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SYMMETRY BREAKING IN $d = 11$ SUPERGRAVITY
ON THE PARALLELIZED SEVEN-SPHERE

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ABSTRACT

In the vacuum solution proposed by Englert, in which the three-index photon of $d = 11$ supergravity plays the role of a non-vanishing torsion which parallelizes the seven-sphere, all supersymmetries are broken while a bosonic symmetry group $SO(3,2) \otimes G_2$ remains. These results are derived by constructing Killing vectors in the group manifold.

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It has been known for a long time that the $N = 8$ (or $N = 4$) supergravity models in $d = 4$ dimensions can be most easily obtained by first considering $N = 1$ supergravity in $d = 11$ (or $d = 10$) and by considering as spacetime not a Minkowski spacetime M^{11} (or M^{10}), but rather a direct product of an M^4 with a torus T^7 (or T^6). Expanding all fields on T^7 in a Fourier series, and keeping only the zero modes, one gets after integration over the torus a massless supergravity theory in $d = 4$ with the maximal number of local supersymmetries, namely 8 [1].

Recently, there has been renewed interest in such Kalusza-Klein ideas, because other exact solutions of the $d = 11$ (or $d = 10$) field equations have been found which describe vacua with less trivial geometries. In these solutions all fermionic fields still vanish, but the photon curls^{*)} $F_{\alpha\beta\gamma\delta}$ (or $F_{\alpha\beta\gamma}$) are non-vanishing and are the sources which drive the Einstein equations to these non-trivial spacetimes. In what follows, we restrict ourselves to the case $d = 11$ [2].

The first non-trivial solution [3] has F_{abcd} proportional to ϵ_{abcd} but all other components of $F_{\alpha\beta\gamma\delta}$ vanishing. In this sense the 3-index photon predicts the emergence of a final $d = 4$ spacetime. The energy momentum tensor satisfies $T_{ab} \sim \eta_{ab}$ and $T_{\alpha\beta} \sim \delta_{\alpha\beta}$ while $T_{a\alpha} = 0$. Thus one ends up with Einstein spaces ($R^a_b \sim \delta^a_b$ and $R^\alpha_\beta \sim \delta^\alpha_\beta$). A special case arises when one assumes that space-time is a direct product of maximally symmetric 4- and 7-spaces

$$R^a_b = \lambda \delta^a_b, \quad R^{\alpha\beta}_{\gamma\delta} = \nu \delta^{\alpha\beta}_{\gamma\delta} \quad (1)$$

In this case the Einstein equations show that the 4-space is an anti-deSitter space and the 7-space a deSitter space, namely a 7-sphere. This case is especially interesting because it leads to a 4-dimensional supergravity theory with eight unbroken supersymmetries [4]; in fact, its massless sector probably reproduces [4] the gauged $N = 8$ model [5]. It has been shown [6] that this vacuum has an $Osp(4/8)$ symmetry, which is also the symmetry group one expects for the gauged $N = 8$ model in 4 dimensions [7].

*) Curved indices in $d = 11$ are denoted by $\hat{\alpha}, \hat{\beta}, \dots$. Flat indices in $d = 4$ are denoted by $a, b, \dots$, in $d = 7$ by $\alpha, \beta, \dots$ and in $d = 11$ by $\hat{a}, \hat{b}$. The 11-bein is decomposed as $V^{\hat{a}} = (V^a, B^\alpha)$ and the 11-connection as $\omega^{\hat{a}\hat{b}} = (\omega^{ab}, B^{\alpha\beta}, K^{\alpha\beta})$. The Dirac matrices in $d = 11, 7, 4$ are written as $\Gamma^{\hat{a}}, \tau^\alpha, \gamma^a$ and the τ^α are 8×8 matrices.

A second solution [8] has not only F_{abcd} but also $\hat{F}_{\hat{a}\hat{b}\hat{c}\hat{d}}$ non-vanishing, again in such a way that T_{ab} and $T_{\alpha\beta}$ are proportional to the 4- and 7-metrics while $T_{a\alpha} = 0$. Most interestingly, the 3-index photon $A_{\alpha\beta\gamma}$ can be interpreted as the torsion one must add to the metric 7-connection in order that the resulting 7-curvature vanishes. In that case the 7-sphere is parallelized [9].

In this letter, we will determine which symmetries the second (Englert [8]) vacuum solution has. In other words, we want to determine which supersymmetry and other bosonic transformation exist which leave the vacuum expectation values of all fields invariant up to gauge transformations. The basic idea is that Kaluza-Klein compactification of the $d = 11$ model on a space which is topologically but not geometrically equal to a 7-sphere, leads to a spontaneous broken version of the gauged $N = 8$ model [10]. To implement this programme we note that local supersymmetry transformations in x -space correspond to general fermionic coordinate transformations with transport terms (Lie derivatives) in a bigger space, namely superspace or the group-manifold. Thus we shall first "lift" the x -space vacuum fields to the bigger space. This we shall do by using the field equations of the bigger space. (We could also have used the gauge completion program for the on-shell theory [11].) It is here that we choose the group manifold approach [12] over the superspace method, because in superspace one has to impose constraints on torsions and curvatures before one can begin defining an action, whereas in the group manifold approach these constraints follow as (part of) the field equations. Having lifted all x -space fields to the bigger space, one can consider from then on this big space as superspace and study the symmetries of these fields.

The Lie derivative of a field is the difference between this field at $x^{\hat{\mu}} + \xi^{\hat{\mu}}(x)$ in the coordinate system $\bar{x}^{\hat{\mu}} = x^{\hat{\mu}} + \xi^{\hat{\mu}}(x)$ and the field at $x^{\hat{\mu}}$ in the coordinate system $x^{\hat{\mu}}$. We have a physical symmetry when this difference is equal to a gauge transformation; in this case $\xi^{\hat{\mu}}(x)$ are the Killing vectors [13]. Since all coordinate transformations are already accounted for by $\xi^{\hat{\mu}}$, the only gauge

transformations left are local $SO(1,10)$ Lorentz rotations and Maxwell transformations. Hence, we will investigate for which fermionic vectors $\vec{q}$ the Lie derivative along $\vec{q}$ of all vacuum fields is equal to a local Lorentz rotation with parameters $(W_{\vec{q}})^m_n$. For example, for the 11-bein $V^{\hat{a}}(x, z, \theta)$ we must solve

$$\ell_{\vec{q}} V^{\hat{a}} = \underline{[\vec{q}]} dV^{\hat{a}} + d \underline{[\vec{q}]} V^{\hat{a}} = (W_{\vec{q}})^{\hat{a}}_{\hat{b}} V^{\hat{b}} \quad (2)$$

Since $V^{\hat{a}}$ is known from the lifting process, one finds an equation for the Killing vector $\vec{q}$. The bosonic symmetries will be investigated in a similar way, except that the Killing vectors $\vec{k}$ will be bosonic in this case. We shall find that no fermionic $\vec{q}$ exists, i.e., all supersymmetries are broken. The bosonic symmetry will be explicitly shown to be $SO(3,2) \times G_2$ as anticipated on general grounds in [8]. We now turn to a derivation of our results.

The fields of $d = 11$ supergravity are the elfbein $V^{\hat{a}}$, the gravitino ψ and the 3-form A . They can be either considered as forms on the x -space manifold, in which case they are expanded only in $dx^{\hat{\mu}}$ differentials, or as forms in superspace, in which case they have both $dx^{\hat{\mu}}$ and $d\theta$ parts. In the first case, the elfbeins $V^{\hat{a}}$ form a complete basis for the expansion of all other forms while in the second case a complete basis includes both $V^{\hat{a}}$ and ψ .

In the group-manifold formulation, one directly obtains the structure of a generic solution of the theory in superspace from the variational principle. This structure exhibits the property of rheonomy which amounts to saying that the complete solution in superspace is uniquely determined (up to gauge and coordinate transformations) by the solution of the field equations at x -space only.

Explicitly, for the most general solution in superspace, the curvatures 2-forms defined by [14]

$$R^{\hat{a}\hat{b}} = d\omega^{\hat{a}\hat{b}} - \omega^{\hat{a}\hat{c}} \wedge \omega^{\hat{c}\hat{b}} \quad (3a)$$

$$R^{\hat{a}} = dV^{\hat{a}} - \omega^{\hat{a}\hat{b}} \wedge V^{\hat{b}} - \frac{i}{2} \bar{\psi} \wedge \Gamma^{\hat{a}} \psi \quad (3b)$$

$$\rho = d\psi - \frac{1}{4} \omega^{\hat{a}\hat{b}} \wedge V^{\hat{b}} - \frac{i}{2} \bar{\psi} \wedge \Gamma^{\hat{a}} \psi \quad (3c)$$

$$R^{\hat{a}} = dA - \frac{1}{2} \bar{\psi} \wedge \Gamma_{\hat{a}\hat{b}} \psi \wedge V^{\hat{a}} \wedge V^{\hat{b}} \quad (3d)$$

obey the following expansion in the anholonomic basis of $(V^{\hat{a}}, \psi)$ [14]:

$$R^{\hat{a}} = 0 \equiv R_{(sol)}^{\hat{a}} \quad (4a)$$

$$R^{\hat{a}} = F_{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4} V^{\hat{a}_1} \wedge V^{\hat{a}_2} \wedge V^{\hat{a}_3} \wedge V^{\hat{a}_4} \equiv R_{(sol)}^{\hat{a}} \quad (4b)$$

$$\rho = \rho_{(sol)} \equiv \rho_{\hat{a}\hat{b}} V^{\hat{a}} \wedge V^{\hat{b}} - \frac{i}{3} \left(\Gamma^{\hat{a}_1 \hat{a}_2 \hat{a}_3} \psi \wedge V^{\hat{a}_4} + \frac{1}{8} \Gamma^{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{m}} \psi \wedge V_{\hat{m}} \right) F_{\hat{a}_1 \dots \hat{a}_4} \quad (4c)$$

$$\begin{aligned} R^{\hat{a}\hat{b}} = R_{(sol)}^{\hat{a}\hat{b}} \equiv R^{\hat{a}\hat{b}}_{\hat{m}\hat{n}} V^{\hat{m}} \wedge V^{\hat{n}} + i \rho_{\hat{m}\hat{n}} \left(\frac{1}{2} \Gamma^{\hat{a}\hat{b}\hat{m}\hat{n}} - \frac{2}{9} \Gamma^{\hat{m}\hat{n}} [\hat{a}^{\hat{b}}]_{\hat{\delta}} \hat{c}^{\hat{\delta}} \right. \\ \left. + 2 \Gamma^{\hat{a}\hat{b}} [\hat{m}^{\hat{n}}]_{\hat{\delta}} \hat{c}^{\hat{\delta}} \right) \psi \wedge V_{\hat{c}} + \bar{\psi} \wedge \Gamma^{\hat{m}\hat{n}} \psi F_{\hat{m}\hat{n}}^{\hat{a}\hat{b}} \\ + \frac{1}{24} \bar{\psi} \wedge \Gamma^{\hat{a}\hat{b}\hat{c}_1 \dots \hat{c}_4} \psi F_{\hat{c}_1 \dots \hat{c}_4} \end{aligned} \quad (4d)$$

The objects $F_{\hat{a}_1 \dots \hat{a}_4}$, $\delta_{\hat{a}\hat{b}}$, $R^{\hat{a}\hat{b}}_{\hat{m}\hat{n}}$, which correspond to the VVVV and VV parts of the expansion and which can be identified with the supercovariant field strengths of the x-space approach [15], obey the following anholonomized propagation equations

$$\Gamma^{\hat{a}\hat{b}\hat{c}} \rho_{\hat{b}\hat{c}} = 0 \quad (5a)$$

$$\mathcal{D}_{\hat{m}} F^{\hat{m}\hat{a}_1 \dots \hat{a}_3} + \frac{1}{96} \epsilon^{\hat{c}_1 \dots \hat{c}_3 \hat{a}_1 \dots \hat{a}_8} F_{\hat{a}_1 \dots \hat{a}_4} F_{\hat{a}_5 \dots \hat{a}_8} = 0 \quad (5b)$$

$$R^{\hat{a}\hat{m}}_{\hat{b}\hat{m}} - \frac{1}{2} \delta_{\hat{b}}^{\hat{a}} R^{\hat{m}\hat{n}}_{\hat{m}\hat{n}} - 6 F^{\hat{a}\hat{c}_1 \dots \hat{c}_3} F_{\hat{b}\hat{c}_1 \dots \hat{c}_3} + \frac{3}{4} \delta_{\hat{b}}^{\hat{a}} F^{\hat{c}_1 \dots \hat{c}_4} F_{\hat{c}_1 \dots \hat{c}_4} = 0 \quad (5c)$$

The aforementioned property of rheonomy corresponds to the fact, visible in (4), that all components of the curvatures are expressed in terms of the coefficients of

the VVVV and VV parts. If one knows these objects in x-space, namely if one knows an explicit solution of the field equation (5) restricted to the x-manifold, one can insert them into (4) obtaining in this way the derivatives of all fields in θ -directions and thus one is allowed to lift the x-space fields to the whole superspace.

Note that the whole stock of eqs. (4) is obtained from the equations of motion on the group-manifold. However, (4c) and (4d) are consequences of (4a) and (4b) if one inserts them in Bianchi identities in superspace.

In the conventional superspace approach one would consider eqs. (4a) and (4c) as constraints defining the geometry and derive the rest. In the group-manifold all eqs. (4) are on the same footing and come from the variational principle.

We recall the classical solution of $d = 11$ supergravity in x-space with non-trivial torsion on the 7-sphere, using covariantly constant spinors η [10]

$$F_{a_1 \dots a_4} \big|_{\theta=0} = e \epsilon_{a_1 \dots a_4}, \quad F_{\alpha_1 \dots \alpha_4} \big|_{\theta=0} = a \bar{\eta} \tau_{\alpha_1 \dots \alpha_4} \eta \quad (6a)$$

$$F_{\hat{a}_1 \dots \hat{a}_4} \big|_{\theta=0} = 0 \quad \text{otherwise} \quad (6b)$$

$$R^{ab}{}_{cd} \big|_{\theta=0} = \lambda \delta_{cd}^{[ab]} \quad (6c)$$

$$R^{\alpha\beta}{}_{\gamma\delta} \big|_{\theta=0} = \nu \delta_{\gamma\delta}^{[\alpha\beta]} \quad (6d)$$

$$R^{\hat{a}\hat{b}}{}_{\hat{c}\hat{d}} \big|_{\theta=0} = 0 \quad \text{otherwise} \quad (6e)$$

$$\psi_{\hat{a}} \big|_{\theta=0} = 0 \Rightarrow \rho_{\hat{a}\hat{b}} \big|_{\theta=0} = 0 \quad (6f)$$

$$\omega_{\hat{a}}{}^{\alpha\beta} \big|_{\theta=0} \equiv K_{\hat{a}}{}^{\alpha\beta} \big|_{\theta=0} = 0 \quad (6g)$$

where e and a are constants, one finds again for $a = 0$ the special solution of [4], which has the $Osp(4/8)$ symmetry [6]. For $a \neq 0$, one has the solution of [8]. In (6a)

η is a commuting 8-component Majorana (real) spinor 0-form in 7-space which satisfies $\bar{\eta}\eta = 1$ and

$$\mathcal{D}^{SO(7)} \eta \equiv d\eta - \frac{1}{4} B^{\alpha\beta} \tau_{\alpha\beta} \eta = b B^{\alpha} \tau_{\alpha} \eta \quad (7)$$

Substituting (6) and (7) into (5), and using the identity

$$1 = \eta \bar{\eta} + \tau_{\alpha} \eta \bar{\eta} \tau^{\alpha}, \quad \bar{\eta} = \eta^{\dagger} \quad (8)$$

one obtains:

$$\lambda = -16e^2 - 56a^2, \quad \nu = 4e^2 + 20a^2, \quad b = -\frac{3}{2}e \quad (9)$$

The consistency of (7) relates b to η and yields $4b^2 = \nu$, hence

$$a = \alpha \frac{e}{2}, \quad \alpha = \pm 1, \quad \lambda = -30e^2, \quad \nu = 9e^2 \quad (10)$$

Calling x^a and z^{α} the coordinates on 4-space and 7-space, respectively, it is very easy to write down anti-deSitter 4-beins and deSitter 7-beins, and their associated metric connections satisfying (6c-6d) with values in (10)

$$V^a|_{\theta=0} \equiv V^a_o(x) = \frac{1}{e} \sqrt{\frac{2}{15}} \frac{dx^a}{1+x^2}, \quad \omega^{ab}|_{\theta=0} \equiv \omega^{ab}_o = \frac{+4 dx^{[a} x^{b]}}{1+x^2} \quad (11a)$$

$$B^{\alpha}|_{\theta=0} \equiv B^{\alpha}_o(z) = -\frac{1}{e} \frac{2}{3} \frac{dz^{\alpha}}{1-z^2}, \quad B^{\alpha\beta}|_{\theta=0} \equiv B^{\alpha\beta}_o = \frac{-4 dz^{[\alpha} z^{\beta]}}{1-z^2} \quad (11b)$$

At this point we can use the rheonomy of (4) in order to determine the first derivatives of all fields in θ on the surface $\theta = 0$ which is identified with our x -space.

Splitting the indices and substituting (6) into (4) we get

$$R^a|_{\theta=0} = (dV^a - \omega^a{}_b V^b - \frac{i}{2} \bar{\Psi} \gamma^a \Psi - K^{a\beta} B^\beta)|_{\theta=0} = 0 \quad (12a)$$

$$R^\alpha|_{\theta=0} = (dB^\alpha - B^\alpha{}_\beta B^\beta - \frac{i}{2} \bar{\Psi} \gamma_5 \tau_\alpha \Psi + K^{m\alpha} V_m)|_{\theta=0} = 0 \quad (12b)$$

$$\begin{aligned} R^{ab}|_{\theta=0} &= (d\omega^{ab} - \omega^a{}_c \omega^c{}_b + K^\alpha{}_\gamma K^b{}_\gamma)|_{\theta=0} = \\ &= 30e^2 V_0^a V_0^b + 4e \bar{\Psi}_0 \gamma_5 \Sigma_{ab} \Psi_0 \\ &\quad - \frac{ie}{24} \bar{\Psi}_0 \tau^{\lambda_1 \dots \lambda_4} \Sigma_{ab} \Psi_0 \bar{\eta} \tau^{\lambda_1 \dots \lambda_4} \eta \end{aligned} \quad (12c)$$

$$\begin{aligned} R^{\alpha\beta}|_{\theta=0} &= (dB^{\alpha\beta} - B^\alpha{}_\gamma B^\beta{}_\gamma + K^{m\alpha} K_m{}^\beta)|_{\theta=0} = \\ &= -ge^2 B_0^\alpha B_0^\beta - ie \bar{\Psi}_0 \tau_{\alpha\beta} \gamma_5 \Psi_0 + \\ &\quad + \frac{\alpha e}{2} \bar{\Psi}_0 \tau^{\gamma\delta} \Psi_0 \bar{\eta} \tau^{\alpha\beta} \gamma_\delta \eta + \frac{\alpha e}{2} \bar{\Psi}_0 \tau_\gamma \Psi_0 \bar{\eta} \tau^{\alpha\beta} \gamma^\gamma \eta \end{aligned} \quad (12d)$$

$$\begin{aligned} e|_{\theta=0} &= (d\psi + \frac{i}{2} \omega^a{}_b \Sigma_{ab} \psi - \frac{1}{4} B^{\alpha\beta} \tau_{\alpha\beta} + \frac{1}{2} K^{m\beta} \gamma_5 \gamma_m \tau_\beta \psi)|_{\theta=0} = \\ &= e B_0^\alpha \tau_\alpha \psi_0 - 2e \gamma_5 \gamma_m \psi_0 V_0^m - \frac{ie}{6} \gamma_5 \tau^{\alpha\beta} \gamma_\alpha \psi_0 B_0^\beta \bar{\eta} \tau_{\alpha\beta} \gamma_\delta \eta \\ &\quad - \frac{ie}{48} e \gamma_m \tau^{\lambda_1 \dots \lambda_4} \psi_0 V_0^m \bar{\eta} \tau^{\lambda_1 \dots \lambda_4} \eta - \frac{ie}{4} \gamma_5 \tau_{\alpha\beta} \psi_0 B_0^\alpha \bar{\eta} \tau^{\alpha\beta} \gamma_\gamma \eta \end{aligned} \quad (12e)$$

$$\begin{aligned} R^\square|_{\theta=0} &= (dA + i \bar{\Psi} \Sigma_{ab} \Psi V^a V^b - \bar{\Psi} \tau_\alpha \gamma_5 \gamma_b \Psi B^\alpha V^b - \frac{1}{2} \bar{\Psi} \tau_{\alpha\beta} \Psi B^\alpha B^\beta)|_{\theta=0} = \\ &= e \epsilon_{abcd} V_0^a V_0^b V_0^c V_0^d + \frac{\alpha e}{2} \bar{\eta} \tau_{\alpha\beta} \gamma_\delta \eta B_0^\alpha B_0^\beta B_0^\gamma B_0^\delta \end{aligned} \quad (12f)$$

$$\begin{aligned} R^{a\beta}|_{\theta=0} &= (dK^{a\beta} - \omega^a{}_b K^{b\beta} - B^\beta{}_\gamma K^{a\gamma})|_{\theta=0} = \\ &= -\frac{\alpha e}{4} \bar{\Psi}_0 \gamma_5 \gamma^a \tau_\beta \gamma_\delta \Psi_0 \bar{\eta} \tau^{\beta\gamma} \gamma^\delta \eta \end{aligned} \quad (12g)$$

In the aforementioned formulae, $\psi_0 = \psi|_{\theta=0}$ denotes the value of the gravitino potential at $\theta = 0$ in superspace. In general we could write

$$\psi_0 \equiv \psi|_{\theta=0} = \psi_{\hat{\mu}} dx^{\hat{\mu}} + 0 d\theta \quad (13)$$

where 0 is a 32×32 (x,z) dependent matrix, which, however, can always be reduced to unity by a redefinition of $\theta' = 0\theta$. Since, moreover, for our solution

$\psi_{\hat{\mu}}|_{t=0} = 0$, we conclude

$$\psi_0 = d\theta \quad (14)$$

(We exclude the case $\theta = 0$ since in that case there would be no θ -dependence of any field.)

In the case of the solution with $a = 0$, all curvature components in (12) are constants, and transporting the right-hand sides of (12) to the left-hand sides, one recognizes the left-invariant 1-forms which satisfy an $Osp(4/8)$ algebra in superspace [6]. These forms obviously admit $10 + 28$ bosonic Killing vectors of $SO(3,2) \otimes SO(8)$ and 8 fermionic Killing vectors which together form an $Osp(4/8)$ algebra.

In the present case the curvature components are no longer constants; rather they become z-dependent. Nevertheless we can perform the lifting order by order in θ obtaining all forms V^a , B^a , ω^{ab} , B^{ab} , ψ , K^{ab} , A as polynomials in θ .

In order to decide whether the lifted potentials admit symmetry (Killing vectors), it is sufficient to know the linear part of the θ -expansion. This is so because of rheonomy: the solution is completely determined by its restriction to the $\theta = 0$ surface. However, to compute the θ -derivatives at $\theta = 0$, one needs to know the terms linear in θ of the forms. For our purposes we just need the explicit lifting of the following two potentials

$$V^a = V_0^a + \frac{i}{2} \bar{\theta} \gamma^a d\theta + \mathcal{O}(\theta^2) \quad (13a)$$

$$K^{m\beta} = \frac{\alpha e}{4} (\bar{\theta} \gamma_5 \gamma^a \tau_{\beta\delta} d\theta) (\bar{\eta} \tau^{\beta\gamma\delta} \eta) + \mathcal{O}(\theta^2) \quad (13b)$$

If the solution preserves some supersymmetries, there must be at least one fermionic Killing vector leaving all the potentials invariant, in particular (13a) and (13b). Let us denote this Killing vector by $\vec{q}$. At order θ , it can be written as follows:

$$\vec{q} = H \frac{\partial}{\partial \theta} + F^a \theta \frac{\partial}{\partial x^a} + F^\alpha \theta \frac{\partial}{\partial z^\alpha} + \mathcal{O}(\theta^2) \quad (14)$$

where H , F^a , F^α are 32×32 x, z -dependent matrices.

The Killing equation states that the Lie derivative of the potential considered along $\vec{q}$ has to be an $SO(1,10)$ rotation $\hat{W}_{(\vec{q})}^{\hat{a}\hat{b}}$ on the same potential. Since $\vec{q}$ is fermionic, $\hat{W}_{(\vec{q})}^{\hat{a}\hat{b}}$ starts at order θ and therefore at $\theta = 0$ it vanishes. The invariance equations of V^a and $K^{m\beta}$ at $\theta = 0$ are therefore

$$\ell_{\vec{q}} V^a \Big|_{\theta=0} = \left(\underline{\vec{q}} dV^a + d\underline{\vec{q}} V^a \right) \Big|_{\theta=0} = 0 \quad (15a)$$

$$\ell_{\vec{q}} K^{m\beta} \Big|_{\theta=0} = \left(\underline{\vec{q}} dK^{m\beta} + d\underline{\vec{q}} K^{m\beta} \right) \Big|_{\theta=0} = 0 \quad (15b)$$

From eq. (15a) and (15b), using (11a), (13a), (13b) and (14), one respectively obtains:

$$\frac{i}{2} H \gamma^a + \sqrt{\frac{2}{15}} \frac{1}{e} F^a \frac{1}{1+x^2} = 0 \quad (16a)$$

$$\frac{\alpha e}{4} H \gamma_5 \gamma^m \tau_{\gamma\delta} (\bar{\psi} \tau^{\beta\gamma\delta} \psi) = 0 \quad (16b)$$

Equations (16) are just ordinary matrix equations which can be solved yielding $H = 0 = F^a$ so that $\vec{q} = 0$: the vacuum has therefore no local supersymmetry left.

Let us now turn to the bosonic symmetries of solution (6). We first solve the symmetry equation

$$\ell_{\vec{k}} V^{\hat{a}} = \left(W_{\vec{k}} \right)^{\hat{a}}_{\hat{b}} V^{\hat{b}} \quad (17)$$

for the elfbein $V^{\hat{a}} = (V_0^a, B_0^\alpha)$ given in eq. (11). The most general Killing vector $\vec{k}$ of bosonic type, which satisfies (17), is a linear combination with constants coefficients

$$\vec{k} = c^a \vec{k}_a + c^{ab} \vec{k}_{ab} + c^\alpha \vec{k}_\alpha + c^{\alpha\beta} \vec{k}_{\alpha\beta} \quad (18)$$

where $\{\vec{k}_a, \vec{k}_{ab}\}$ and $\{\vec{k}_\alpha, \vec{k}_{\alpha\beta}\}$ satisfy $SO(2,3)$ and $SO(8)$ algebras respectively

$$\begin{aligned} [\vec{k}_{ab}, \vec{k}_{cd}] &= -2\eta_{[c} \vec{k}_{b]d}, \quad [\vec{k}_{\alpha\beta}, \vec{k}_{\gamma\delta}] = -2\eta_{[\gamma} \vec{k}_{\beta]\delta} \\ [\vec{k}_{ab}, \vec{k}_c] &= -\eta_c [a \vec{k}_{b]} \quad , \quad [\vec{k}_{\alpha\beta}, \vec{k}_\gamma] = -\eta_\gamma [\alpha \vec{k}_{\beta}] \\ [\vec{k}_a, \vec{k}_b] &= -60e^2 \vec{k}_{ab} \quad , \quad [\vec{k}_\alpha, \vec{k}_\beta] = 18e^2 \vec{k}_{\alpha\beta} \end{aligned} \quad (19)$$

and $\eta_{ab} = (+, -, -, -)$, $\eta_{\alpha\beta} = (-, -, -, -, -, -, -)$. The explicit form of the Killing vectors $\vec{k}$ is

$$\begin{aligned} \vec{k}_{ab} &= x_{[a} \partial_{b]} \quad , \quad \vec{k}_{\alpha\beta} = z_{[\alpha} \partial_{\beta]} \\ \vec{k}_a &= \sqrt{\frac{15}{2}} e \left\{ (1-x^2) \partial_a + 2x_a x \cdot \partial \right\}, \quad \vec{k}_\alpha = -\frac{3}{2} e \left\{ (1+z^2) \partial_\alpha - 2z_\alpha z \cdot \partial \right\} \end{aligned} \quad (20)$$

In this case, the Lorentz matrices $\hat{W}_{\vec{k}}^{\hat{a}\hat{b}}$ are non-vanishing elements of the $SO(3,1) \otimes SO(7)$ subalgebra of $SO(1,10)$

$$\begin{aligned} (W_{\vec{k}_c})^{ab} &= -4\sqrt{\frac{15}{2}} e \delta_c^{[a} x^{b]} \quad , \quad (W_{\vec{k}_\gamma})^{\alpha\beta} = -6e \delta_\gamma^{[\alpha} z^{\beta]} \\ (W_{\vec{k}_{cd}})^{ab} &= -\delta_{cd}^{[ab]} \quad , \quad (W_{\vec{k}_{\gamma\delta}})^{\alpha\beta} = -\delta_{\gamma\delta}^{[\alpha\beta]} \end{aligned} \quad (21)$$

The requirement that also the 3-index photon A be invariant, which is necessary for the invariance of the total solution, translates into $l_{\vec{k}}^A = d W_{\vec{k}}$ where $W_{\vec{k}}$ is a two-form associated with the Killing vector.

Since the Lie derivative commutes with the exterior derivative, the same condition is more easily written as

$$\mathcal{L}_{\vec{k}} R^\square = 0 \quad (22)$$

because $R^\square$ is gauge invariant. It is obvious from the fact that $R^\square$ depends only on the z -coordinates that eq. (22) is satisfied by all $SO(2,3)$ Killing vectors. To investigate the 7-space symmetries we note that for a $\vec{k}$ in the $SO(8)$ algebra we have

$$\mathcal{L}_{\vec{k}} R^\square = \mathcal{L}_{\vec{k}} \left(F_{\alpha\beta\gamma\delta} B^\alpha_\lambda B^\beta_\lambda B^\gamma_\lambda B^\delta_\lambda \right) = 0 \quad (23)$$

Recalling $\mathcal{L}_{\vec{k}} B^\alpha = W_{\vec{k}}^{\alpha\beta} B^\beta$ and (6), (23) becomes

$$\bar{\eta}^\tau \alpha\beta\gamma\delta \mathcal{L}_{\vec{k}} \eta = -2 W_{\vec{k}}^\alpha{}_\mu \bar{\eta}^\tau \beta\gamma\delta\mu \eta \quad (24)$$

The Lie derivative of a 0-form being equal to the differential operator $\vec{k}$ acting on a function, one evaluates $\vec{k}_\alpha \eta$ and $\vec{k}_{\alpha\beta} \eta$ and substitutes the result into (23).

At this point, all objects in (23) to order θ are known, except the constants $c^{\alpha\beta}$, c^α .

To proceed, one uses the identity $I = \xi \bar{\xi} + \tau_\alpha \xi \bar{\xi} \tau^\alpha$ for constant ξ normalized to one, and applying this relation to η one obtains

$$\eta = X \xi + \tau_\alpha \xi Y^\alpha, \quad X = \bar{\xi} \eta, \quad Y^\alpha = \bar{\xi} \tau^\alpha \eta \quad (25)$$

Since $\bar{\eta} \eta = 1$, $X^2 - Y^\alpha Y_\alpha = 1$ which shows that X and Y^α are coordinates on the 7-sphere. In fact, the Y^α are equal to $z^\alpha (1-z^2)^{-1/2}$ and $X = (1-z^2)^{-1/2}$ as one may verify, inserting (25) and (11) into (7). Replacing all η 's in (24) by

$$\eta = (1-z^2)^{-1/2} \left[\xi + \tau_\alpha \xi z^\alpha \right] \quad (26)$$

one can expand (24) into powers of z^α . The terms cubic in z^α yield $c^\alpha = 0$, which reduces $SO(8)$ to $SO(7)$. The remaining terms yield:

$$c^{\alpha\beta} \left\{ \tau_{\alpha\beta\gamma} \right\} = 0 \quad (27)$$

This is the equation which restricts the $SO(7)$ parameters $c^{\alpha\beta}$ to those of G_2 because the 3-index constant antisymmetric tensor $\bar{\xi} \tau_{\alpha\beta\gamma} \xi$ can be identified as the G_2 -invariant structure constants of the multiplication table of the octonions [16]. Thus $R^\square$ is invariant only under G_2 , not $SO(8)$, and the bosonic symmetry group of the solution is $SO(3,2) \times G_2$.

As a final remark, we mention a fact which might be related to the $SU(8)$ symmetry of $N = 8$ supergravity in 4-dimensions and which we are studying further.

If we consider the 7-bein $B_{(0)}^\alpha$ and its associated metric connection $B_{(0)}^{\alpha\beta}$ as given in (11) and also the η -spinor satisfying (7) we can construct the following 1-forms on the 7-sphere:

$$\hat{B}^\alpha = B_o^\alpha \quad (28a)$$

$$\hat{B}^{\alpha\beta} = B_o^{\alpha\beta} + e \bar{\eta} \tau^{\alpha\beta\gamma\delta} \eta B_{o,\gamma} \quad (28b)$$

$$\hat{B}^{\alpha\beta\gamma} = \pm \frac{1}{6\sqrt{2}} e \bar{\eta} \tau^{\alpha\beta\gamma\delta} \eta B_{o,\delta} \quad (28c)$$

These 1-forms can be interpreted as the components of an $SU(8)$ connection. Writing for instance, in the complex 8-representation:

$$M = -\frac{1}{4} \hat{B}^{\alpha\beta} \tau_{\alpha\beta} + \hat{B}^\alpha \tau_\alpha + i \hat{B}^{\alpha\beta\gamma} \tau_{\alpha\beta\gamma} \quad (29)$$

the result is that the above defined connection is $SU(8)$ flat, namely the $SU(8)$ Maurer-Cartan equation is satisfied

$$dM + M \wedge M = 0 \quad (30)$$

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