EFFECTIVE ACTIONS IN NON-ABELIAN THEORIES

A. D'Adda *)

CERN - Geneva

A. C. Davis

Institute for Advanced Study
Princeton

and

P. Di Vecchia

Physics Department, University of Wuppertal
Wuppertal

A B S T R A C T

In two-dimensional $U(\ell)$ non-Abelian gauge theories coupled to fermions, the fermionic sector possesses an additional, local $U(\ell)_R \otimes U(\ell)_L$ invariance. This chiral invariance is broken to $U(\ell)$ vector by a non-Abelian axial anomaly. Using a method invented by Wess and Zumino we show that the anomaly determines the low energy effective action almost entirely. Possibilities of extending the argument to four dimensions are briefly discussed.

*) On leave of absence from :
INFN, Sezione di Torino, Torino, Italy.

The two-dimensional $\mathbb{CP}^{n-1}$ model with fermions ¹⁾ has proved useful as a testing ground for ideas in four-dimensional gauge theories ²⁾. For instance, in both QCD and the $\mathbb{CP}^{n-1}$ model with fermions, the structure of the axial current anomaly is well known and independent of the details of the model. The corresponding anomalous Ward identities have been studied ³⁾ and effective actions which satisfy them either explicitly derived in the case of the $\mathbb{CP}^{n-1}$ model ⁴⁾ or constructed in the QCD case ^{5),6)}. Such effective actions show explicitly the effect of the axial anomaly. This has led to a deeper understanding and simple resolution of the $U(1)$ problem ⁶⁾.

Recently, effective actions for the $N=1$ supersymmetric Yang-Mills theory ⁷⁾ and for the supersymmetric $\mathbb{CP}^{n-1}$ model have also been constructed ⁸⁾. In supersymmetric theories the axial anomaly appears in the same supermultiplet as the scale and γ trace anomaly ⁹⁾. Thus, effective actions show explicitly the effect of the anomaly supermultiplet, while preserving supersymmetry.

In these examples the axial anomaly is an Abelian, $U(1)$, anomaly. The consequences of the Ward identities with non-Abelian anomalies have been studied long ago by Wess and Zumino ¹⁰⁾. In particular, they constructed a local functional satisfying the anomalous Ward identities for the case of a hadronic system in the presence of external $SU(3) \otimes SU(3)$ vector and axial vector fields.

Here we apply the method of Wess and Zumino in a rather different framework, in the context of a two-dimensional theory. In our case, the non-Abelian chiral symmetry is local and is a symmetry not of the whole Lagrangian, but only of the fermionic sector. This local chiral symmetry of the fermion determinant is then broken by a non-Abelian anomaly.

The fermionic Lagrangian we are considering is the one of model I of Ref. 11). It is given by the Lagrangian

$$\begin{aligned} \mathcal{L} = & \bar{\Psi}_L^j \alpha D_-^\alpha \Psi_{Lj}^\beta + \bar{\Psi}_R^j \alpha D_+^\alpha \Psi_{Rj}^\beta + \\ & + \frac{g}{m} (\bar{\Psi}_L^i \alpha \Psi_{Ri}^\beta) (\bar{\Psi}_R^j \beta \Psi_{Lj}^\alpha) \end{aligned} \quad (1)$$

where Ψ_L and Ψ_R are left and right moving fermions. They are in the ℓ representation of the local $U(\ell)$ gauge group, characterized by the index α, β , and in n representation of the global $U(n)$, characterized by the indices i, j . The covariant derivatives are given by

$$D_{\pm} = D_0 \pm i D_1 \quad (2)$$

$$D_{\mu}^{\alpha}{}_{\beta} = \partial_{\mu} \delta^{\alpha}{}_{\beta} + \frac{i}{\sqrt{m}} A_{\mu}^{\alpha}{}_{\beta} \quad (3)$$

Now, the gauge group of the Lagrangian in (1) is actually bigger than $U(\ell)$. Because the left moving fermions couple only to A_- and the right moving fermions only to A_+ , A_- and A_+ can be transformed independently. This means that the gauge invariance is $U(\ell)_L \otimes U(\ell)_R$.

If the fermionic Lagrangian is not considered independently, but coupled with a bosonic sector, for instance, either with the Grassmanian model as in Ref. 11) or with $\mathcal{L}_{\text{boson}} = \frac{1}{4} F_{\mu\nu}^{\alpha}{}_{\beta} F^{\beta}{}_{\mu\nu}{}^{\alpha}$, then only $U(\ell)_{\text{vector}}$ survives as a local invariance of the whole Lagrangian. Despite this the $U(\ell) \otimes U(\ell)$ invariance of $\mathcal{L}_{\text{fermion}}$ has important consequences for the features of this model, as was seen in Ref. 11) and will be demonstrated later.

To analyze the model, one can introduce the auxiliary fields

$$\begin{aligned} \Phi^{\alpha}{}_{\beta} &= \frac{g}{\sqrt{m}} \psi_{Rj}^{\alpha} \bar{\psi}_{Lj}{}_{\beta} \\ \bar{\Phi}^{\alpha}{}_{\beta} &= \frac{g}{\sqrt{m}} \psi_{Lj}^{\alpha} \bar{\psi}_{Rj}{}_{\beta} \end{aligned} \quad (4)$$

The resulting Lagrangian is now quadratic in fermion fields, allowing the functional integral over fermions to be performed. This gives the effective action

$$\begin{aligned} S_{\text{eff}}^{\text{Fermion}} &= -m \text{Tr} \log \begin{bmatrix} D_-^{\alpha}{}_{\beta} & -\frac{1}{\sqrt{m}} \Phi^{\alpha}{}_{\beta} \\ -\frac{1}{\sqrt{m}} \bar{\Phi}^{\alpha}{}_{\beta} & D_+^{\alpha}{}_{\beta} \end{bmatrix} + \\ &+ \frac{1}{g} \int d^2x \bar{\Phi}^{\alpha}{}_{\beta} \Phi^{\beta}{}_{\alpha} \end{aligned} \quad (5)$$

The properties of this effective action have been extensively studied in Refs. 11) and 12). Here we summarize the general features, referring the reader to Refs. 11) and 12) for further details.

a) The infra-red divergences of two-dimensional theories require a shift of the fields ϕ and $\bar{\phi}$ given by

$$\phi^\alpha_\mu = \varphi^\alpha_\mu + \sqrt{m} M^\alpha_\mu \quad (6)$$

$$\bar{\phi}^\alpha_\mu = \bar{\varphi}^\alpha_\mu + \sqrt{m} \bar{M}^\alpha_\mu$$

with

$$M^\alpha_\mu \bar{M}^\alpha_\nu = M^2 \delta^\alpha_\nu \quad (7)$$

b) The saddle-point condition determines the dependence of g on the ultra-violet cut-off :

$$\frac{1}{g(\Lambda)} - \int_0^\Lambda \frac{d^2 p}{(2\pi)^2} \frac{1}{p^2 + M^2} = 0 \quad (8)$$

c) Equation (8) automatically eliminates the divergences in the ϕ propagators. However, to ensure transversality of the A_μ propagator, one introduces a Pauli-Villars regulator, by defining

$$S_{eff} = S_{eff}^{Ferm.} + S_{eff}^{P.V.} \quad (9)$$

where

$$S_{eff}^{P.V.} = m \text{Tr} \log \begin{pmatrix} D_-^\alpha{}_\beta & -m d^\alpha{}_\mu \\ -m d^\alpha{}_\mu & D_+^\alpha{}_\beta \end{pmatrix} \quad (10)$$

and the limit $m \rightarrow \infty$ is taken at the end of the calculation.

S_{eff} is now a finite, well-defined functional of the fields $\phi, \bar{\phi}$ and A_μ . However, due to the axial anomaly, S_{eff} is only invariant under $U(1)_{\text{vector}}$. More precisely, if $\theta_R(x)$ and $\theta_L(x)$ are infinitesimal parameters of right and left local transformations, we obtain

$$\delta_\theta S_{eff} = \frac{\sqrt{m}}{4\pi} \int d^2 x \text{Tr} [\theta_R(x) - \theta_L(x)] F \quad (11)$$

where

$$F = \partial_+ A_- - \partial_- A_+ + \frac{i}{\sqrt{m}} [A_+, A_-] \quad (12)$$

For a vector gauge transformation $\theta_R(x) = \theta_L(x)$, giving

$$\delta_{\theta}^{(\text{vector})} S_{\text{eff}} = 0 \quad (13)$$

whilst, for an axial transformation $\theta_R(x) = -\theta_L(x) \equiv \theta(x)$, giving

$$\delta_{\theta}^{(\text{axial})} S_{\text{eff}} = \frac{\sqrt{m}}{2\pi} \int d^2x \text{Tr} \theta(x) F \quad (14)$$

resulting in the usual axial anomaly in two dimensions. Now the infinitesimal transformation (11) satisfies the group integrability conditions :

$$[\delta_{\eta}, \delta_{\theta}] S_{\text{eff}} = \delta_{\eta \otimes \theta} S_{\text{eff}} \quad (15)$$

Hence, the finite transformation can be obtained simply by exponentiating (11). Thus,

$$\begin{aligned} S_{\text{eff}}(\phi', \bar{\phi}', A'_\mu) &= e^{\delta_{\theta}} S_{\text{eff}}(\phi, \bar{\phi}, A_\mu) = \\ &= S_{\text{eff}}(\phi, \bar{\phi}, A_\mu) - \frac{m}{4\pi} \int d^2x \text{Tr} \left[e^{i\theta_R} D_+ e^{-i\theta_R} \right. \\ &\quad \left. e^{i\theta_L} D_- e^{-i\theta_L} - D_+ D_- \right] \end{aligned} \quad (16)$$

where ϕ' , $\bar{\phi}'$ and A'_μ are the transformed fields,

$$\begin{aligned} \phi &\rightarrow \phi'^\alpha{}_\beta = \left(e^{i\theta_R} \phi e^{-i\theta_L} \right)^\alpha{}_\beta \\ \bar{\phi} &\rightarrow \bar{\phi}'^\alpha{}_\beta = \left(e^{i\theta_L} \bar{\phi} e^{-i\theta_R} \right)^\alpha{}_\beta \\ A_+ &\rightarrow A'_+{}^\alpha{}_\beta = \left(e^{i\theta_R} A_+ e^{-i\theta_R} - i\sqrt{m} e^{i\theta_R} \partial_+ e^{-i\theta_R} \right)^\alpha{}_\beta \\ A_- &\rightarrow A'_-{}^\alpha{}_\beta = \left(e^{i\theta_L} A_- e^{-i\theta_L} - i\sqrt{m} e^{i\theta_L} \partial_- e^{-i\theta_L} \right)^\alpha{}_\beta \end{aligned} \quad (17)$$

It should be noted that ϕ and $\bar{\phi}$ are the unshifted fields.

Now, since ϕ and $\bar{\phi}$ can be parametrized as

$$\begin{aligned}\Phi^\alpha_\mu &= \left(e^{-i\chi} \rho \quad e^{-i\chi} \right)^\alpha_\mu \\ \bar{\Phi}^\alpha_\mu &= \left(e^{i\chi} \rho \quad e^{i\chi} \right)^\alpha_\mu\end{aligned}\quad (18)$$

with $\rho = \rho^\dagger$ and $\chi = \chi^\dagger$, then, by choosing $\chi = \theta_R = -\theta_L$ we obtain $\phi' \equiv \bar{\phi}' = \rho$ and (16) becomes

$$\begin{aligned}S_{\text{eff}}(\Phi, \bar{\Phi}, A_\mu) &= S_{\text{eff}}(\rho, \rho, A'_\mu) - \\ &- \frac{m}{4\pi} \int d^2x \text{Tr} \left[e^{-2i\chi} D'_+ e^{2i\chi} D'_- - D'_+ D'_- \right]\end{aligned}\quad (19)$$

Hence, we conclude that the anomaly completely determines the dependence of the effective action on χ . This can be checked in the Abelian limit. In this case (19) reduces to

$$\begin{aligned}S_{\text{eff}}(\Phi, \bar{\Phi}, A_\mu) &= S_{\text{eff}}(\rho, \rho, A'_\mu) - \\ &- \frac{\sqrt{m}}{4\pi} \int d^2x \, 2\chi F' - \frac{m}{2\pi} \int d^2x \, \partial_+ \chi \partial_- \chi\end{aligned}\quad (20)$$

The term χF is the well-known $U(1)$ anomaly coupling in two dimensions ⁴⁾, i.e.,

$$\chi F \sim F \log \left(\frac{\bar{\Phi}}{\Phi} \right) \quad (21)$$

Of course, in four dimensions, this term would be

$$\chi F \tilde{F} = F \tilde{F} \log \left(\frac{\bar{\Phi}}{\Phi} \right) \quad (22)$$

as obtained in the QCD effective action ⁶⁾. In the non-Abelian case the situation is more complicated. Indeed, we may write (19) as

$$\begin{aligned}S_{\text{eff}} &= - \int d^2x \left[\frac{\sqrt{m}}{2\pi} \chi^a F^a - \frac{m}{4\pi} \epsilon^{\mu\nu} D_\mu \chi^a \left(\frac{\sin 2\chi \cdot T - 2\chi T}{(\chi \cdot T)^2} \right)_{ab} \right. \\ &\cdot D_\nu \chi^b + \frac{m}{4\pi} D_\mu \chi^a \left(\frac{\cos 2\chi \cdot T - 1}{(\chi \cdot T)^2} \right)_{ab} D_\mu \chi^b \left. \right]\end{aligned}\quad (23)$$

where we have written $\chi = \chi^a T^a$, T^a being the $U(\ell)$ matrices. Besides the usual anomaly term, $\chi \cdot F$, (23) contains higher order interaction terms, absent in the Abelian case.

The dynamics of the fermionic Lagrangian (1), as revealed in Eq. (19), can be described as follows : by writing the fields ϕ and $\bar{\phi}$ in terms of the two fields, ρ and χ , (the "modulus" and the "phase"), the parameters of the axial part of the $U_R(\ell) \otimes U_L(\ell)$ gauge invariance have been completely absorbed, being replaced by the field χ . The χ dependence in the effective action is completely determined by the axial anomaly. The original invariance of the Lagrangian can still be seen in the dependence on the phase χ . Indeed, to the first two orders in the $1/n$ expansion the vacuum energy is independent of the vacuum expectation value of χ , showing that the vacuum is unaligned to this order ^{11),12)}. Thus the vacuum expectation value of χ is arbitrary and undetermined for first non-leading order in $1/n$ perturbation theory. A detailed investigation is underway to try to ascertain whether or not the vacuum is aligned to all orders in $1/n$ ¹³⁾. The only part of ϕ whose vacuum expectation value is determined is ρ , this being given by (7). Indeed, the effective action $S_{\text{eff}}(\rho, \rho, A'_\mu)$ appearing in (19) is essentially given by (5) with ϕ and $\bar{\phi}$ being replaced by ρ in the $\text{Tr} \log$ term. This is because $\bar{\phi}^\alpha_\beta \phi^\beta_\alpha$ is identically equal to $\text{Tr} \rho^2$. Minimizing the effective potential gives

$$\langle \rho^\alpha_\mu \rangle = M \delta^\alpha_\mu \sqrt{n} \quad (24)$$

Expanding ρ around the vacuum expectation value

$$\rho^\alpha_\mu = \sqrt{n} M \delta^\alpha_\mu + \sigma^\alpha_\mu \quad (25)$$

one finds that σ is a massive field of mass $2M$. Thus, in the low momentum regime ($q \ll M$) the only surviving term in $S_{\text{eff}}(\rho, \rho, A'_\mu)$ is the dynamically generated term for A'_μ . Hence, we obtain

$$S_{\text{eff}}(\phi, \bar{\phi}, A_\mu) \sim \frac{1}{48\pi M^2} \int d^2x F^2 - \frac{n}{4\pi} \int d^2x \text{Tr} \left[e^{-2i\chi} D_+ e^{2i\chi} D_- - D_+ D_- \right] \quad (26)$$

This is the full effective action in the low momentum regime for the purely fermionic Lagrangian.

The situation is rather more complicated when the fermionic Lagrangian is coupled to a bosonic sector. Since the bosonic Lagrangian is not invariant under $U(\ell)_R \otimes U(\ell)_L$ then the local chiral transformation which eliminates the x dependence in ϕ and $\bar{\phi}$ actually introduces a x dependence in the bosonic part of the Lagrangian. For example, the usual kinetic term

$$S_{\text{Boson}} = \frac{\text{Tr}}{4\pi m^2} \int d^2x F^2 \quad (27)$$

becomes

$$S'_{\text{Boson}} = \frac{n \text{Tr}}{4\pi m^2} \int d^2x \left[e^{-ix} D_+ e^{ix}, e^{ix} D_- e^{-ix} \right] \times \\ \times \left[e^{-ix} D_+ e^{ix}, e^{ix} D_- e^{-ix} \right] \quad (28)$$

We can summarize the results of our investigations as follows : in this paper we have given a two-dimensional example of a purely fermionic theory. Classically, the fermionic Lagrangian possesses a local, non-Abelian chiral symmetry, broken by an axial anomaly. As a result, some degrees of freedom which are gauge parameters in the classical Lagrangian become dynamical degrees of freedom in the effective action, their dynamics being completely determined by the axial anomaly. These degrees of freedom are relevant in the low energy regime. Indeed, in the Abelian case, they are coupled to the gauge field, as shown in (21); this coupling has been extensively used in the literature to provide a solution to the $U(1)$ problem⁶⁾. Similar conclusions can be drawn when the fermionic Lagrangian is only part of a Lagrangian that contains, for example, kinetic terms for the gauge field.

Do our results extend beyond two-dimensions ? There are important differences between the two and four-dimensional situation. For instance, there are two typical two-dimensional features in the fermionic model we have studied : the quartic interaction which naturally gives origin to the scalar composite field $\phi, \bar{\phi}$; the local $U(\ell)_R \otimes U(\ell)_L$ invariance which is a consequence of the left and right moving fermions being coupled respectively to the two light cone components of A_μ . Neither of these features are present in four dimensions. However, it is still possible in four dimensions to introduce two gauge fields coupling to ψ_L and ψ_R . To avoid gauging the anomaly $A_\mu^{(R)}$ and $A_\mu^{(L)}$ need to be related by a $U(\ell)$ gauge transformation, by introduction of a constraint. The four-dimensional theory is then rather similar to the two-dimensional one. The properties of such a theory are currently under investigation.

ACKNOWLEDGEMENTS

We thank M.E. Peskin for very useful discussions, in particular Eq. (23) has been derived during one of them. One of us (A.D'A.) wishes to thank the Department of Mathematics of the University of Durham for the hospitality extended to him during the time when most of this paper was written.

REFERENCES

- 1) E. Witten, Nuclear Phys. B144 (1979) 285;
A. D'Adda, P. Di Vecchia and M. Lüscher, Nuclear Phys. B152 (1979) 125.
- 2) See, for example,
P. Di Vecchia, Proceedings Schladming Institute; Acta Phys. Austriaca Suppl. (1980) 341 and references therein.
- 3) See, for example,
R.J. Crewther, Rivista del Nuovo Cimento 2 (1979) 63.
- 4) P. Di Vecchia, Phys.Letters 85B (1979) 357;
F. Riva, Nuovo Cimento 61A (1981) 69.
- 5) E. Witten, Nuclear Phys. B156 (1979) 269; and
G. Veneziano, Nuclear Phys. B159 (1979) 357.
- 6) P. Di Vecchia and G. Veneziano, Nuclear Phys. B171 (1980) 253;
E. Witten, Annals of Phys. 130 (1980) 363.
- 7) G. Veneziano and S. Yankielowicz, Phys.Letters 113B (1982) 231.
- 8) A. D'Adda, A.C. Davis, P. Di Vecchia and P. Salomonson, CERN-TH preprint in preparation.
- 9) S. Ferrara and B. Zumino, Nuclear Phys. B87 (1975) 207.
- 10) J. Wess and B. Zumino, Phys.Letters 37B (1971) 95.
- 11) A. D'Adda, A.C. Davis and P. Di Vecchia, Nuclear Phys. B198 (1982) 209.
- 12) A. D'Adda, A.C. Davis, P. Di Vecchia, Institute for Advanced Study, Princeton, preprint (1982)
- 13) A. D'Adda, A.C. Davis, P. Di Vecchia and M.E. Peskin, in progress.