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ITS PHYSICS CONTENT

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ON THE SUPERPOSITION PRINCIPLE AND ITS PHYSICS CONTENT

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Abstract: What is commonly denoted the superposition principle is shown to consist of three different physical assumptions: conservation of probability, completeness, and some phase conditions. The latter conditions form the physical assumptions of the superposition principle. These phase conditions are exemplified by the Kobayashi-Maskawa matrix. Some suggestions for testing the superposition principle are given.

The superposition principle is so well integrated into the basic framework of quantum mechanics that hardly anybody, not even authors of textbooks on quantum mechanics pause to ask the question: what exactly is the physical content of the superposition principle? Surely, it must have a physical content.

The first explicit formulation of the principle seems to have been given in 1930 by Dirac in the first edition of his book (1). In the later editions of the book, Dirac becomes less explicit in this respect.

The present article is based on a rather detailed study of the superposition principle which I published in a very little read journal (2). The essential mathematical formalism (3) also seems to be little known.

The notion of superposition of states enters the standard quantum mechanics as follows. Suppose an observation A on a physical system can have the possible outcomes $a_1, a_2, \dots, a_n$. If the system is prepared to yield a unique result a_i with certainty whenever the observation A is made, we say that the system is in a pure state $|a_i\rangle$.

The states $|a_i\rangle$ form a linear vector space in the sense that

- an arbitrary state $|\psi\rangle$ is a vector

$$|\psi\rangle = \alpha_i |a_i\rangle \quad (1)$$

in this space, (summation implied)

- the α_i are complex numbers,
 - the real numbers $|\alpha_i|^2$ give the probability for the outcome a_i .
 Departure from linear vector spaces generally imply some assumption of non-linearity. It is not our purpose here to test models of non-linear quantum mechanics which violate the superposition principle.

For simplicity, let us assume that the outcomes a_i form a discrete spectrum, and that the sum in Eq. (1) contains a finite number of terms, n .

Suppose now that another possible observation, B , is made on the same system, with n possible outcomes $b_1, b_2, \dots, b_n$. Then, in the linear vector space of the pure states $|b_i\rangle$, the arbitrary state $|\psi\rangle$ in Eq. (1) is the linear combination

$$|\psi\rangle = \beta_i |b_i\rangle \quad i = 1, \dots, n. \quad (2)$$

It follows from the above that

$$\alpha_i \alpha_i^* = \beta_i \beta_i^* = 1. \quad (3)$$

Moreover, since the physical system was assumed to be the same, regardless of whether the observation was A or B , it follows that

$$\alpha_i = U_{ij} \beta_j \quad (4)$$

where $U = (U_{ij})$ is a unitary matrix.

The physical meaning of Eq. (4) is obviously that the probability to obtain the outcome b_j in an observation of B on the system in a state $|a_i\rangle$ is

$$p(a_i, b_j) = |U_{ij}|^2 \quad (5)$$

How much physics has gone into this?

(i) Conservation of probability is implied by the n conditions

$$\sum_{i=1}^n p(a_i, b_j) = 1 \quad j = 1, \dots, n. \quad (6)$$

(ii) Completeness of the set of states is implied by the further $n-1$ independent conditions

$$\sum_{j=1}^n p(a_i, b_j) = 1 \quad (7)$$

(iii) The superposition principle may or may not imply some

further restrictions on the remaining set of $(n-1)^2$ independent real probabilities $p(a_i, b_j)$. This we shall see below.

Recall that a unitary matrix depends on n^2 real independent parameters. If the moduli of its elements are fixed, one still has the freedom to choose $2n-1$ phases arbitrarily. Thus the unitary matrices U and U' have the same moduli, if

$$U' = D_\varphi U D_\psi, \quad (8)$$

where D_φ is the diagonal matrix with elements $\delta_{jk} \exp(i\varphi_k)$, and D_ψ is the diagonal matrix with elements $\delta_{jk} \exp(i\psi_k)$. The φ_k , $k = 1, \dots, n$ and ψ_k , $k = 2, \dots, n$ are $2n-1$ arbitrary real phases, and $\psi_1 = 0$. Thus, subtracting the number of arbitrary phases, there are at most $n^2 - (2n-1) = (n-1)^2$ independent moduli $|U_{ij}|$.

We have learned that the number of available independent probabilities $p(a_i, b_j)$ is the same as the number of independent moduli $|U_{ij}|$ in a unitary matrix.

Consider now an experimental setup measuring all the quantities $p(a_i, b_j)$. A Gedanken-experiment may consist of two consecutive Stern-Gerlach magnets, rotated with respect to each other. Given a molecular beam of spin j , the first magnet, A, splits the beam up into $n = 2j+1$ components, corresponding to the emergence of the pure states $|a_i\rangle$, $i = 1, \dots, n$ behind the magnet. Subsequently the second magnet, B, splits each of these beams into n further components, corresponding to the emergence of n beams of pure states $|b_1\rangle$, n beams of $|b_2\rangle$, etc. The intensity of these latter beams will be proportional to $p(a_i, b_j)$.

Can one always form a unitary matrix with elements of moduli $|U_{ij}| = \sqrt{p(a_i, b_j)}$, given the experimental data set $p(a_i, b_j)$? If yes, the superposition principle is always valid; i.e. it has no physics content beyond Eqs. (6) and (7). If no, there are conditions which can serve as tests of the superposition principle.

Let's decompose U as follows (4,3):

$$U = D_\varphi \prod_{k=2}^n \prod_{j=1}^{k-1} S_{jk}(\theta_{jk}, \sigma_{jk}). \quad (9)$$

Here D_φ is the diagonal matrix in Eq. (8) and the S_{jk} are $n \times n$ unitary unimodular matrices, each a function of two parameters only: θ_{jk} and σ_{jk} . Only four elements in each matrix S_{jk} are different from δ_{jk} : the jj and kk elements have the value $\cos \theta_{jk}$, the jk element has the value $-\sin \theta_{jk} \exp(-i\sigma_{jk})$, and the kj element is the negative of the jk element. In this way the matrix U is expressed very conveniently in terms of the n phases φ_i , the $\frac{1}{2}n(n-1)$ parameters θ_{jk} , and the $\frac{1}{2}n(n-1)$ parameters σ_{jk} .

The reason why this parametrization is convenient is the following. The moduli of U will depend on the θ and on a well-defined subset of $\frac{1}{2}(n-1)(n-2)$ parameters γ which are linear

combinations of the σ . The remaining $n-1$ independent parameters σ correspond to the ψ in Eq. (8). Thus we can determine the $(n-1)^2$ parameters θ and γ from the independent experimental quantities $p(a_i, b_j)$, and ignore the $2n-1$ phases ϕ and ψ .

However, one can show (3) that there are $\frac{1}{2}(n-1)(n-2)$ inequalities of the type

$$|\cos \gamma| \leq 1 \quad (10)$$

which also must be satisfied by the data! These represent the physical content of the superposition principle.

Consider the first few cases with n small. When $n = 2$ there are no conditions of the kind of Eq. (10). Thus the superposition principle, as defined here, lacks physical content.

For $n = 3$ there is one inequality to be respected. In ref. (2) I have given this condition for the case of the double Stern-Gerlach experiment. Let me use here a more modern example: the 3×3 Kobayashi-Maskawa matrix (5) for the mixing of three quark flavors. Let's fill in only the moduli of the matrix:

$$\begin{pmatrix} A & B & \cdot \\ C & D & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad (11)$$

Here the dots denote dependent elements which can be obtained from Eqs. (6) and (7). In terms of the usual parametrization of the Kobayashi-Maskawa matrix, one has (with C_i for $\cos \theta_i$ and S_i for $\sin \theta_i$):

$$A = C_1; \quad B = S_1 C_3; \quad C = -S_1 C_2 \quad (12)$$

$$D^2 = (C_1 C_2 C_3 + S_2 C_3 \cos \delta)^2 + (S_2 C_3 \sin \delta)^2$$

The quantities A, B, C, D can in principle be determined independently. The superposition principle imposes the condition

$$-1 \leq \cos \delta \leq 1, \quad (13)$$

where

$$\cos \delta = \frac{D^2(1-A^2)^2 - (ABC)^2 - B^2(1-A^2-C^2)}{-2AB^2C \sqrt{1-A^2-C^2}}. \quad (14)$$

So far the experimental information on A, B, C, D does not allow one to test the superposition principle. Rather, one uses A, B, C and the superposition principle to set limits on D .

The case $n = 4$ has been solved in ref. (3). It is too laborious to report here.

In conclusion, we note that what usually is referred to as

the superposition principle really consists of three physical assumptions: conservation of probability, completeness of the set of states, and some phase conditions in unitary matrices. These phase conditions have not to my knowledge ever been tested. A way to test them would be to measure the intensities of the $(2j+1)^2$ beams of spin j molecules passing through a double Stern-Gerlach magnet. Assuming conservation of probability and completeness, only $(2j)^2$ of the intensities are independent. Of the remaining $4j+1$ intensities, one would serve as normalization and $4j$ as additional constraints.

The superposition principle would be best tested in a beam of large j , since the number of phase conditions grows as $j(2j-1)$, and the number of extra constraints available grows as $4j$.

In a test of the superposition principle there is no obvious need to go to high energies. Since quantum mechanics works so well, we have no reason to anticipate a violation of the superposition principle in any particular area of physics. However, we note that CP-violation 18 years after its discovery has not been satisfactorily explained, and that one obvious but unattractive explanation may still be a non-linear quantum mechanics (2,6).

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