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SUPERGRAVITY, THE SEVEN-SPHERE, AND SPONTANEOUS SYMMETRY BREAKING ⁺)

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A B S T R A C T

$N = 1$ supergravity in $d = 11$ dimensions spontaneously compactifies on S^7 to an $N = 8$ supergravity in $d = 4$ with a local $SO(8) \times SO(8)$ invariance, probably enlargeable to $SO(8) \times SU(8)$. Apart from group manifolds, S^7 is the only compact manifold to admit an absolute parallelism. This permits (A) a "squashing" of S^7 which gives expectation values to the scalar fields and (B) a parallelizing "torsion" which gives expectation values to the pseudoscalars. This correspondence between extrema of the $d = 4$ effective potential and solutions of the $d = 11$ field equations provides a Kaluza-Klein origin for the spontaneous breakdown of gauge symmetries, discrete symmetries, and supersymmetries. It also puts a new perspective on the puzzle of the cosmological constant.

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PREFACE

The work described here on supergravity and S^7 was carried out mainly in collaboration with C.N. Pope [1]. Also mentioned briefly are some results on "squashing" spheres and symmetry breaking obtained with R. Coquereaux [2]. Neither work is yet published though preliminary results have been announced in [3] and [4]. We shall then make contact with some recent work of F. Englert [5] on S^7 and parallelizing torsion. The ramifications of these results are presently being investigated and will appear in future papers together with B. de Wit, F. Englert and H. Nicolai [6] for $d = 11$ supergravity and with P. Townsend and P. van Nieuwenhuizen [7] for $d = 10$ supergravity. Only the barest outline is given in this paper and more details may be found in a forthcoming review [8].

KALUZA-KLEIN THEORIES

Supersymmetric models are unique among field theories in that they are formulated most naturally in space-time dimensions $d > 4$, namely $d_{\max} = 10$ for rigid supersymmetry and $d_{\max} = 11$ for local supersymmetry. Up until recently, however, the predominant interpretation of these extra dimensions has been merely one of a mathematical device for obtaining four-dimensional theories via "dimensional reduction", i.e., by requiring the fields to be independent of the extra co-ordinates. In this scheme, no physical significance need be attributed to the extra dimensions. For a review, see Cremmer [9].

By contrast, we shall here explore the consequences of taking the extra dimensions seriously. We shall not discuss in detail the way in which these old Kaluza-Klein [10] ideas work from a modern point of view. See, for example, Salam and Strathdee [11]. Nor shall we repeat all the reasons why $d = 11$ supergravity is an especially attractive candidate. See Witten [12] and Duff and Toms [3]. Instead we shall plunge straight into the $d = 11$ field equations and look for solutions in which extra dimensions are "spontaneously compactified". For the impatient reader, a summary of the results is given in Table I.

SUPERGRAVITY

In the $d = 11$ theory [13] all fields are gauge fields: the gravitational elfbein $e_M^A(x)$ (44 degrees of freedom), a single (hence $N = 1$ supersymmetry) Rarita-Schwinger Majorana spinor $\psi_M(x)$ (128 degrees of freedom), and a rank-three antisymmetric tensor gauge field $A_{MNP}(x)$ (84 degrees of freedom). We use a signature $(-++++++)$. The Lagrangian is now uniquely fixed: no other matter couplings are allowed, nor are we permitted to include a cosmological constant. It is given in Table II, and the action is invariant under the following symmetries:

(dotted lines represent calculations presently under construction)

TABLE I - (OVER) SIMPLIFIED SUMMARY

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{4} e e_B^N e_A^M R_{MN}{}^{AB}(\omega) \\
 & - \frac{i}{2} e \bar{\Psi}_M \Gamma^{MNP} D_N \left[\frac{1}{2} (\omega + \hat{\omega}) \right] \Psi_P \\
 & - \frac{1}{48} e F_{MNPQ} F^{MNPQ} + c \varepsilon^{M_1 M_2 \dots M_{11}} F_{M_1 \dots M_4} F_{M_5 \dots M_8} A_{M_9 \dots M_{11}} \\
 & + \frac{3}{4} e d \left[\bar{\Psi}_M \Gamma^{MNPQRS} \Psi_N + 12 \bar{\Psi}^W \Gamma^{XYZ} \Psi \right] \left[F_{WXYZ} + \hat{F}_{WXYZ} \right]
 \end{aligned}$$

where

$$e = \det e_M^A \quad \bar{\Psi} = \Psi^\dagger \Gamma_0 \quad F_{MNPQ} = 4 \partial_{[M} A_{NPQ]}$$

$$\omega_{MAB} = \overset{\circ}{\omega}_{MAB}(e) + K_{MAB}$$

$$\overset{\circ}{\omega}_{ABM} = \frac{1}{2} (-\Omega_{MAB} + \Omega_{ABM} - \Omega_{BMA}) \quad \Omega_{MN}{}^A = 2 \partial_{[N} e_M^A]$$

$$K_{MAB} = \frac{i}{4} \left[-\bar{\Psi}_N \Gamma_{MAB}{}^{NP} \Psi_P + 2 (\bar{\Psi}_N \Gamma_B \Psi_A - \bar{\Psi}_M \Gamma_A \Psi_B + \bar{\Psi}_B \Gamma_M \Psi_A) \right]$$

$$D_M(\omega) = \partial_M + \frac{1}{4} \omega_M{}^{AB} \Gamma_{AB}$$

$$\hat{\omega}_{MAB} = \omega_{MAB} + \frac{i}{4} \bar{\Psi}_N \Gamma_{MAB}{}^{NP} \Psi_P$$

$$\hat{F}_{MNPQ} = F_{MNPQ} - 3 \bar{\Psi}_{[M} \Gamma_{NP} \Psi_{Q]}$$

$$c = \frac{2}{(12)^4} \quad d = \frac{1}{(12)^2} \quad [\Gamma_A, \Gamma_B] = -2 \eta_{AB}$$

TABLE II - THE LAGRANGIAN OF $d = 11$ SUPERGRAVITY

- a) $d = 11$ general co-ordinate transformations ;
- b) local $SO(1,10)$ Lorentz transformations ;
- c) $N = 1$ supersymmetry transformations

$$\delta e_M^A = -i \bar{\epsilon} \Gamma^A \psi_M$$

$$\delta \psi_M = D_M(\hat{\omega}) \epsilon + i d (\Gamma^{NPQR}{}_M - 8 \Gamma^{PQR} \delta_M^N) \hat{F}_{NPQR} \epsilon \equiv \bar{D}_M \epsilon$$

$$\delta A_{MNP} = \frac{3}{2} \bar{\epsilon} \Gamma_{[MN} \psi_{P]} \quad (1)$$

- d) Abelian gauge transformations

$$\delta A_{MNP} = \partial_{[M} \Lambda_{NP]} \quad (2)$$

- e) time-reversal together with $A_{MNP} \rightarrow -A_{MNP}$.

To see whether $d = 11$ supergravity admits spontaneous compactification, we examine the field equations and look for solutions which might be candidates for a ground state. Although this analysis is strictly classical we use the language of ground-states, vacua, vacuum expectation values, etc., in anticipation of quantization (though with the caution that this tree approximation may not accurately describe the "true" vacuum). Accordingly, we set

$$\langle \psi_M \rangle = 0 \quad (3)$$

and focus our attention on the equations for the boson fields g_{MN} and A_{MNP}

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3} \left[F_{MPQR} F_N{}^{PQR} - \frac{1}{8} g_{MN} F_{PQRS} F^{PQRS} \right] \quad (4)$$

$$\nabla_M F^{MNPQ} = -18 c \epsilon^{M_1 M_2 \dots M_9 N P Q} F_{M_1 \dots M_4} F_{M_5 \dots M_9} \quad (5)$$

We look for solutions of the direct product form $M_1 \times M_2$ where M_1 is space-time with the usual signature and co-ordinates x^μ (but whose dimension is not yet specified) and M_2 is a compact space of Euclidean signature and co-ordinates y^m . Thus we set

$$\begin{aligned} \langle g_{\mu\nu} \rangle &= g_{\mu\nu}(x) , & \langle F_{\mu\nu\sigma\tau} \rangle &= F_{\mu\nu\sigma\tau}(x) \\ \langle g_{mn} \rangle &= g_{mn}(y) , & \langle F_{mnpq} \rangle &= F_{mnpq}(y) \end{aligned} \quad (6)$$

but

$$\langle g_{\mu n} \rangle = 0 , \quad \langle F_{\mu\nu\sigma\tau} \rangle = \langle F_{\mu\nu\rho\tau} \rangle = \langle F_{\mu n p q} \rangle = 0 \quad (7)$$

Bearing in mind that the field strength $F_{\mu\nu\sigma\tau}$ may give rise to a cosmological constant [14] we follow Freund and Rubin [15] and look initially for solutions with

$$e F_{\mu\nu\sigma\tau} = \pm 3 m \epsilon_{\mu\nu\sigma\tau} \quad (8)$$

$$F_{mnpq} = 0 \quad (9)$$

where m is a real constant and the factor 3 is chosen for future convenience. Note that from (8) the dimension of space-time $11-7=4$ is now output rather than input since $F_{\mu\nu\rho\sigma}$ has rank 4. (Another solution with the indices interchanged and space-time dimension $11-4=7$ is also possible [15]. We shall return to this possibility later.) Substituting (8) and (9) into (4) and (5) we find that (5) is trivially satisfied while (4) yields the product of a four-dimensional Einstein space-time

$$R_{\mu\nu} = -12 m^2 g_{\mu\nu} \quad (10)$$

with the usual signature and a seven-dimensional Einstein space

$$R_{mn} = 6 m^2 g_{mn} \quad (11)$$

with Euclidean signature and hence compact since $m^2 > 0$.

Of course, there are still infinitely many solutions of (10) and (11) but the ground state should presumably be distinguished by its symmetries. With this in mind, Duff and Pope [1] proceeded to restrict the solution further by demanding that the vacuum be supersymmetric, i.e., by requiring the existence of covariantly constant spinors ϵ for which

$$\langle \delta \psi_M \rangle = 0 = \langle \bar{D}_M \epsilon \rangle \quad (12)$$

where the covariant derivative $\bar{D}_M$ is given in (1). For solutions of the type given in (6), (7), (8) and (9) we have [choosing the + sign in (8)]

$$\bar{D}_\mu = D_\mu + m \gamma_\mu \gamma_5 \quad (13)$$

$$\bar{D}_m = D_m - m \Gamma_m / 2 \quad (14)$$

Here we have used the decomposition of the $d = 11$ Γ matrices

$$\Gamma_A = (\gamma_\mu \otimes 1, \gamma_5 \otimes \Gamma_a) \quad (15)$$

The requirement of unbroken supersymmetry (12) becomes

$$\bar{D}_m \eta^I = 0 \quad (16)$$

with

$$\bar{D}_\mu \epsilon^I = 0 \quad (17)$$

where

$$\epsilon(x, y) \equiv \epsilon^I(x) \eta^I(y) \quad (18)$$

and the index I runs over solutions of (16). Equations (16) and (17) further require

$$[\bar{D}_m, \bar{D}_n] \eta = \left[-\frac{1}{4} R_{mnab} \Gamma^a \Gamma^b + \frac{m^2}{4} [\Gamma_m, \Gamma_n] \right] \eta = 0 \quad (19)$$

$$[\bar{D}_\mu, \bar{D}_\nu] \varepsilon = \left[-\frac{1}{4} R_{\mu\nu\alpha\beta} \gamma^\alpha \gamma^\beta - m^2 [\gamma_\mu, \gamma_\nu] \right] \varepsilon = 0 \quad (20)$$

and hence we obtain the (maximally symmetric) special case of (10) and (11) for which space-time is anti-de-Sitter space (AdS) = SO(3,2)/SO(3,1) with curvature

$$R_{\mu\nu\sigma\rho} = -4m^2 (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) \quad (21)$$

and the compact extra dimensions correspond to the seven-sphere with its standard metric $S^7 = SO(8)/SO(7)$ and radius m^{-1} :

$$R_{mnpq} = m^2 (g_{mp} g_{nq} - g_{mq} g_{np}) \quad (22)$$

In seven dimensions a spinor has 8 components and if the 8×8 matrix $[\bar{D}_a, \bar{D}_b]$ has all 8 eigenvalues zero as in (19), then the index I in (18) runs over 1 to 8 and we have $N = 8$ supersymmetry. D'Auria and Fré [16] have recently observed that the bosonic symmetry of $SO(3,2) \times SO(8)$ and the $N = 8$ fermionic symmetry may be combined by the supergroup $OSp(4|8)$. Let us tentatively adopt (3), (8), (9), (21) and (22) as our ground-state and examine the consequences.

THE SEVEN-SPHERE

To obtain the effective $d = 4$ theory, we next perform a Fourier expansion of all the fields, denoted generically by $\Phi(x, y)$, thus

$$\Phi(x, y) = \sum_n \phi_n(x) Y_n(y) \quad (23)$$

$$M^2 Y_n(y) = m_n^2 Y_n(y) \quad (24)$$

The expansion coefficients $\phi_n(x)$ are then interpreted as physical fields in $d = 4$ and the Y_n are eigenfunctions of the "mass matrix" M^2 . This mass matrix will be given by some appropriate differential operators on S^7 determined by expanding the field equations to first order in fluctuations about the ground state. Integration over the extra y^m co-ordinates then yields an effective $d = 4$ theory describing an infinite tower of massive states, classified by $OSp(4|8)$ multiplets with masses quantized in units of m , the inverse radius of S^7 . There will also be a finite number of massless states coming from the zero-eigenvalue modes. All states have spins ≤ 2 .

The rules for determining the spins and internal quantum numbers of these massless modes in such a Kaluza-Klein reduction are non-trivial. To isolate the massless states in the "dimensional reduction" approach one simply takes the fields to be independent of the extra y^m co-ordinates, but this works only for the flat torus. For other spaces, in particular S^7 , one must make an ansatz like (18) which takes some of y dependence into account. The Kaluza-Klein literature is very confusing and contradictory on this subject, especially when it comes to counting spin 0 fields. Indeed, the very word "ansatz" has connotations of subjectivity about it, with each author imposing his own geometrical prejudices. We emphasize that for us the correct "ansatz" will be, by definition, that which selects all the massless states and only the massless states. This means discarding all but the lowest modes in the Fourier expansion (23). We must bear in mind, however, that in AdS a massless spin $\frac{3}{2}$ particle (with two degrees of freedom) nevertheless requires an apparent mass term, i.e.,

$$\gamma^{\mu\nu\lambda} \bar{D}_\nu \psi_\lambda = 0 \quad (25)$$

where $\bar{D}_\nu$ is given by (13), and that massless scalars (propagating on the light cone) satisfy the conformal wave equation

$$-\square \phi + \frac{R}{6} \phi = 0 \quad (26)$$

Isolating the massless states, therefore, is a fairly complicated exercise requiring a knowledge of the spectra of various differential operators on S^7 . This will be dealt with in detail in [1] and [6], where we found that one can indeed give an a posteriori geometrical interpretation of these zero-mass modes in terms of harmonic p forms, Betti numbers, Killing spinors, Killing vectors, Killing vector-spinors and Killing tensors on S^7 . It is a remarkable fact, not to be found in maths textbooks, that all the relevant eigenmodes on S^7 may be built out of products of the 8 covariantly constant commuting spinors (Killing spinors) of Eq. (16)

$$\bar{D}_m \eta^I = (D_m - m \Gamma_m / 2) \eta^I = 0 \quad (27)$$

For example, the ansatz which yields the 28 massless gauge bosons $B_{\mu}^{IJ}(x)$ is given by

$$g_{\mu m}(x, y) = B_{\mu}^{[IJ]}(x) K_m^{[IJ]}(y) \quad (28)$$

where K_m^{IJ} are zero-modes of the spin 1 mass matrix :

$$M^2 K_m \equiv -\square K_m - R_{mn} K^n = 0 \quad (29)$$

The K_m^{IJ} are therefore just the 28 Killing vectors on S^7 given by

$$K_m^{IJ} = \bar{\eta}^I \Gamma_m \eta^J \quad (30)$$

The ansätze for the other spins work in a similar way. In particular we note that 35 massless scalar fields $S^{IJKL}(x)$ come from

$$g_{mn}(x,y) = \gamma_{mn}(y) + S^{[IJKL]}(x) K_m^{[IJ]}(y) K_n^{KL]}(y) \quad (31)$$

where $\gamma_{mn}(y)$ is the metric on S^7 and $S^{IJKL}(x)$ is self-dual

$$S^{IJKL} = \frac{1}{4!} \epsilon^{IJKLMNPQ} S_{MNPQ} \quad (32)$$

and 35 massless pseudoscalars come from

$$A_{mnp}(x,y) = P^{[IJKL]}(x) K_{[mnp]}^{[IJKL]}(y) \quad (33)$$

where K_{mnp}^{IJKL} are anti self-dual in the sense of (32) and are defined to be the 35 solutions of the linear self-dual equation

$$\nabla^R K_{Rlmn} = \frac{m}{4} \epsilon_{lmnpqrs} K^{pqrs} \quad (34)$$

where

$$K_{Rlmn} = 4 \partial_{[R} A_{lmn]} \quad (35)$$

[Remarkably Eq. (34) is precisely the Englert [5] equation satisfied by F_{klmn} which occurs in solution II of Table V even though we are still in the framework of Solution I. We shall return to this point shortly.]

The results of [1] are that the massless states on S^7 are given by 1 spin 2, 8 spin $\frac{3}{2}$, 28 spin 1, 56 spin $\frac{1}{2}$ and 70 spin 0 (35 scalars + 35 pseudoscalars). A comparison with the counting of massless states on T^7 is given in Table III.

In addition to $N = 8$ local supersymmetry, the Lagrangian of the massless sector will be invariant under $G = \text{local } SO(8) \times \text{"local" } SO(8)$ where local refers to the 28 elementary gauge bosons and "local" to the "hidden symmetry" described by 28 "composite gauge bosons" formed from the scalars and their derivatives. It is the diagonal subgroup of the $SO(8) \times SO(8)$ which is to be identified with the $SO(8)$ vacuum symmetry, H .

Obtaining the complete $d = 4$ Lagrangian with all its interactions is a gargantuan task which has not yet been completely carried out but all the above, together with the observation that the $SO(8)$ coupling constant e and the cosmological constant Λ are related by

$$\Lambda = -12 m^2 = -3 e^2 \quad (36)$$

leads us to suspect that the massless sector may be identified with the $N = 8$ supergravity of de Wit and Nicolai [17] and hence that the local $SO(8) \times \text{"local" } SO(8)$ may be enlarged to local $SO(8) \times \text{"local" } SU(8)$. Their theory was obtained by taking the global $E_7 \times \text{"local" } SU(8)$ theory

$d = 11$		$d = 4$	spin	T^7		S^7
g_{MN}	$g_{\mu\nu}$	2	1			1
	B_μ	1	7	}	Killing vectors	{ 28
	S	0	28	}	$\frac{GL(7,R)}{SO(7)}$ Cosets $\frac{SL(8,R)}{SO(8)}$ + Killing tensors	{ 35
ψ_M	ψ_μ	$\frac{3}{2}$	8	}	Killing spinors	{ 8
	χ	$\frac{1}{2}$	56	}	Killing vector-spinors	{ 56
A_{MNP}	$A_{\mu\nu\rho}$	-	1	}	Betti numbers	1 0 0
	$A_{\mu\nu}$	0	7			
	A_μ	1	21			
	A	0	35		Betti / Cartan	{ 35

$N = 8$ spin content :

spin	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0
number	1	8	28	56	70

TABLE III - THE SEVEN-TORUS VERSUS THE SEVEN-SPHERE

of Cremmer and Julia [13] and gauging the $SO(8)$ subgroup of E_7 . The Cremmer-Julia version was, in its turn, derived by dimensional reduction from the $d = 11$ theory. Within a Kaluza-Klein framework, this dimensional reduction is equivalent to first choosing a ground-state solution for which space-time is Minkowski space R^4 and the extra dimensions correspond to the seven-torus $T = S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1 \times S^1$, and then discarding the massive modes. This chain of interconnections is summarized in Table IV. Since the Cremmer-Julia theory and the de Wit-Nicolai theory are believed to be the only four-dimensional massless $N = 8$ invariant supergravity theories, one suspects that $R^4 \times T^7$ and anti-de Sitter space $\times S^7$ exhaust all possible solutions of the $d = 11$ field equations admitting $N = 8$ supersymmetry. This is indeed the case [6].

It should be emphasized, however, that since the masses of massive states are multiples of m , the inverse radius of S^7 , and since the gauge coupling e is proportional to $m/M(\text{Planck})$ the massless sector

TABLE IV - ORIGIN OF HIDDEN SYMMETRIES

is in no sense a limiting case of the full theory. Indeed, the full theory has more supersymmetry than conventional $N = 8$ supergravity in the following sense. Consider the Fourier expansion of the $d = 11$ supersymmetry parameter

$$\varepsilon(x, y) = \sum_n \varepsilon_n(x) \gamma_n(y) \quad (37)$$

where $\gamma_n(y)$ are the eigenmodes of the Dirac operator on S^7

$$\Gamma^m \bar{D}_m \gamma_n = m_n \gamma_n \quad (38)$$

which is also the mass matrix for the gravitino. As we have already seen in (16), the zero eigenvalue mode (which is 8-fold degenerate) corresponds to the usual $N = 8$ local supersymmetry which transforms one component of a supermultiplet into another, whereas the rôle of the other ϵ_n is to move up and down between levels of the infinite tower of massive $OSp(4|8)$ supermultiplets (which will require central charges since the maximum spin is 2). This seems to correspond to a superextension of the non-compact symmetries discussed by Salam and Strathdee [11]. The full theory may, however, be consistently truncated to the massless theory, if so desired, since one can show [1] that massless states transform correctly amongst themselves under the conventional $N = 8$ supersymmetry.

Although lack of space prevents a more complete description of the S^7 compactification, there is one other aspect which should be emphasized concerning the scalar fields. These scalars, which are typically described by a non-linear σ model, are commonly ignored in much of the traditional Kaluza-Klein literature but from our point of view their importance cannot be overemphasized. First, it is inconsistent to set the scalars to zero in the $d = 4$ field equations since it would lead to unacceptable constraints on the Yang-Mills fields like $F_{IJ}F_{\mu\nu}K_L = 0$. Secondly, these σ model scalars are vital for understanding the aforementioned "hidden symmetries". Einstein's Lagrangian in $d=11$ may be written in σ model form

$$e \left[D_M e_A^M D_N e^{AN} - D_N e_A^M D_M e^{AN} \right] \quad (39)$$

and the $SO(1,10)$ spin connection ω_M^{AB} is expressible in terms of e_A^M and its derivatives via the equations of motion, i.e., it is a "composite gauge field". In $d = 4$, the usual Lorentz correction is provided by $\omega_\mu^{\alpha\beta}(x)$ while the composite gauge fields of the hidden symmetry are contained in $\omega_\mu^{ab}(x,y)$. Finally, as we shall now discuss, the scalars and pseudoscalars are vital for spontaneous symmetry breaking.

SQUASHING S^7 : HIGGS INTERPRETATION, $\langle S^{IJKL}(x) \rangle \neq 0$

Requiring that the gauge coupling constant e in (36) be of order unity or thereabouts fixes m^{-1} to be of Planck dimensions, and the massive states, having masses in excess of 10^{19} GeV, cannot be identified with any known particles. As far as low energy phenomenology is concerned, therefore, one can discard the massive states and focus one's attention on the massless sector. [It should be stressed, however, that the inclusion of an infinite tower of massive states will drastically affect the ultra-violet behaviour of the quantum theory [4,19]].

Consequently, in any realistic Kaluza-Klein theory most of the gauge symmetries, discrete symmetries and supersymmetries must eventually be broken. In particular, the massless spin $\frac{1}{2}$ fermions and W bosons will have to acquire masses small compared to the Planck mass.

One viewpoint would be first to obtain a symmetric $d = 4$ theory with scalars, and then to look for a conventional spontaneous symmetry breaking via a $d = 4$ Higgs mechanism. Another viewpoint, more in keeping with the spirit of Kaluza-Klein, would be to seek a higher-dimensional geometrical origin for the symmetry breaking by compactifying on a space which deviated slightly from its maximally symmetric geometry. Here we shall espouse the point of view that these are merely different ways of understanding the same mechanism !

Before discussing S^7 and supergravity, let us examine a simpler Kaluza-Klein model in $d = 7$ where the extra dimensions are taken to be a three-sphere, S^3 . This is the subject of the forthcoming paper by Coquereaux and Duff [2]. Some preliminary results were announced in [4], and most of the relevant mathematics may be found in an interesting paper by Hitchin [18].

The three sphere with its standard metric $S^3 = SO(4)/SO(3)$ admits 6 Killing vectors corresponding the three left-invariant vector fields L_a which generate the algebra of one $SU(2)$

$$[L_a, L_b] = \epsilon_{abc} L^c \quad (40)$$

and the three right-invariant vector fields R_a which generate the other $SU(2)$

$$[R_a, R_b] = -\epsilon_{abc} R^c \quad (41)$$

with

$$[L_a, R_b] = 0 \quad (42)$$

The contravariant metric on S^3 (with radius m^{-1}) may be written

$$m^2 g = L_1 \otimes L_1 + L_2 \otimes L_2 + L_3 \otimes L_3 \quad (43)$$

and has vanishing Lie derivative with respect to all 6 Killing vectors. Starting with pure gravity in $d = 7$ (possibly with a rank-two anti-symmetric tensor A_{MN} to effect the compactification [7], see final section) and compactifying on such a three-sphere would lead, after discarding the massive states, to an effective $d = 4$ theory describing gravity coupled to an $SU(2) \times SU(2)$ Yang-Mills system (i.e., 6 massless vector bosons) and a non-linear σ model with 9 scalar fields belonging to the coset $SL(4, \mathbb{R})/SO(4)$. [cf. the 35 scalars belonging to $SL(8, \mathbb{R})/SO(8)$ on S^7 .] Note that this differs from previous authors, e.g., Cho and Freund [21] who obtain only 3 massless vector bosons of $SU(2)$ and correspondingly only a subset of the scalars. Our result follows from the dictum that our ansatz selects all the massless states and only the massless states. (One more spin 0 field would come from A_{MN} .)

Now consider a "squashed" three sphere [18]

$$m^2 g = L_1 \otimes L_1 + L_2 \otimes L_2 + \lambda^2 L_3 \otimes L_3 \quad (44)$$

where $\lambda^2 \neq 1$ is a constant. This metric has only $SU(2) \times U(1)$ invariance and the corresponding Kaluza-Klein theory would describe only 4 massless vector bosons. In other words, two of the 6 vector bosons of the "round sphere" have acquired a mass M which turns out to be given by $M^2 \sim m^2(\lambda^2 - 1)^2$. This seems somewhat reminiscent of spontaneous symmetry breaking and a little thought reveals that it is spontaneous symmetry breaking because squashing the sphere as in (44) corresponds to giving a vacuum expectation value to the scalars : $\langle S \rangle \neq 0$.

One would not expect the squashed sphere to be an Einstein space, i.e.

$$R^a_b = 2m^2 \delta^a_b, \quad R = 6m^2$$

except for $\lambda^2 = 1$. What is remarkable is that it is still homogeneous, and R^a_b has constant eigenvalues even after squashing

$$R^a_b = 2m^2 \begin{bmatrix} 1-\lambda^2 & 0 & 0 \\ 0 & 2-\lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \quad (45)$$

and that, contrary to one's intuition, the scalar curvature

$$R = 2m^2(4 - \lambda^2) \quad (46)$$

can pass through zero and become negative as λ^2 is increased. (This provides one of the ways of circumventing Lichnerowicz's theorem and obtaining massless fermions in $d = 4$ starting from spin $\frac{1}{2}$, as opposed to spin $\frac{3}{2}$, in $d > 4$ [4]. Hitchin's paper [18] is in fact devoted to counting zero modes of the Dirac operator on squashed S^3 , which turns out to be a non-trivial exercise in number theory. Within a Kaluza-Klein context, one can now amuse oneself by constructing electroweak $SU(2) \times U(1)$ theories using the prime numbers.) A further breaking of $SU(2) \times U(1)$ down to $SU(2)$ is also possible by squashing some more in a different direction and the same properties apply. The introduction of the right invariant vectors R_a can break this surviving $SU(2)$.

The fact that the eigenvalues of the Ricci tensor remain constant even after squashing may be traced to the existence on S^3 of the three globally defined (everywhere non-vanishing) linearly independent vector fields L_1, L_2 and L_3 , i.e., S^3 is "parallelizable". A d dimensional manifold is said to be parallelizable if there exist d such vector fields, and it follows that all group manifolds have this property. S^1 and S^3 are the only spheres which are group manifolds. The mechanism we have just described would not work for S^2 , for example.

The truly amazing fact from the point of view of supergravity is that S^7 , though not a group manifold, is also parallelizable !

The parallelizability of S^7 is well known to mathematicians, of course, but was first brought to my attention by N. Warner. The sequence S^1, S^3 and S^7 is intimately connected with the complex numbers, quaternions and octonions, respectively. One can now proceed

to squash S^7 in a similar fashion using the 7 globally defined vector fields which may be identified with the 7 Killing vectors of $SO(8)$ which do not correspond to generators of $SO(7)$. See Duff and Pope [1]. This corresponds to a non-vanishing expectation value for the scalar fields appearing in (31).

$$\langle S^{ijkl}(x) \rangle \neq 0 \quad (47)$$

Several questions now arise. Why should the theory prefer a squashed sphere to a round one? In other words, is there a solution of the $d = 11$ field equations corresponding to a squashed S^7 , and is it stable? Are the masses of the fermions and vector bosons small compared to the Planck mass, i.e., are the squashing parameters λ^2 close to unity? How many of the gauge symmetries, discrete symmetries and supersymmetries remain unbroken? The value of the cosmological constant in the symmetric vacuum is, by Eq. (36), far too big. What is the new value of the cosmological constant?

To begin with, we note that with the ansätze (8) and (9) the seven sphere has to be an Einstein metric in order to solve (11) but from the S^7 analogue of (45), it ceases to be Einstein as soon as we start to squash. However, one can now invoke the truly remarkable result (communicated to my colleague Pope by the French mathematician, Bourguignon) that S^7 has not one but two Einstein metrics! A little calculation reveals that the other one corresponds to a squashed S^7 with critical values for the squashing parameters. Its Riemann tensor and other details will be published elsewhere [1]. We simply remark that the squashing parameters are not close to unity and so the breaking is at the Planck scale. The cosmological constant remains large.

The number of unbroken supersymmetries will be given by the number of Killing spinors, i.e., the number of solutions to (16) with the covariant derivative appropriate to squashed S^7 . This number, which also dictates the number of massless spin $\frac{3}{2}$, is therefore given by the zero eigenvalue modes of the commutator $[\bar{D}_a, \bar{D}_b]$. We are presently investigating this but it would be miraculous if any supersymmetries remain unbroken. This and the residual gauge symmetry (determined by the Killing vectors) will also be discussed elsewhere.

Thus we have seen how squashing the extra dimensions in a Kaluza-Klein theory gives a geometrical interpretation to the Higgs and super-Higgs effect. In order to acquire a mass, however, it is necessary that the massless spin $\frac{3}{2}$ and spin 1 absorb Goldstone fermions and Goldstone bosons. This provides an interesting rule of thumb in counting massless states in Kaluza-Klein theories, namely the number of massless spin $\frac{3}{2}$ and massless spin $\frac{1}{2}$ coming from ψ_A in the symmetric vacuum must obey

$$N_{1/2} \gg N_{3/2} \quad (48)$$

and the number of massless spin 1 and massless spin 0 coming from g_{AB} must obey

$$N_0 \gg N_1 \quad (49)$$

When the extra dimensions are squashed, of course, we must relax our previous definition of the correct ansatz to include those massive states which were massless in the symmetric vacuum.

Finally, to avoid confusion, we stress that this spontaneous symmetry breaking via Kaluza-Klein is totally distinct from the symmetry breaking mechanism introduced by Scherk and Schwarz [21] in the context of dimensional reduction. Their scheme is tailored to (a) the existence of an internal symmetry and (b) the extra dimensions being circles, neither of which applies to $d = 11$ supergravity on S^7 .

PARALLELIZING TORSION : HIGGS INTERPRETATION, $\langle P \rangle \neq 0$

In Ref. [4] we posed the question "Seven-spheres hold great fascination for mathematicians. Will all this differential geometry eventually be important for supergravity?". Any lingering doubts that the answer is "Yes !" must surely be dispelled by the new and beautiful solution to the $d = 11$ equations of motion (4) and (5) found by Englert [5]. He replaces (8) and (9) by

$$e F_{\mu\nu\sigma} = \pm 2m \epsilon_{\mu\nu\sigma} \quad (50)$$

$$F_{mnpq} \neq 0 \quad (51)$$

so that (5) yields

$$\nabla_m F^{mnpq} = \pm \frac{m}{4} \epsilon^{rstwnpq} F_{rstw} \quad (52)$$

Note the factor of 2 in (50) as compared to 3 in (8). The reason for this choice will shortly become clear. Furthermore (4) splits into

$$R_{\mu\nu} = \frac{1}{3} \left[F_{\mu\alpha\beta\gamma} F_{\nu}{}^{\alpha\beta\gamma} - \frac{1}{12} g_{\mu\nu} (F_4^2 + F_7^2) \right] \quad (53)$$

$$R_{mn} = \frac{1}{3} \left[F_{mabc} F_n{}^{abc} - \frac{1}{12} g_{mn} (F_4^2 + F_7^2) \right] \quad (54)$$

where

$$F_4^2 = F_{\mu\nu\sigma} F^{\mu\nu\sigma} \leq 0, \quad F_7^2 = F_{mnpq} F^{mnpq} \gg 0 \quad (55)$$

We also note that

$$R^{\mu}_{\mu} = \frac{1}{36} \left[8 F_4^2 - 4 F_7^2 \right] \leq 0 \quad (56)$$

$$R^m_m = \frac{1}{36} \left[-7 F_4^2 + 5 F_7^2 \right] \geq 0 \quad (57)$$

so that the extra dimensions are still compact and that (unfortunately) the cosmological constant for space-time is still non-zero (except in the trivial case $F_4 = F_7 = 0$).

To solve (52), (53) and (54) Englert goes back to 1926, to the paper by Cartan and Schouten [22] dealing with parallelizable manifolds. Subject to some technical provisos which need not concern us here, the definition of parallelizability given in the previous section is equivalent to the existence of a torsion tensor S_{mnp} which, when added to the Riemannian connection Γ_{mnp} , yields a vanishing curvature

$$R^m_{npq} (\Gamma^s_{tu} + S^s_{tu}) = 0 \quad (58)$$

If we further insist that $\Gamma+S$ yield the same geodesics as Γ , then $S_{mnp} = S_{[mnp]}$ and (58) implies

$$R_{mnpq} = S_{t[mn} S^t_{p]q} - S_{t[mn} S^t_{p]q} \quad (59)$$

$$\nabla_q S_{mnp} = \partial_{[q} S_{mnp]} = S_{t[mn} S^t_{p]q} \quad (60)$$

on using the cyclic identities. According to Cartan and Schouten, Riemannian geometries permitting such an "absolute parallelism" fall into two categories :

- (A) Group manifolds,
- (B) S^7 .

Thus S^7 is not merely a non-group-manifold which is parallelizable, it is the only non-group manifold to admit an absolute parallelism ! In the case of group manifolds, (59) and (60) are solved by

$$S_{mnp} = \pm \frac{1}{2} f_{mnp} \quad (61)$$

where f_{mnp} are the structure constants.

In the case of S^7 , Englert establishes a correspondence between (59)/(60) and (54)/(52) by seeking solutions of the form

$$F_{mnpq} \propto \partial_m S_{npq} \quad (62)$$

i.e., in the "Landau gauge" $\nabla^m A_{mnp} = 0$

$$A_{mnp} \propto S_{mnp} \quad (63)$$

[This correspondence is most elegantly established using the Killing spinors again [1]. The self-dual totally antisymmetric 35-dimensional representation of $SO(8)$:

$$\bar{\eta}^{\pm} \Gamma_{mn} \eta^{\mp} \bar{\eta}^{\pm} \Gamma_p \eta^{\pm} \quad (64)$$

and the trace-free symmetric 35-dimensional representation of $SO(8)$:

$$\bar{\eta}^{\pm} \Gamma_{mnp} \eta^{\pm} \quad (65)$$

provide the 70 solutions to

$$\Delta A_{mnp} = 16 m^2 A_{mnp} \quad (66)$$

where Δ is the Laplacian on 3-forms, the former satisfying the self-dual linear equation (52) and the latter the anti-self dual.] Taking the space-time to be $AdS \times S^7$ (with its standard metric) one finds that

$$A_{mnp} = \frac{1}{4m} S_{mnp} \quad (67)$$

with m^{-1} once again the radius of S^7 , provides the desired solution. This yields [5]

$$F_{mabc} F_n{}^{abc} = 24 m^2 g_{mn} \quad (68)$$

and hence

$$R_{\mu\nu} = -10 m^2 g_{\mu\nu} \quad (69)$$

We recall that the round sphere with no torsion (Vacuum I) had a cosmological constant $\Lambda = -12m^2$, and this is to be compared with the round sphere with torsion (Vacuum II) which, from (69), has $\Lambda = -10m^2$.

As with the squashed sphere with no torsion (Vacuum III), the commutator $[\bar{D}_a, \bar{D}_b]$ no longer vanishes in Vacuum II when the appropriate covariant derivative, given by (1), is employed [6] and hence $N = 8$ supersymmetry is again broken. Moreover, the vacuum expectation values (6) no longer have vanishing Lie derivative with respect to the 28 Killing vectors of $SO(8)$, so the gauge symmetry is also broken.

Guided by our experience with Vacuum III, it is natural to seek a Higgs interpretation for the non-vanishing Cartan-Schouten torsion form on S^7 and this is readily supplied by Eq. (33). In Vacuum I, the criterion that the 35 pseudoscalars $p^{IJKL}(x)$ be massless was provided by the linear self-dual equation (34). But since $\langle A_{mnp} \rangle$ is zero in Vacuum I, we had $\langle p^{IJKL}(x) \rangle = 0$. In Vacuum II, however, $\langle A_{mnp} \rangle$ itself satisfies the same linear self-dual equation (52), and hence

I. - The round seven-sphere. Duff and Pope [1]

Solution :

$$F_{\mu\nu\sigma} = \pm 3m \epsilon_{\mu\nu\sigma} \quad R_{\mu\nu\sigma} = -4m^2 (g_{\mu\sigma} g_{\nu} - g_{\mu\nu} g_{\sigma})$$

$$F_{mnpq} = 0 \quad R_{mnpq} = m^2 (g_{mp} g_{nq} - g_{mq} g_{np})$$

N = 8 supersymmetry : Yes

II. - The round seven-sphere with torsion. Englert [5]

Solution :

$$F_{\mu\nu\sigma} = \pm 2m \epsilon_{\mu\nu\sigma} \quad R_{\mu\nu\sigma} = -\frac{10}{3}m^2 (g_{\mu\sigma} g_{\nu} - g_{\mu\nu} g_{\sigma})$$

$$\nabla^2 F_{\ell mnp} = \pm \frac{m}{4} \epsilon_{lmnpqrst} F_{qrst} \quad F_{npqr} F_n{}^{pqr} = 24m^2 g_{mn}$$

$$R_{mnpq} = m^2 (g_{mp} g_{nq} - g_{mq} g_{np})$$

N = 8 supersymmetry : No

III. - The squashed seven-sphere. Duff and Pope [1]

Solution :

$$F_{\mu\nu\sigma} = \pm 3m \epsilon_{\mu\nu\sigma} \quad R_{\mu\nu\sigma} = -4m^2 (g_{\mu\sigma} g_{\nu} - g_{\mu\nu} g_{\sigma})$$

$$F_{mnpq} = 0 \quad R_{mn} = 6m^2 g_{mn}$$

N = 8 supersymmetry : No

TABLE V - KNOWN SOLUTIONS WITH S^7 TOPOLOGY

$$\langle P^{ijkl}(x) \rangle \neq 0 \quad (70)$$

Once again, it seems unlikely that any supersymmetries will survive unbroken but, as discussed in a forthcoming paper [6] the maximal unbroken subgroup of $SO(8)$ seems to be the exceptional group G_2 whose representations demand that one gravitino will have a different mass from the other seven. (The group G_2 was also suggested independently by P. Fré.)

TOWARDS A REALISTIC THEORY

Assuming that the massless sector of our effective $d = 4$ theory based on the round S^7 of Vacuum I really is the de Wit-Nicolai theory, then solutions of the $d = 11$ field equations given by Vacua II and III, which have different geometry but the same topology as Vacuum I must correspond to non-symmetric extrema of the de Wit-Nicolai effective potential in $d = 4$. Finding such extrema is a very difficult task by conventional four-dimensional methods and, to date, only one has been found this way very recently by Warner [23]. Using the known $N = 5$ solution [24] embedded in $N = 8$, he finds all supersymmetries broken, $SO(8)$ broken down to $SO(3) \times SO(3)$ and $\Lambda = -14e^2$. We conclude therefore that this Vacuum Number IV is distinct from I, II and III.

What can be said about other solutions with S^7 topology? As far as field configurations are concerned, there would seem to be an upper limit of 42 parameters corresponding to the $35 + 35$ spin 0 fields less 28 for $SO(8)$ rotations, e.g., 7 squashing parameters plus 35 torsion parameters. Solutions on the other hand, i.e., critical points of the $d = 4$ effective potential, will probably not be described by several parameters but will rather be discrete in character (I am grateful for discussions on this point with P. Breitenlohner). In any event, we expect solutions with both squashing and non-zero torsion, hence the flow diagram in Table I.

Freedman and Breitenlohner [24] have established the positivity of the energy functional in gauged extended supergravity even though the effective potentials are unbounded below and the vacua correspond to critical points which are maxima or saddle points. Further discussion of the $N = 8$ effective potential may be found in the papers of de Wit and Nicolai [17].

The Cartan-Schouten torsion given by A_{mnp} is not the physical torsion which appears in the supergravity field equations and which is described by fermionic bilinears which have been set equal to zero in the tree approximation. Consequently, as may be seen from (56) and (57), no amount of squashing S^7 or A_{mnp} torsion will yield a zero cosmological constant, in spite of the parallelizability. Now parallelizable manifolds have also been advocated in the context of vanishing Λ in Kaluza-Klein theories by Orzalesi and collaborators. See [25] for a recent discussion. They point out that if fermionic bilinears were to acquire non-zero expectation value at the quantum level then a zero cosmological constant would result from flattening the physical connection by a parallelizing torsion. Their discussions were devoted to group manifolds and spin $\frac{1}{2}$ fermions in non-supersymmetric theories but it would be interesting to extend them to S^7 , spin $\frac{3}{2}$ and supergravity. Could it be that the "true" vacuum of $d = 11$ supergravity has $\Lambda = 0$?

The questions of the "true" vacuum and stability are perhaps the most urgent in Kaluza-Klein supergravity but also, it seems, the most difficult. The concept of "energy" in spaces which are products of space-time with $\Lambda \neq 0$ and compact extra dimensions is very obscure and

it is not clear whether energy is anyway the right criterion for stability. Some pertinent references are Christensen and Duff [26], Coleman and De Luccia [27], Witten [28], Abbott and Deser [29], Weinberg [30] and Breitenlohner and Freedman [24]. When the massive states are included we also have to worry about the enormous Casimir energies, and while there is a Bose-Fermi cancellation due to supersymmetry in the case that the extra dimensions are circles [19,4], the case of spheres, even when round, is much less obvious (this is being studied by David Pollard, a student at Imperial College). We are encouraged by the vanishing β function properties of gauged extended supergravities for $N > 4$ when there are no massive states [3] and that the spin-sum-rule derivation of these results would seem to carry over when the massive $OSp(4|8)$ multiplets are included (I am grateful to T. Curtwright for discussions on this).

In the 1920's Kaluza and Klein attempted to unify gravity with electromagnetism by giving an extra fifth dimension to space-time which was compactified to a circle. Nowadays, of course, we are more ambitious and would like to include the strong and weak interactions as well. Witten [12] has suggested that the extra seven dimensions of $d = 11$ supergravity might be enough to do this. Though similar in spirit, our approach has differed significantly from Witten's. He assumes a ground state metric of the form Minkowski space $\times G/\mathcal{H}$ where $G/\mathcal{H}$ is a seven-dimensional compact manifold and $G \supset SU(3) \times SU(2) \times U(1)$. Unfortunately, as Witten himself points out, there are two serious drawbacks: (i) there seems no reason why the theory should single out such a ground state; (ii) there seems to be no reason why the fermion representations should be chiral. By contrast, we have looked to the field equations to determine the ground state and have focused our attention on the maximally symmetric compactified solution, namely $AdS \times S^7$, and other less symmetrical solutions which are nevertheless topologically equivalent. Of course, there are other solutions, [4], like $S^5 \times S^2$ (yielding an $SU(4) \times SU(2)$ gauge theory) which are topologically distinct and which have nothing to do with extrema of the de Wit-Nicolai potential, but one has the feeling that the ground state should favour S^7 just as one has the feeling that space-time ground-state should be AdS rather than some other four-dimensional Einstein space with less symmetry. Since $S^7 = SO(8)/SO(7)$ and since $SO(8) \not\supset SU(3) \times SU(2) \times U(1)$ it will be necessary to appeal to the hidden $SU(8)$ for which the gauge bosons are composite if we are to make contact with particle physics phenomenology. Moreover the 56 spin $\frac{1}{2}$ fermions in the de Wit-Nicolai theory will be chiral with respect to this $SU(8)$. It is not clear how known quarks and leptons could be identified with these fundamental fermions and it may be necessary to invoke bound states. Note, however, that the breaking of 8 supersymmetries would mean that 8 spin $\frac{1}{2}$ would be absorbed in giving masses to the spin $\frac{3}{2}$. This would leave $56 - 8 = 48$ helicity states: just enough for 3 families.

Witten also points out that non-vanishing expectation values for $F_{\mu\nu\rho\sigma}$ and F_{mnpq} will give rise to C, P and CP violation in $d = 4$. For the round or squashed sphere with torsion, therefore, all three discrete symmetries will be broken. [Discrete symmetries in the $d = 11$ theory of Table II are rather curious: the theory is trivial under C

since each particle is its own antiparticle, trivial under P_{10} (reversal of all 10 spatial co-ordinates) since this is connected to the identity, and time reversal T requires simultaneous $A \rightarrow -A$ inversion. It is P_3 and P_7 for which $P_{10} = P_3 P_7$ which acquire significance in $d = 4$; P_3 as the usual parity operator and P_7 as $d = 4$ charge conjugation [31]. However, we disagree with Witten's conclusion [12] that the strong interaction angle θ will inevitably vanish at the tree level. The term in the $d = 11$ Lagrangian of Table II corresponding to the ϵ AFF interaction yields terms in the massless $d = 4$ sector of the form

$$f_{IJKL}(P) F_{\mu\nu}^{IJ} F_{\sigma\tau}^{KL} \epsilon^{\mu\nu\sigma\tau}$$

where f is a function of the pseudoscalars P . When $\langle P \rangle \neq 0$ one can obtain a

$$\theta F_{\mu\nu} F_{\sigma\tau} \epsilon^{\mu\nu\sigma\tau}$$

term.

In Vacua II and III all the symmetry breaking effects, whether gauge symmetry, discrete symmetry or supersymmetry, are at the Planck scale, and it remains to be seen whether more realistic solutions may be found. Perhaps quantum corrections could induce squashing parameters close to unity.

It has been known for some time [9] that $d > 4$ supergravities compactified on S^1 's given consistent supersymmetric theories in $d = 4$ and now we have learned [1] that compactification on S^7 also gives a consistent supersymmetric theory in $d = 4$. In the same spirit, Duff, Townsend and van Nieuwenhuizen [7] have recently shown that the two index antisymmetric tensor A_{MN} of $N = 1$ supergravity in $d = 10$ gives rise to a spontaneous compactification on three-spheres. It may not have escaped the reader's notice that these three known examples involve S^1 , S^3 and S^7 : the three parallelizable spheres! Is this coincidence, or is parallelizability a necessary criterion for consistency? If the latter, then this would resolve the ambiguity discussed earlier as whether $d = 11$ supergravity can compactify on S^4 to give space-time 7 dimensions as well as on S^7 to give space-time four dimensions. This is presumably equivalent to the question raised by Salam and Sezgin [33] of whether $d = 7$ supergravity can be gauged.

Will the mathematical mysteries of S^7 yield any other physical implications for supergravity? One thinks of octonions, E_7 , triality and the fact that S^7 has several inequivalent differentiable structures. S^7 is also the $SU(2)$ Yang-Mills instanton over S^4 .

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