \text{Tr} U e^{\tilde{\beta} N \text{Tr}(U+U^\dagger-2)} \\ &= \frac{1}{2N} \frac{\partial}{\partial(\tilde{\beta} N)} Z_0(\tilde{\beta} N) + Z_0(\tilde{\beta} N) \end{aligned} \quad (\text{A.26})$$

where, in the last step, we have used the fact that Z_1 is real. Therefore,

$$\frac{Z_1}{Z_0} = 1 + \frac{1}{2N^2} \frac{\partial}{\partial \tilde{\beta}} \ln Z_0(\tilde{\beta} N) \quad (\text{A.27})$$

Thus, only $Z_0(\beta N)$ is needed, and it was computed before ¹⁵⁾ for any $U(N)$ in the form of a determinant of rank N , except for a new over-all coefficient in the definition

$$Z_0(\tilde{\beta} N) = e^{-2\tilde{\beta} N^2} \begin{vmatrix} I_0 & I_{-1} & I_{-2} & \dots & I_{-N+1} \\ I_1 & I_0 & I_{-1} & \dots & \vdots \\ I_2 & I_1 & I_0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{N-1} & \dots & \dots & \dots & I_0 \end{vmatrix} \quad (\text{A.28})$$

The Bessel functions $I_n(2\tilde{\beta} N) = I_{-n}(2\tilde{\beta} N)$ have arguments $2\tilde{\beta} N$. The large N limit of this expression is also known and it exhibits the phase transition of Gross and Witten ¹⁵⁾

$$-\frac{1}{N^2} \ln Z_0(\tilde{\beta} N) \xrightarrow{N \rightarrow \infty} \begin{cases} \frac{1}{2} \ln 2\tilde{\beta} + \frac{3}{4} = \frac{1}{2} \ln(1/g^2 \alpha) + \frac{3}{4} & , \quad \tilde{\beta} > \frac{1}{2} \\ \tilde{\beta}(2-\tilde{\beta}) = \frac{1}{g^2 \alpha} \left(1 - \frac{1}{4g^2 \alpha}\right) & , \quad \tilde{\beta} < \frac{1}{2} \end{cases} \quad (\text{A.29})$$

These cut-off dependent forms are compared to the reduced model in Section 3.

To establish the continuum result, we need $a \rightarrow 0$, or $\tilde{\beta} \rightarrow \infty$ for any N . This limit is more easily obtained directly from the integral representation of $Z_0(\beta N)$ in Eq. (A.26), rather than the determined form of Eq. (A.28). Diagonalizing U in the integrand and using Weyl's parametrization, we have

$$Z_0(\tilde{\beta} N) = \tilde{C}_N \int_0^{2\pi} \frac{d\phi_1}{2\pi} \dots \frac{d\phi_N}{2\pi} \prod_{i>j} \sin^2\left(\frac{\phi_i - \phi_j}{2}\right) e^{2\tilde{\beta} N \sum_i (\cos \phi_i - 1)} \quad (A.30)$$

For $N = \text{fixed}$ and $\beta \rightarrow \infty$, this integral is evaluated with the steepest descent method. One finds at the saddle point

$$\phi_i = \frac{\alpha_i}{\sqrt{2\tilde{\beta} N}} \quad (A.31)$$

with α_i satisfying

$$\alpha_i = \sum_{j \neq i} \frac{1}{\alpha_i - \alpha_j} \quad (A.32)$$

Substituting this form gives an over-all factor proportional to $\tilde{\beta}^{-\frac{1}{2}N(N-1)}$. With the additional contribution from the saddle point integration, one obtains, for any N ,

$$Z_0(\tilde{\beta} N) \xrightarrow{\tilde{\beta} \rightarrow \infty} C_N \tilde{\beta}^{-N^2/2} \left[1 + O(1/\tilde{\beta}) \right] \quad (A.33)$$

where C_N is an N dependent constant. Substituting this form in Eq. (3.26) and evaluating, in Eq. (A.25),

$$\lim_{a \rightarrow 0} \frac{1}{a} \ln(Z_0/Z_1) \xrightarrow{a \rightarrow 0} g^2/2 \quad (A.34)$$

we obtain the correct continuum result, Eq. (3.14), for any N

$$G(\tau) = N e^{-g^2 \tau/2} \quad (A.35)$$

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FIGURE CAPTIONS

- Fig. 1 : Index loops and momenta
- Fig. 2 : Quenching in two dimensions
- Fig. 3 : Angle between nearest neighbour vectors

Fig. 1

P_1	P_2	P_3	P_4
P_5	P_6	P_7	P_8
P_9	P_{10}	P_{11}	P_{12}
P_{13}	P_{14}	P_{15}	P_{16}

Fig. 2

Fig. 3