REF.TH.3450-CERN

MINIMIZING CONFIGURATIONS OF THE ACTION IN THE
TWISTED EGUCHI KAWAI MODEL

Y. Brihaye

CERN -- Geneva

ABSTRACT

We obtain the most general twist matrix compatible with the minimization condition of the twisted Eguchi Kawai model. For all values of the twist tensor, we propose configurations of matrices which realize the minimum of the action.

REF.TH.3450-CERN

27 October 1982

Recently, it has been stressed that twisted solutions of the Yang-Mills equations in a finite and periodic box¹⁾ can play a role in the large N lattice gauge theories. Indeed, A. Gonzales Arroyo and M. Okawa²⁾ using such a solution have modified the Eguchi-Kawai model³⁾ in such a way that the twisted version of the model (T.E.K.) become valid in the weak coupling region; this was not the case for the original model of Eguchi and Kawai⁴⁾.

The authors of Ref. 2) used a twisted Yang-Mills solution valid in $SU(N)$ gauge theory only if $N = 4k$ ($k = 1, 2, 3, \dots$); their solutions are not in the t'Hooft ansatz¹⁾ and they raise the problem of finding the most general twisted Yang Mills solution in $SU(N)$.

In this note, we obtain a set of solutions in $SU(N)$ including the solutions of Refs. 1) and 2) as particular cases; furthermore, our solutions can reproduce the most general twist matrix; at least in the absence of background fields $F_{\mu\nu}$ through the finite box.

First, let us briefly describe the problem; from the point of view of Ref. 1), we have to find $4SU(N)$ matrices Ω_μ ($\mu = 1, 2, 3, 4$) which are space-time independent and which satisfy the following commutation relations.

$$\Omega_\mu \Omega_\nu = Z_{\mu\nu} \Omega_\nu \Omega_\mu \quad (1)$$

In (1) the $Z_{\mu\nu}$ denote elements of the centre of $SU(N)$. They are related to the so-called twist numbers $n_{\mu\nu}$ (defined modulo N) by

$$Z_{\mu\nu} = \exp\left(\frac{2i\pi}{N} n_{\mu\nu}\right) \quad ; \quad n_{\mu\nu} = -n_{\nu\mu}. \quad (2)$$

In order to obtain configurations Ω_μ which minimize the action of the T.E.K. model proposed in Ref. 2), we have to consider only $n_{\mu\nu}$ satisfying the relation

$$K = \frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = 0 \mod N \quad (3)$$

where $K = n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23}$.

The relation (3) means absence of any background gauge fields $F_{\mu\nu}$ in the periodic box. In Ref. 1), the author proposes several configurations Ω_μ for arbitrary values of K (i.e. in the presence of background gauge fields) but we will not consider such solutions in this note.

So far, a first problem is to find the most general $n_{\mu\nu}$ satisfying (3), then a second one is to construct configurations of Ω_μ obeying (1) and (2). As pointed out by t'Hooft, the equation (3) is invariant under the transformation

$$\eta_{\mu\nu} \rightarrow \eta'_{\mu\nu} = U_{\mu\rho} U_{\nu\sigma} \eta_{\rho\sigma} \quad (4)$$

for U belonging to $SL[N, Z(N)]$, [the set of unit determinant matrices with coefficients in $Z(N)^5$]. Using (4) one can put n in a canonical form. The best we can do is

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & l & m \\ -k & -l & 0 & 0 \\ 0 & -m & 0 & 0 \end{pmatrix} \quad (5)$$

with

$$k m = 0 \text{ mod } N \quad (6)$$

in order to satisfy (3). The solutions Ω_μ proposed by t'Hooft are only for the case $k = 0$ in (5). As pointed out in Ref. 1), it is the most general case as long as N is a prime number, but if N is not a prime number, we have to consider matrices like (5). In fact we have proved the following property

Property

Let m and k be different from 1 and satisfy (6).

Then the matrix (5) cannot be reduced to the $k = 0$ case if and only if l in (5) is a common multiple of m and k .

To prove the property, write the matrix (5) as

$$n_{\mu\nu} = a_{\mu} b_{\nu} - a_{\nu} b_{\mu} + c_{\mu} d_{\nu} - c_{\nu} d_{\mu}$$

in all the possible ways with a, b, c, d four quadruples of $Z(N)$ numbers.

Then if k, l, m are such as in the hypothesis, no one of the four quadruples is a linear combination of the three others (as would be necessary to reach the t'Hooft ansatz).

So we have solved the first part of the problem; we have now to construct Ω_μ matrices satisfying (1) and (2). Therefore, let us introduce a lemma and some definitions which will be useful.

Lemma

Given three integers N, k, m . If

$$k m = 0 \pmod{N} \quad (6)$$

Then there exist integers p, q, p_1, q_1 such that

$$k = p p_1, \quad m = q q_1 \quad (7)$$

$$p q \equiv N \quad (8)$$

The proof of this lemma follows trivially from elementary arithmetic.

Definition 1⁶⁾

For any integer n , we define P_n and Q_n , two matrices of $SU(n)$, by

$$P_n = \mu_n \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}, \quad Q_n = \mu_n \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & a_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & a_n \end{pmatrix} \quad (9)$$

where

$$a_n = \exp \frac{2i\pi}{n}, \quad \mu_n = \exp \frac{i\pi(1-n)}{n} \quad (10)$$

Definition 2

Given N, k, m satisfying (6); p, q as constructed in the lemma; we define four $SU(N)$ matrices by

$$\begin{aligned} P &= P_p \otimes \mathbb{1}_q, & R &= \mathbb{1}_p \otimes P_q \\ Q &= Q_p \otimes \mathbb{1}_q, & S &= \mathbb{1}_p \otimes Q_q \end{aligned} \quad (11)$$

where $\otimes$ denote the usual tensor product of matrices.

Let us remark that the matrices (11) commute with each other except for the following:

$$PQ = a_p QP \quad (12)$$

$$RS = a_q SR \quad (13)$$

We are now in position to construct the matrices Ω_μ of (1) and (2); to this aim, let us consider the following ansatz

$$\Omega_\mu = P^{a_\mu} Q^{b_\mu} R^{c_\mu} S^{d_\mu} \quad (14)$$

where a_μ, b_μ are defined modulo p while c_μ, d_μ are defined modulo q . From (14), the commutation relations (1) can easily be derived; it appears that

$$m_{\mu\nu} = q(a_\mu b_\nu - a_\nu b_\mu) + p(c_\mu d_\nu - c_\nu d_\mu) \quad (15)$$

Taking

$$\begin{aligned} a_\mu &= (0, 1, 0, 0) \quad , \quad c_\mu = (1, 0, 0, 0) \\ b_\mu &= (0, 0, 1, q) \quad , \quad d_\mu = (0, 0, p, 0) \end{aligned} \quad (16)$$

we recover the canonical form (5). Let us give an explicit example: if $N = 4, m = k = p = q = 2$ then the configurations

$$\Omega_1 = P, \Omega_2 = QA, \Omega_3 = PQ, \Omega_4 = QS \quad (17)$$

is the new solution exhibited in Ref. 2), all Ω_μ anticommute with each other; they form a representation of the Dirac matrices in Euclidean space. If p (resp. q) is equal to 1, then from (11) P and Q (resp. R and S) are equal to $\mathbb{I}_N$; consequently (14) reduces to the t'Hooft ansatz. On the other hand, the matrices (14) furnish minimizing configurations to action of the T.E.K. model

$$S = -\beta \sum_{\mu \neq \nu=1}^4 z_{\mu\nu} \text{tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) \quad (18)$$

for any values of $z_{\mu\nu}$ compatible with (3). Indeed taking $U_\mu = \Omega_\mu^t$ gives $S_{\text{Min}} = -12\beta N$ which is indeed the minimum²⁾.

So far, we have obtained here a general⁷⁾ class of twisted Yang-Mills solutions in the absence of background gauge fields $F_{\mu\nu}$ [i.e. submitted to the condition (3)]. It would be interesting to apply the method above to find new solutions (if any) with

$$\frac{1}{4} \eta_{\mu\nu} \tilde{\eta}_{\mu\nu} \neq 0 \mod N \quad (19)$$

Of course this would imply the presence of non-zero gauge potential A_μ and gauge fields $F_{\mu\nu}$ in the box. At another level it would be interesting to discover some relevance of the solutions (11) and (14) in the large N lattice gauge theories.

ACKNOWLEDGEMENTS

I would like to thank R. Petronzio and E. Onofri for carefully reading the manuscript. I am grateful to E. Onofri for interesting me in the subject.

REFERENCES

- 1) G. t'Hooft, Comm. Math. Phys. 81 (1981) 267.
- 2) A. Gonzales-Arroyo and M. Okawa, Brookhaven Nat. Lab. 31689 (July 1982).
- 3) T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063.
- 4) G. Bhanot, Urs. M. Heller and H. Neuberger, The Institute for Advanced Study. Princeton (February 1982).
- 5) Here, $Z(N)$ is the set of integers $\{0,1,\dots,N-1\}$ with addition and multiplication defined modulo N .
- 6) This definition has been introduced in Ref. 1).
- 7) The solutions (14) may be multiplied by elements of the centre of $SU(N)$ and by global gauge transformations.