Ref. TH.3463-CERN

ONE-DIMENSIONAL EXACTLY INTEGRABLE DYNAMICAL SYSTEMS
(CLASSICAL AND QUANTUM REGIONS)

I.A. Fedoseev and A.N. Leznov
IHEP - Serpukhov, USSR

and

M. V. Saveliev *)
CERN -- Geneva

ABSTRACT

We develop an approach for the construction of explicit solutions to a wide class of essentially non-linear exactly integrable one-dimensional systems in classical and quantum regions. The systems describe, in particular, monopole configurations for an arbitrary embedding of $SU(2)$ in a gauge group with a Higgs field in the adjoint representation and with vanishing self-coupling. The approach is based on the application of the usual perturbation theory methods in the framework of the Yang-Feldman formalism and gives an exact solution of the classical equations and the Heisenberg operators in the quantum case as finite polynomials over a coupling constant. An investigation of the systems in the Schrödinger picture is also given. In this, the wave functions are expressed directly via the matrix elements of the principal continuous series of the unitary irreducible representations of the real forms of the semi-simple Lie groups between the states, which are the generalized Whittaker vectors.

*) On leave of absence from IHEP, Serpukhov, USSR.

1. INTRODUCTION

We proposed, in Ref. 1), a general approach for the construction of exactly and completely integrable classical dynamical systems associated by means of a Lax-type representation in two-dimensional (2d) space with an arbitrary graded Lie algebra or superalgebra $\mathfrak{g} = \Sigma \oplus \mathfrak{g}_a$, $\dim \mathfrak{g}_a = d_a < \infty$, $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$, and developed a group method to find general solutions to these systems (in the sense of the Goursat problem). For the case when the Lax-pair operators take the values in the local part $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of an arbitrary Lie algebra $\mathfrak{g}$, whose grading is consistent with an arbitrary integral embedding of 3d subalgebra (A_1) in $\mathfrak{g}$, the corresponding equations can be formulated in explicit closed form ²⁾

$$\partial(g^{-1} \partial g / \partial z_-) / \partial z_+ = \lambda [J_-, g^{-1} J_+ g] \quad (1.1)$$

Here g is an element of the complex hull of the Lie group G_0 with Lie algebra $\mathfrak{g}_0$; 3d subalgebra is generated by the elements $J_{\pm}$, $H([J_+, J_-] = H$, $[H, J_{\pm}] = \pm 2J_{\pm})$; $2z_{\pm} \equiv t \pm r$ are the co-ordinates in 2d space; λ is a coupling constant. Note that there a grade of $\mathfrak{g}$ is defined by the Cartan element H of an embedding, $[H, \mathfrak{g}_a] = 2a \mathfrak{g}_a$, with respect to which elements of $\mathfrak{g}$ are arranged into multiplets with fixed values of the A_1 irreducible representation weight; in other words, values of the angular momentum. The right-hand side of (1.1) takes the values in $\mathfrak{g}_0^f \equiv \mathfrak{g}_0 / \mathfrak{g}_0^0$, where $\mathfrak{g}_0^0$ is invariant under A_1 , $[J_{\pm}, \mathfrak{g}_0^0] = 0$. So it means that $g^{-1} \partial g / \partial z_-$ also takes values in $\mathfrak{g}_0^f$.

System (1.1) has a non-trivial internal symmetry group and just this fact allows one to find explicit expressions for the solutions. There, as $\mathfrak{g}$ is finite-dimensional, the system is exactly integrable in the sense that it admits solutions with a number of arbitrary functions $\varphi_{\pm\alpha}(z_{\pm})$, $1 \leq \alpha \leq d_1$, sufficient for statement of the Goursat problem.

Particular cases of system (1.1) are the equations for the cylindrically symmetric self-dual Yang-Mills fields for an arbitrary embedding of $SU(2)$ in a gauge group ¹⁾; monopole, dyon and vortex systems ³⁾⁻⁵⁾; dual models ⁶⁾; relativistic strings and minimal surfaces ^{2), 7), 8)}; multicomponent Lund-Regge model ⁹⁾; generalized Toda lattice ¹⁰⁾ (with fixed end-points as well as the periodic one), etc.

The present paper is devoted to a constructive investigation in classical and quantum regions of the 1d version of an exactly integrable system (1.1),

$$d(g^{-1}dg/dt)/dt = \lambda [J_-, g^{-1}J_+g] \quad (1.2)$$

At quantum level, dynamical systems can be considered in various representations, the Schrödinger and Heisenberg ones being the most universal. For system (1.2), we managed to obtain here explicit solutions in both representations. In the Schrödinger picture the wave functions are the matrix elements of the principal continuous series of the unitary representations of non-compact real forms of semi-simple Lie groups, which are taken between states with definite quantum numbers (generalized Whittaker vectors). At the same time, the presence of a Hamiltonian formalism for the considered dynamical systems allows us to apply the usual perturbation methods of classical and quantum mechanics for its investigation. There, the first term in the Hamiltonian of Eq. (1.2)

$$\mathcal{H} = Sp[1/2 (g^{-1}\dot{g})^2 + \lambda (g^{-1}J_+gJ_-)], \dot{g} \equiv dg/dt \quad (1.3)$$

plays the rôle of the free part, whereas the second one describes an interaction in the system with some coupling constant λ . Perturbation series turn out to be a finite polynomial over λ and reproduce the exact solution of the corresponding systems in classical, as well as quantum, regions. [Another completely equivalent form for them can also be obtained from the known classical solutions ²⁾ of system (1.1) if one succeeds in finding an appropriate choice of arbitrary functions $\varphi_{\pm\alpha}(z_{\pm})$, which leads to a dependence on t only. Note that the last procedure is not so trivial in the general case.]

It should be stressed once more that here the perturbation method is used not for establishing some approximate results, but for the direct construction of explicit expressions of the exact solutions in the form of a finite series to the corresponding equations describing quantum and classical systems.

2. MATHEMATICAL PRELIMINARIES

We begin with a summary of some definitions and notations needed for what follows.

Denote by $X_{\alpha}^{\pm a}$ the elements of subspaces $\mathfrak{G}_{\alpha}$ of $\mathfrak{G}$, $1 \leq \alpha \leq d_a$. The elements X_{α}^0 and $X_{\alpha}^{\pm 1} (\equiv X_{\alpha}^{\pm})$ of $\mathfrak{G}_0$ and $\mathfrak{G}_{\pm 1}$ satisfy the commutation relations

$$\begin{aligned} [X_{\alpha}^0, X_{\beta}^0] &= \sum_{\gamma} B_{\alpha\beta}^{\gamma} X_{\gamma}^0, [X_{\alpha}^0, X_{\beta}^{\pm}] = \sum_{\gamma} C_{\alpha\beta}^{\pm\gamma} X_{\gamma}^{\pm}, \\ [X_{\alpha}^{\pm}, X_{\beta}^{\mp}] &= \sum_{\gamma} A_{\alpha\beta}^{\gamma} X_{\gamma}^0 \end{aligned} \quad (2.1)$$

and the structure constants A, B, C completely define the structure of the $\mathfrak{G}$ algebra as a whole. The reconstruction procedure of the $\mathfrak{G}$ algebra is based on the introduction of a bilinear symmetric form $(,)$ on $\hat{\mathfrak{G}}$ with its subsequent extension on the $\mathfrak{G}$ algebra. In this $(\mathfrak{G}_i, \mathfrak{G}_j) = 0$ when $i+j \neq 0$ ($i, j = 0, \pm 1$), and the invariance property gives $\sum_{\gamma} A_{\alpha\beta}^{\gamma} (X_{\rho}^0, X_{\gamma}^0) = -\sum_{\gamma} C_{\rho\beta}^{-\gamma} (X_{\alpha}^+, X_{\gamma}^-)$. In the particular case of simple Lie algebra supplied by the canonical grading, subspaces $\mathfrak{G}_0$ and $\mathfrak{G}_{\pm 1}$ are generated by the elements $h_j (=X_j^0)$ of the Cartan subalgebra η of $\mathfrak{G}$ and root vectors $X_{\pm j}^{\pm} (=X_j^{\pm})$ corresponding to the simple roots of $\mathfrak{G}$, $1 \leq j \leq n \equiv \text{rank } \mathfrak{G}$; $d_0 = d_{\pm 1} = n$. Here $B_{ij}^{\ell} = 0$, $C_{ij}^{\pm\ell} = \pm \delta_{j\ell}^k \delta_{ji}$, $A_{ij}^{\ell} = \delta_{ij} \delta_{i\ell}$, $1 \leq i, j, \ell \leq n$, where k is the Cartan matrix of $\mathfrak{G}$. In what follows we denote by $\bar{\alpha}, \alpha_0, \alpha, \underline{\alpha}$ and α the positive roots corresponding to the subspaces $\mathfrak{G}_0, \mathfrak{G}_0^0, \mathfrak{G}_0^f, \sum_{a \geq 1} \mathfrak{G}_a$ and $\mathfrak{G}_1$, respectively. The sets of roots $\bar{\alpha}, \alpha_0, \underline{\alpha}$ and α are R_0, R_0^0, R_+ and R_+ .

An arbitrary regular element $\hat{g}$ of a non-compact real form $\hat{G}$ of a semi-simple Lie group G with the Lie algebra $\mathfrak{G}$ supplied by the grading $\mathfrak{G} = (\sum_{a \geq 1} \mathfrak{G}_{-a}) \oplus \mathfrak{G}_0 \oplus (\sum_{a \geq 1} \mathfrak{G}_{+a})$ can be parametrized by a modified Gauss decomposition, $\hat{g} = \hat{z}_- g \hat{z}_+$. Here $g \in \hat{G}_0$ and $\hat{z}_{\pm}$ are generated by the elements $X_{\pm \alpha}$. Then, applying the technique developed in Ref. 11), it is easy to show that the infinitesimal operators of the left shifts on $\hat{G}$ (left generators) corresponding to the tangent transformations $\mathcal{F}$ of $\mathfrak{G}$ are the following

$$\begin{aligned} \hat{F} &= \sum_{\underline{\alpha}} (X^{\underline{\alpha}}, \hat{z}_-^{-1} \mathcal{F} \hat{z}_-) Z_{-\underline{\alpha}}^R + \sum_{\underline{\alpha}, \beta} (X^{\underline{\alpha}}, g X_{-\beta} g^{-1}) (X^{\beta}, \hat{z}_-^{-1} \mathcal{F} \hat{z}_-) Z_{\underline{\alpha}}^L \\ &+ \sum_{\mu_0} (X^{\mu_0}, \hat{z}_-^{-1} \mathcal{F} \hat{z}_-) F_{\mu_0}^R \end{aligned} \quad (2.2)$$

Here $Z_{-\underline{\alpha}}^R$ ($Z_{\underline{\alpha}}^L$) are right (left) generators corresponding to the root vectors $X_{-\underline{\alpha}} (X_{\underline{\alpha}})$, $F_{\mu_0}^R$ are the right generators on $\hat{G}_0$ corresponding to the Cartan elements h_j , $1 \leq j \leq n$, and root vectors $X_{\pm \bar{\alpha}}$ ($X_{\mu_0} \equiv \{h_j, X_{\pm \bar{\alpha}}\}$). The elements $X^{\pm \alpha}$ and X^{μ_0} generate a basis of dual subspaces $\mathfrak{G}_{\pm \bar{\alpha}}^{\pm}$ and $\mathfrak{G}_0^0$, which can be normalized by the relations $(X^{\underline{\alpha}}, X_{-\underline{\beta}}) = \delta_{\underline{\alpha}\underline{\beta}}$, $(h^i, h_j) = \delta_{ij}$, $(X^{\bar{\alpha}}, X_{-\bar{\beta}}) = \delta_{\bar{\alpha}\bar{\beta}}$, etc.

The Casimir operators of $\hat{G}$ are defined as the traces of successive powers of the generator matrix $\hat{F}$. Thus the quadratic Casimir operator (up to the canonical transformation) is

$$K = K_0 + \sum_{\alpha, \beta} (X^\alpha, g X^{-\beta} g^{-1}) Z_{-\beta}^R Z_\alpha^L \quad (2.3)$$

We omit the details of the calculations which are a direct generalization of those of Ref. 11) for the canonical grading of $\mathfrak{g}$.

In what follows we need also an explicit realization of the representation of $\hat{G}$ on a space of functions defined on the maximal compact subgroup $\mathcal{K}$ of $\hat{G}$. To achieve this, it is convenient to use the Iwasawa decomposition for an arbitrary (excluding a set of lower dimension) element $\hat{g}$ of $\hat{G}$,

$$\hat{g} = \hat{k} T(\tau) \hat{z}_+ \quad (2.4)$$

Here $T(\tau) = \exp \sum_j h_j \tau_j$, $1 \leq j \leq \text{rank } \mathcal{A}$, belongs to the maximal non-compact Abelian subgroup $\mathcal{A}$ of $\hat{G}$; $\hat{k} = k \cdot s \in \mathcal{K}$ with s being an element of the centralizer S of $\mathcal{A}$ in $\mathcal{K}$. Then the generators of an operator irreducible representation of $\hat{G}$ can be obtained in accordance with the method of Ref. 11) as the asymptotic values of the infinitesimal shift operators with $\alpha(\tau) \rightarrow \infty$ in the positive Weyl chamber. Here, the eigenvalues ρ_j of the operators $\partial/\partial \tau_j$ and the weights $\{\lambda_v\}$, $1 \leq v \leq \text{rank } S$, of irreducible representations of S define completely the irreducible representation of $\hat{G}$, $n \equiv \text{rank } \mathcal{A} + \text{rank } S$. Note that for the principal continuous series of unitary representations of $\hat{G}$ $\rho_j = -\frac{1}{2}\Delta_j - i\sigma_j$ with $\Delta_j \equiv \sum_a (h_j [X^a, X_{-a}])$, σ_j are continuous real parameters. As an example, consider natural real forms $\hat{G}$, for which left shift operators

$$\begin{aligned} \hat{F}^L = & - \sum_{\alpha} (X^\alpha, \hat{k}^{-1} \mathcal{F} \hat{k}) \mathcal{K}_\alpha^R + \sum_{\alpha} \exp(-\alpha(\tau)) (X^\alpha + X^{-\alpha}, \hat{k}^{-1} \mathcal{F} \hat{k}) Z_\alpha^L \\ & + \sum_j (h^j, \hat{k}^{-1} \mathcal{F} \hat{k}) \partial/\partial \tau_j \end{aligned} \quad (2.5)$$

have the following asymptotic values when $\alpha(\tau) \rightarrow \infty$

$$\hat{F}^L = - \sum_{\alpha} (X^\alpha, \hat{k}^{-1} \mathcal{F} \hat{k}) \mathcal{K}_\alpha^R + \sum_j (h^j, \hat{k}^{-1} \mathcal{F} \hat{k}) \rho_j \quad (2.6)$$

Here $\mathcal{K}_\alpha^R$ are the right generators of $\mathcal{K}$ corresponding to the elements $X_\alpha - X_{-\alpha}$; $\hat{k} = k$; the parameters ρ_j are just the weights of the irreducible representations of $\hat{G}$.

A finite transformation of an irreducible representation of $\hat{G}$ is defined as

$$T_g^{i\ell, \ell'} f_{i\ell}^{(k)} = \prod_j [R_j(g, k)]^{f_j} \sum_{i\ell'} D_{i\ell, i\ell'}^{i\ell}(N) f_{i\ell'}^{(\tilde{k})} \quad (2.7)$$

$R_j \equiv \exp(\tilde{\epsilon}_j - \epsilon_j)$, where $\tilde{k}$ and $\tilde{\epsilon}_j$ are obtained from the original element k and the non-compact parameters ϵ_j of $\mathcal{A}$ under the action of a transformation $\hat{g} \in \hat{G}$. Here $D_{\{m\}, \{m'\}}^{\{l\}}$ is a matrix element of an irreducible representation $\{l\}$ of S between the states with quantum numbers $\{m\}$ and $\{m'\}$; $\tilde{s} = sN(k\hat{g})$ and $N \in S$ is independent of group parameters of s . The relations connecting the transformed $\tilde{k}$ and $\tilde{\epsilon}_j$ with the original parameters k and ϵ_j are especially simple for the natural forms if $\hat{g} = \exp \sum_j h_j \tau_j$,

$$R_j^2(k, \tau) = \xi^{ij}(k^{-1} \exp(2 \sum_i h_i \tau_i) k),$$

$$\xi^{ij}(\tilde{k}) = \exp \tau_j \xi^{ij}(k) [\xi^{ij}(k^{-1} \exp(2 \sum_i h_i \tau_i) k)]^{-1/2} \quad (2.8)$$

where $\xi^{\{j\}}$ is the restricted (on $\mathcal{K}$) highest vector of the j fundamental representation of $\hat{G}$.

3. THE HAMILTONIAN FORMALISM (CLASSICAL REGION)

We give here the Hamiltonian formulation of the considered system (1.2). To achieve this, consider the operator

$$\mathcal{H} = 1/2 K_0 + \lambda (\mathcal{I}_-, g^{-1} \mathcal{I}_+ g) \quad (3.1)$$

with the quadratic Casimir operator K_0 of G_0 , $K_0 = \sum_{\mu_0} (F_{\mu_0}^L)^2$. A Poisson bracket of $\mathcal{H}$ with a group element g in some representation of G_0 has the form

$$\{\mathcal{H}, g\} = \sum_{\mu_0} F_{\mu_0}^L X_{\mu_0} g \quad \text{or} \quad \{\mathcal{H}, g\} g^{-1} = \sum_{\mu_0} F_{\mu_0}^L X_{\mu_0}$$

A Poisson bracket of $\mathcal{H}$ with the latter element is

$$\begin{aligned} \{\mathcal{H}, \sum_{\mu_0} F_{\mu_0}^L X_{\mu_0}\} &= \lambda \text{Sp} \{g^{-1} \mathcal{I}_+ g \mathcal{I}_-, \sum_{\mu_0} F_{\mu_0}^L X_{\mu_0}\} = \\ &= -\lambda \sum_{\mu_0} X_{\mu_0} \text{Sp} \{X^{\mu_0}, [g^{-1} \mathcal{I}_+ g, \mathcal{I}_-]\} = -\lambda [g^{-1} \mathcal{I}_+ g, \mathcal{I}_-] \end{aligned}$$

Therefore if one identifies operator $\mathcal{H}$ with the Hamiltonian of a dynamical system, then the equations describing it coincide with (1.2).

The elements $F_{\mu_0}^L$ play the rôle of generalized momenta, while the appropriate group parameters of g are the generalized co-ordinates, $\{F_{\mu_0}^L, g\} = X_{\mu_0} g$. Because we are now interested in the classical problem, consider the whole $\hat{G}$ as a Lie-Engel functional group G^f with generators $F_{\mu_0}^L$, $Z_{\underline{\alpha}}^L$ and $Z_{-\underline{\alpha}}^R$ entering (2.2) and satisfying the relations

$$\begin{aligned} \{F_{\mu_0}^L, F_{\nu_0}^L\} &= B_{\mu_0 \nu_0}^{\sigma_0} F_{\sigma_0}^L, \{F_{\mu_0}^L, Z_{\underline{\alpha}}^L\} = \{F_{\mu_0}^L, Z_{-\underline{\alpha}}^R\} = \{Z_{\underline{\alpha}}^L, Z_{-\underline{\beta}}^R\} = 0, \\ \{Z_{\underline{\alpha}}^L, Z_{\underline{\beta}}^L\} &= N_{\underline{\alpha} \underline{\beta}} Z_{\underline{\alpha} + \underline{\beta}}^L, \{Z_{-\underline{\alpha}}^R, Z_{-\underline{\beta}}^R\} = N_{-\underline{\alpha}, -\underline{\beta}} Z_{-(\underline{\alpha} + \underline{\beta})}^R \end{aligned}$$

In this $\{Z_{\underline{\alpha}}^L, g\} = \{Z_{-\underline{\alpha}}^R, g\} = 0$. Then, if we consider the operator $K^f \equiv K(G^f)$ as a Hamiltonian of a corresponding dynamical system, the usual equation $\dot{Q} = \{K^f, Q\}$ with $Q = \{Z_{-\underline{\alpha}}^R, Z_{\underline{\alpha}}^L\}$ leads to

$$\begin{aligned} \dot{Z}_{\underline{\alpha}}^L &= \sum_{\underline{\beta}, \underline{\gamma}} (X^{\underline{\beta}}, g X^{-\underline{\gamma}} g^{-1}) N_{\underline{\beta} \underline{\gamma}} Z_{-\underline{\gamma}}^R Z_{\underline{\alpha} + \underline{\beta}}^L, \\ \dot{Z}_{-\underline{\alpha}}^R &= \sum_{\underline{\beta}, \underline{\gamma}} (X^{\underline{\beta}}, g X^{-\underline{\gamma}} g^{-1}) N_{-\underline{\gamma}, -\underline{\alpha}} Z_{\underline{\beta}}^L Z_{-(\underline{\alpha} + \underline{\gamma})}^R \end{aligned}$$

Thus the quantity $\dot{Z}_{\underline{\alpha}}^L$ with $\underline{\alpha} \in R_+$ is expressed via the generators $Z_{\underline{\alpha} + \underline{\beta}}^L$ corresponding to the elements $X_{\underline{\alpha} + \underline{\beta}}$ of subspaces of $\mathfrak{G}$ with higher grading indices due to the property $[\mathfrak{G}_{\underline{a}}, \mathfrak{G}_{\underline{b}}] \subset \mathfrak{G}_{\underline{a} + \underline{b}}$. For this reason if one chooses initial conditions as $Z_{\underline{\alpha}}^L(t=0) = 0$ with $\underline{\alpha} \in R_+/R_+^1$ then $Z_{\underline{\alpha}}^L(t) = 0$ for $\underline{\alpha} \in R_+/R_+^1$ and $Z_{\underline{\alpha}}^L(t)$ with $\underline{\alpha} \in R_+^1$ are equal to some constants. The same is true for the right generators, i.e.,

$$Z_{-\underline{\alpha}}^R(t) = 0 \quad \forall \underline{\alpha} \in R_+/R_+^1 \quad \text{and} \quad Z_{-\underline{\alpha}}^R(t) = \text{const.}$$

Define these constants as follows: $Z_{\underline{\alpha}}^L = v_{\underline{\alpha}} c_{\underline{\alpha}}$ and $Z_{-\underline{\alpha}}^R = v_{\underline{\alpha}} c_{-\underline{\alpha}}$ with $v_{\underline{\alpha}} \equiv (X_{\underline{\alpha}}, X_{-\underline{\alpha}})$ and $c_{\pm \underline{\alpha}}$ coincide with those in the decompositions $J_{\pm} = \sum_{\underline{\alpha} \in R_{\pm}^1} c_{\pm \underline{\alpha}} X_{\pm \underline{\alpha}}$. Then the remaining equations for $Q = \{F_{\mu_0}^L, g\}$ lead to system (1.2).⁺ Note that we must also require $F_{\mu_0}^L = 0$ for $\mu_0 = \{\alpha_0, i_0\}$ corresponding to the elements of $\mathfrak{G}_0^0$ to provide the condition $g^{-1}g \in \mathfrak{G}_0^f$. It is obvious that expression (2.3) in this case is identically reduced to the Hamiltonian (1.3) of the system. Therefore, an embedding of $A_1 = \{H, J_{\pm}\}$ in $\mathfrak{G}$ is directly related to the initial conditions.

The presence of the Hamiltonian formulation allows us to apply the standard perturbation scheme for the construction of exact solutions for system (1.2) using the well-known Schwinger formula. But it seems more convenient to do it in the framework of the Yang-Feldman formalism, in which the solution of a system of differential equations is reduced to a calculation of multi-fold integrals in each order of perturbation series. Here the coupling constant λ plays the rôle of a small parameter.

Let us look for the solution of Eq. (1.2) in the form

$$g(t) = \left(\sum_{m=0}^{\infty} \lambda^m u_m(t) \right) g_0(t) \quad (3.2)$$

with $u_0 \equiv 1$ and u_m take values in $\mathfrak{G}_0^f$. Here $g^{-1} = g_0^{-1} \left(\sum_{m=0}^{\infty} \lambda^m v_m \right)$, where the elements v_m are defined via u_m by the formula $\left(\sum_{m=0}^{\infty} \lambda^m u_m \right) \left(\sum_{\ell=0}^{\infty} \lambda^{\ell} v_{\ell} \right) \equiv 1$ or $\sum_{s=0}^m v_s u_{m-s} \equiv 0$, $m \geq 1$; $v_0 \equiv 1$. Then the substitution of (3.2) into (1.2) leads to a system of differential equations

$$(\lambda^0): \quad d/dt (g_0^{-1} \dot{g}_0) = 0, \quad (3.3)$$

$$\begin{aligned} (\lambda^{m \geq 1}): \quad d/dt \left(g_0^{-1} \sum_{s=0}^{m-1} v_s \dot{u}_{m-s} g_0 \right) = \\ = \left[J_-, g_0^{-1} \sum_{s=0}^{m-1} v_s J_+ u_{m-s-1} g_0 \right] \end{aligned} \quad (3.4)$$

The solution of Eq. (3.3) can be presented in the following form

$$g_0(t) = h_0 \cdot \exp[(X^0 p)/t] \equiv \exp[(X^0 \tilde{p})/t] \cdot h_0 \quad (3.5)$$

because $g_0^{-1} \dot{g}_0$ is independent of t and takes values in $\mathfrak{G}_0^f$, $g_0^{-1} \dot{g}_0 \equiv (X^0 p) \equiv \sum_{\alpha=1}^d X_{\alpha}^0 p_{\alpha}$, $X_{\alpha}^0 \in \mathfrak{G}_0$, $h_0 \equiv \exp(X^0 q)$. Here the requirement that $g^{-1} \dot{g}$ takes values in $\mathfrak{G}_0^f$ is satisfied if one equates to zero the corresponding components of p , i.e., $p_{\alpha_0} \equiv p_{i_0} \equiv 0$.

We will use the following notations :

$$\begin{aligned} g_0^{(t_1, -t_2, \dots, \varepsilon_s t_s)} &\equiv g_0(t_1) g_0^{-1}(t_2) \dots g_0^{\varepsilon_s}(t_s) \equiv \exp[(X^0 \tilde{p})/T_s] \cdot h_0^{(\varepsilon_s+1)/2}, \\ \varepsilon_s &\equiv (-1)^{s-1}, \quad T_s \equiv t_1 - t_2 + \dots + \varepsilon_s t_s; \quad J_{\pm}^{(t_1, -t_2, \dots, \varepsilon_s t_s)} \equiv \\ &\equiv g_0^{(t_1, -t_2, \dots, \varepsilon_s t_s)} J_{\pm} (g_0^{(t_1, -t_2, \dots, \varepsilon_s t_s)})^{-1}; \quad J_{+}^{(\tau_+)} \equiv \exp[(X^0 \tilde{p})/\tau_+] \cdot J_{+} \cdot \\ &\cdot \exp[-(X^0 \tilde{p})/\tau_+], \quad J_{-}^{(\tau_-)} \equiv \exp[-(X^0 \tilde{p})/\tau_-] \cdot J_{-} \cdot \exp[(X^0 \tilde{p})/\tau_-]. \end{aligned}$$

It is easy to show that the solutions of Eq. (3.4) with $m=1,2$ can be written as

$$\begin{aligned} u_1(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \left[J_-^{(t_1)}, J_+^{(t_1-t_2)} \right], \\ u_2(t) &= \int_0^t dt' u_1(t') \frac{d u_1(t')}{dt'} + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \left[J_-^{(t_1)}, \right. \\ &\quad \left. \left[J_+^{(t_1-t_2)}, \left[J_-^{(t_1-t_2-t_3)}, J_+^{(t_1-t_2-t_3-t_4)} \right] \right] \right] \right], \end{aligned} \quad (3.6a)$$

or, after some trivial tricks, as

$$\begin{aligned} u_1(t) &= - \int_0^t dt_1 \int_0^{t_1} d\tau_1 \left[J_+^{(\tau_1)}, J_-^{(t_1)} \right], \\ u_2(t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \left\{ \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \left[J_+^{(\tau_1)}, J_-^{(t_1)} \right] \left[J_+^{(\tau_2)}, J_-^{(t_2)} \right] + \right. \\ &\quad \left. + \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \left[J_+^{(\tau_1)}, \left[\left[J_+^{(\tau_2)}, J_-^{(t_2)} \right], J_-^{(t_1)} \right] \right] \right\} \end{aligned} \quad (3.6b)$$

The algebraic structure of (3.6) is caused by the fact that the elements $u_m(t)$ take values in $\mathfrak{g}_0^f$. For this reason all the elements $X_{\pm\alpha}$, $\alpha \in R_+^1$, entering the decomposition of $J_{\pm}^{(\tau)}$ appear in formulae (3.6) only in the combinations which belong to subspace $\mathfrak{g}_0^f$. Therefore, it is possible to reformulate (3.6b) in terms of the normal ordering product: $J_+^{(\tau_1)} \dots J_+^{(\tau_m)} J_-^{(t_m)} \dots J_-^{(t_1)}$: ($m=1,2$) between the states annihilating under the action of $J_+^{(\tau_i)}$ and $J_-^{(t_i)}$, i.e., $J_+^{(\tau_i)}| \rangle = \langle | J_-^{(t_i)} = 0$. Such a matrix element is reduced to multiple commutators as in (3.6b) if one removes all $J_+^{(\tau_i)}$ ($J_-^{(t_i)}$) in it from left to right (from right to left) and annihilates the states. In fact, it follows from the modified Gauss decomposition for an element $\hat{g}$ of the Lie group with the Lie algebra $\mathfrak{g} = (\sum_{a \geq 1} \mathfrak{g}_{+a}) \oplus \mathfrak{g}_0 \oplus (\sum_{a \geq 1} \mathfrak{g}_{-a})$ that g is equivalent (in the definite sense) to the matrix element $\langle | \hat{g} | \rangle$ between the states $\mathfrak{g}_{+a} | \rangle = 0$ and $\langle | \mathfrak{g}_{-a} = 0 \nmid a \geq 1$.

This circumstance gives us a hint that in the general case the solutions $u_m(t)$ of system (3.4) have the following form

$$\begin{aligned} u_m(t) &= (-1)^m \int \dots \int \prod_{i=1}^m \theta(t_{i-1} - t_i) \theta(\tau_{i-1} - \tau_i) : J_+^{(\tau_1)} \dots J_+^{(\tau_m)} J_-^{(t_m)} \dots J_-^{(t_1)} : \equiv \\ &= (-1)^m \int \dots \int \prod_{i=1}^m \theta(\tau_{i-1}^+ - \tau_i^+) \theta(\tau_{i-1}^- - \tau_i^-) : J_+^{(\tau_1^+)} \dots J_+^{(\tau_m^+)} g_0(t) J_-^{(\tau_m^-)} J_-^{(\tau_1^-)} : g_0^{-1}(t), \\ &\quad t_0 \equiv t, \tau_0^{\pm} \equiv 0. \end{aligned}$$

The proof of this is quite easy and can be obtained using the reduction procedure. Thus the final complete solution to system (1.2) is represented as

$$g(t) = \sum_{m=0}^{\infty} (-\lambda)^m \int \dots \int \prod_{i=1}^m \theta(\tau_{i-1}^+ - \tau_i^+) \theta(\tau_{i-1}^- - \tau_i^-) \times$$

$$\times : J_+^{(\tau_1^+)} \dots J_+^{(\tau_m^+)} g_0(t) J_-^{(\tau_m^-)} \dots J_-^{(\tau_1^-)} : \quad (3.7)$$

or, in terms of chronologically $\hat{\tau}_{\pm}$ ordered exponentials :

$$g(t) = : (\hat{\tau}_+ \exp \lambda \int d\tau_+ J_+^{(\tau_+)}) g_0(t) (\hat{\tau}_- \exp \int d\tau_- J_-^{(\tau_-)}) : \quad (3.8)$$

Here, due to finite-dimensionality of the considered Lie algebras $\mathfrak{g}$, the series in (3.7) contains only a finite number of terms because one can always consider $g(t)$ in a finite-dimensional representation of $\mathfrak{g}$.

In order to carry out the integrations in (3.7), one has to decipher the elements $J_{\pm}^{(\tau_{\pm})}$ using relations (2.1) and the decomposition $J_{\pm} = \sum_{\alpha} c_{\pm\alpha} X_{\pm\alpha}$. Introduce the functions $f_{\alpha\gamma}^{\pm} \equiv \pm \sum_{\beta} c_{\beta\alpha}^{\gamma\pm} p_{\beta}$. Then $[(X^0 p), J_+] = \sum_{\alpha} c_{+\alpha} (f_+^+ X_+)_{\alpha}$, $[(X^0 p), J_-] = \sum_{\alpha} c_{-\alpha} (f_-^- X_-)_{\alpha}$ and therefore $J_{\pm}^{(\tau_{\pm})} = \sum_{\alpha} c_{\pm\alpha} (\exp f_{\pm}^{\tau_{\pm}} X_{\pm})_{\alpha}$. To calculate the integrals in (3.7) we need to diagonalize the matrices $f^{\pm} = S^{\pm} d^{\pm}$.

$$(S^{\pm})^{-1} \text{ with } d_{\alpha\beta}^{\pm} \equiv \delta_{\alpha\beta} d_{\alpha}^{\pm} \text{ being the eigenvalues of the operators } \tilde{X}_{\pm\alpha} \equiv$$

$$\equiv ((S^{\pm})^{-1} X_{\pm})_{\alpha}, \quad [(X^0 \tilde{\rho}), \tilde{X}_{+\alpha}] = d_{\alpha}^+ \tilde{X}_{+\alpha}, \quad [(X^0 \tilde{\rho}), \tilde{X}_{-\alpha}] = -d_{\alpha}^- \tilde{X}_{-\alpha};$$

$$[(X^0 q), \tilde{X}_{-\alpha}] = -e_{\alpha}^- \tilde{X}_{-\alpha}.$$

The final result is

$$g(t) = \sum_m (-\lambda)^m \sum_{\{\alpha_{\pm}\}} G_{\{\alpha_{\pm}\}}^{(m)} : \tilde{X}_{+\alpha_{+1}} \dots \tilde{X}_{+\alpha_{+m}} g_0(t) \tilde{X}_{-\alpha_{-m}} \dots \tilde{X}_{-\alpha_{-1}} : \equiv$$

$$\equiv \sum_m \lambda^m \sum_{\{\alpha_{\pm}\}} G_{\{\alpha_{\pm}\}}^{(m)} : \tilde{X}_{+\alpha_{+1}} \dots \tilde{X}_{+\alpha_{+m}} \tilde{X}_{-\alpha_{-m}} \dots \tilde{X}_{-\alpha_{-1}} : \times \quad (3.9)$$

$$\times \exp(X^0 q) \cdot \exp[(X^0 \tilde{\rho})t] \cdot \exp[-\sum_{s=1}^m (d_{\alpha_{-s}}^- t + e_{\alpha_{-s}}^-)],$$

where

$$G_{\{\alpha_{\pm}\}}^{(m)} \equiv \tilde{C}_{\{\alpha_{\pm}\}} \left[\prod_{i=1}^m \prod_{\varepsilon=\pm} \left(\sum_{\ell_{\varepsilon}=m-i+1}^m d_{\alpha_{\varepsilon}\ell_{\varepsilon}}^{\varepsilon} \right) \right]^{-1},$$

$$\tilde{C}_{\{\alpha_{\pm}\}} \equiv \prod_{j=1}^m \prod_{\varepsilon=\pm} \sum_{\{\beta_{\pm}\}} S_{\alpha_{\varepsilon j} \beta_{\varepsilon j}}^{\varepsilon} c_{\varepsilon \beta_{\varepsilon j}}.$$

Let us illustrate this formula for the simplest example of system (1.2): the generalized Toda lattice with fixed end-points, which corresponds to a finite-dimensional simple Lie algebra with canonical grading. In this case

$$J_{\pm} = \sum_{j=1}^n \delta_j^{\pm 1/2} X_{\pm j}, \quad \delta_j = 2 \sum_i k_{ji}^{-1}; \quad f_{ij}^{\pm} = \sum_{\ell} k_{i\ell} \delta_{ij} / \ell = \delta_{ij} (kp)_j$$

and (3.9) can be rewritten as

$$g(t) = \sum_m \lambda^m \sum_{i_1, \dots, i_m} \exp \left\{ - \sum_{s=1}^m \left[(kp)_{i_s} t + (kq)_{i_s} \right] \right\} \times \\ \times \prod_{\ell=1}^m (\delta_{i_{\ell}} \delta_{j_{\ell}})^{1/2} \left[\sum_{\mu=m-\ell+1}^m (kp)_{j_{\mu}} \right]^{-1} \left[\sum_{\nu=m-\ell+1}^m (kp)_{i_{\nu}} \right]^{-1} \times \quad (3.10) \\ \times : X_{+j_1} \dots X_{+j_m} X_{-i_m} \dots X_{-i_1} : \exp \left[(kp)t + (kq) \right].$$

The solutions of the generalized Toda lattice $\ddot{x}_i = \lambda \exp(kx)_i$ are related to $g(t)$ as $g(t) = \exp \left[- \sum_i h_i (x_i - \sum_j k_{ij}^{-1} \ln \delta_j) \right]$. Hence $\exp(-x_i) = \prod_j \delta_j^{-k_{ij}^{-1}} \langle i | g | i \rangle$, where $|i\rangle$ is the highest state of the i fundamental representation of $\mathfrak{g}$ satisfying the relations $X_{+j} |i\rangle = 0$, $h_j |i\rangle = \delta_{ij} |i\rangle$; $X_{-j} |i\rangle = 0 \quad \forall j \neq i$. The expression for $\exp(-x_i)$ contains the matrix element $\langle i | X_{+j_1} \dots X_{+j_m} \times X_{-i_m} \dots X_{-i_1} | i \rangle$ which, due to the properties of the highest state $|i\rangle$ and commutation relations between $X_{\pm j}$, h_j [see (2.1)], is not equal to zero only if $j_1 = i_1 = i$ and $i_s \equiv j_{\omega(s)}$, $2 \leq s \leq n$. Here ω is an arbitrary permutation of indices $(2, \dots, n)$. It is possible to show that

$$\mathcal{D}_{\{i, j, \omega\}} \equiv \langle i | X_{+j_1} \dots X_{+j_m} X_{-j_{\omega(m)}} \dots X_{-j_{\omega(1)}} | i \rangle = \quad (3.11) \\ = \delta_{ij_1} \prod_{\ell=2}^m \left[\delta_{i j_{\ell}} (1 + \xi_{j_{\ell}}) - k_{i j_{\ell}} - \eta_{j_{\ell}} \right],$$

where

$$\eta_{j_{\ell}} \equiv \sum_{s=2}^{\ell-1} k_{j_{\ell} j_s} \theta(j_{\omega(\ell)} - j_{\omega(s)}), \quad \xi_{j_{\ell}} \equiv \begin{cases} 1 & \text{if } \eta_{j_{\ell}} = 0 \\ 0 & \text{if } \eta_{j_{\ell}} \neq 0 \end{cases}.$$

Note that such an algebraic structure of this matrix element provides a cut-off in the series (3.10) over λ . The final explicit formula for $\exp(-x_i)$

$$\exp(-x_i) = \prod_j \delta_j^{-k_{ij}^{-1}} \sum_m \lambda^m \sum_{i_1, \dots, i_m} \sum_{\omega} \delta_{i_{\ell}} \left[\sum_{\mu=m-\ell+1}^m (kp)_{j_{\mu}} \right]^{-1} \times \quad (3.12) \\ \times \left[\sum_{\nu=m-\ell+1}^m (kp)_{i_{\omega(\nu)}} \right]^{-1} \exp \left\{ - \sum_{s=1}^m \left[(kp)_{i_{\omega(s)}} t + (kp)_{i_{\omega(s)}} \right] \right\} \times \\ \times \exp(p_i t + q_i) \cdot \mathcal{D}_{\{i, i, \omega\}}$$

gives the result of Kostant ¹⁰⁾ and Leznov and Saveliev ^{3),10)} in a somewhat different form.

The problem of a more explicit formulation of expression (3.9) in the general case requires the knowledge of the algebraic structure of the matrix elements $:\tilde{X}_+ \dots \tilde{X}_-:$.

4. QUANTIZATION

4.1 The Schrödinger picture

Usually the quantum-mechanical Hamiltonian is identified with the radial part of the quadratic Casimir operator of a semi-simple Lie group $\hat{G}$ in an appropriate parametrization. Here the choice of the basis functions in the representation space of $\hat{G}$ plays the same rôle as the choice of the initial conditions in the phase space of a functional group G^F for the classical problem. The Poisson brackets defining a classical system are changed by the commutation relations between the dynamical variables of the corresponding quantum system. There is a deep link between the solution of the quantization problem and the theory of group representations which was first discovered by Kostant ¹²⁾ for Toda-type systems [see also Ref. 13]. These papers contain an integral representation for one-component wave functions of the Toda lattice via Whittaker vectors. The last ones are in fact the afore-mentioned basis functions. In the following the approach has been generalized for series of exactly integrable quantum systems with multicomponent wave functions, whose classical limit coincides with the Toda lattice. Here we use the same method for the investigation of more general systems (1.2). Consider the Iwasawa decomposition (2.4), in which the parameters of S are excluded from $\hat{k}$ and committed in the middle,

$$\hat{g} = k \cdot \exp B \cdot \hat{z}_+ \quad (4.1)$$

Thus k contains only group parameters corresponding to the roots of $\mathfrak{g}/\mathfrak{g}_0$, while B takes values in $\mathfrak{g}_0$ and possesses in addition to τ_j the parameters χ_s , $1 \leq s \leq d_0 - n$. In this parametrization the expression for the quadratic Casimir operator takes the form

$$\begin{aligned} K = K_0 - \sum_{\alpha, \beta} (X^\beta, \exp B X_{-\alpha} \exp(-B)) \mathcal{K}_\alpha^R Z_\beta^L + \\ + \sum_{\alpha, \beta} (\exp(-B) X^\alpha \exp B, \exp B X^{-\beta} \exp(-B)) Z_\alpha^L Z_\beta^L \end{aligned} \quad (4.2)$$

where $\mathcal{K}_{\alpha}^R$ are the right generators of $\mathcal{K}$ corresponding to the roots $\alpha \in R_+$.

The action of the operator (4.2) with $\exp 2B \equiv g \in \hat{G}_0$ on its eigenfunctions ψ leads to expression (1.3) for the Hamiltonian of system (1.2) on the quantum level if

$$i) \quad \mathcal{K}_{\alpha}^R \psi = 0, \quad (4.3)$$

$$ii) \quad Z_{\alpha}^L \psi = \sum_{\alpha \in R_+^1} S_{\alpha} v_{\alpha} c_{\alpha} \psi, \quad (4.4)$$

where the constants c_{α} coincide with those in the decomposition of J_+ . Moreover, one also has to exclude from K_0 the terms connected with the elements of $\mathfrak{g}_0^0$.

The functions ψ must be transformed under some unitary irreducible representation of $\hat{G}$ because the quantum mechanical wave functions are normalizable. There the conditions imposed above uniquely define the functions ψ as the matrix elements of the principal continuous series between the states which are invariant under right shifts $\mathcal{K}_{\alpha}^R$.

Note that if one considers the Hamiltonian eigenfunctions ψ satisfying (4.4) and transforming under some finite-dimensional representation of $\mathcal{K}$ which is not identically reduced to a representation of S , then operator (4.2) corresponds to an exactly integrable quantum-mechanical system with multicomponent wave functions.

As it was established above, the wave functions of the considered system (1.2) coincide with the definite form matrix elements of the principal continuous series of unitary representations of $\hat{G}$. For the construction of explicit parametric expressions and the investigation of their asymptotic properties, which define the measure and the scattering amplitude in terms of the Jost functions, one needs some realization of the corresponding representation of $\hat{G}$. It is suitable to realize the representation of $\hat{G}$ on a space of functions defined on $\mathcal{K}$ as (2.7). Then the matrix element of the transition between the states with the vectors $\varphi_{\pm}(\mathbf{k})$ is determined by the formula $\int d\mathbf{k} \varphi_{-}^{*}(\mathbf{k}) \prod_j [\mathbf{R}_j(\mathbf{k}, \hat{g})]^{p_j} \varphi_{+}(\hat{\mathbf{k}})$. Thus, to obtain the required wave functions, one has to find the expressions for the functions $\varphi_{\pm}(\mathbf{k})$ satisfying the above-mentioned conditions with $\varphi_{-}(\mathbf{k})|_{\mathcal{K}/S} \equiv 1$ due to (4.3).

It is obvious that φ_+ must be transformed under some unitary representation $\{\ell\}$ of S being a scalar with respect to the transformations generated by $\mathfrak{G}_0^0$. Let us now first satisfy formally the requirements (4.4) without RHS, $\hat{P}_0^0 \varphi_+^0 = 0 \quad \forall \alpha \in R_+$ and then take into account the inhomogeneity. Evidently for a given representation $\{\rho, \ell\}$ of $\hat{G}$, φ_+^0 are the matrix elements $\langle R |$ between the highest states $|j\rangle$ ($1 \leq j \leq n$) of the fundamental representations of $\hat{G}$, as well as all the basis states of these representations annihilating under the action of $X_\alpha, \alpha \in R_+$. Thus φ_+^0 are just the restricted (on $\mathcal{K}$) matrix elements $\Xi\{\rho, \ell\}$ of the fundamental representations of $\hat{G}$ highest with respect to the elements of $\mathfrak{G}/\mathfrak{G}_0$. The inhomogeneous part of relations (4.4) can be reconstructed if one supplies $\Xi\{\rho, \ell\}$ with a factor $\exp(J_-^L \ln \xi')$, where $J_-^L = \sum_{\alpha} c_{-\alpha} \hat{X}_{-\alpha}^L$ with $\hat{X}_{-\alpha}^L$ being the left generators on $\mathcal{K}$ corresponding to the negative roots $(-\alpha)$;

$$\xi' = \prod_{\substack{i=1 \\ (i \notin I_0)}}^n \xi^{i i}$$

Here I_0 is a subset of indices $1, \dots, n$, which corresponds to the values $i_0 \in I_0$ with $h_{i_0} \in \mathfrak{G}_0^0$. Thus we have

$$\varphi_+^0 = W^{\{\rho, \ell\}} = \Xi^{\{\rho, \ell\}} \exp(J_-^L \ln \xi'), \quad (4.5)$$

which reduces to the Whittaker vector $W^{\{\rho\}} = \xi^{\{\rho\}}(\hat{k}) \exp(J_-^L \ln \xi)$ with $\xi = \prod_{i=1}^n \xi^{i i}(\hat{k})$ for the canonical embedding of A_1 in $\mathfrak{G}$. For this reason we call $W^{\{\rho, \ell\}}$ the generalized Whittaker vector.

Therefore the wave functions $\psi^{\{\rho, \ell\}}(\tau, \chi)$ corresponding to the Hamiltonian of the system (1.2) are expressed by the following integral representation

$$\psi^{\{\rho, \ell\}}(\tau, \chi) = \exp\left(\frac{1}{2} \sum_j \Delta_j \tau_j\right) \int d\hat{k} \mathcal{D}^{\{\rho, \ell\}}(\hat{k}_0) \prod_j [R_j(\tau, \chi; \hat{k})]^{j_j} W^{\{\rho, \ell\}}(\hat{k}). \quad (4.6)$$

Here $\mathcal{D}^{\{\rho, \ell\}}(\hat{k}_0)$ is the matrix element of $\mathcal{K}_0$ satisfying condition (4.3); the normalizing factor $\exp(\Delta\tau/2)$ was introduced to provide the Hermiticity of the Hamiltonian.

4.2 The Heisenberg picture

Let us reformulate the construction given in Section 3 to receive an exact expression for the quantum dynamical variable operator $g(t)$ in the framework of the Heisenberg treatment. Define in accordance with the free

part of the Lagrangian of system (1.2) in the quantum region, $\mathcal{L}_0(g) = \frac{1}{2} \text{Sp}(g_0^{-1} \dot{g}_0)^2$ with $g_0(t) = \exp(-X^0 \hat{q})$, the momenta $\hat{p}_\alpha = \sum_{\beta} c_{\alpha\beta} \hat{q}_\beta$ canonically conjugated to $\hat{q}_\alpha(t)$, $[\hat{p}_\alpha, \hat{q}_\beta] = -i\hbar \delta_{\alpha\beta}$; $\hbar$ is the Planck constant. Remember that the element $(X^0 \hat{q})$ takes values in $\mathfrak{g}_0^f$, i.e., $1 \leq \alpha, \beta \leq d_1 \equiv \dim \mathfrak{g}_0^f$; thus $c_{\alpha\beta} \equiv (X_\alpha^0, X_\beta^0)$ is the symmetrical and non-degenerate $d_1 \times d_1$ matrix. It is convenient to introduce the operators $\hat{f}_\alpha \equiv \sum_{\beta} \mathcal{M}_{\alpha\beta} \hat{q}_\beta$, $\mathcal{M}_{\alpha\beta} \equiv \sum_{\gamma} c_{\beta\gamma}^{-1} Y_{\alpha\gamma}^+$, whose general form is $\hat{f}_\alpha = \hat{A}_\alpha t + \hat{B}_\alpha$. Here $\hat{A}_\alpha$ and $\hat{B}_\alpha$ are t independent arbitrary operators satisfying the commutation relations

$$[\hat{A}_\alpha, \hat{B}_\beta] = -i\hbar R_{\alpha\beta}, [\hat{A}_\alpha, \hat{A}_\beta] = 0, [\hat{B}_\alpha, \hat{B}_\beta] = 0; R \equiv \mathcal{M} C^{-1} \mathcal{M}^T. \quad (4.7)$$

The system (3.3)-(3.4) is still conserved in the quantum case, as well as the expressions for the solutions in a form like (3.6a). But the transition to (3.6b) is not correct, due to the quantum character of the system, which is manifest in the algebraic structure of $J_{\pm}^{(t_1, -t_2, \dots, \varepsilon_s t_s)}$. To understand clearly the problem, we consider the simplest pattern of system (1.2) - the generalized Toda lattice, for which $\hat{f}_i = (k\hat{q})_i$ and $g_0(t) = \exp[-(\hbar k^{-1} \hat{A})t - (\hbar k^{-1} \hat{B})]$. Here $[\hat{f}_i(t_1), \hat{f}_j(t_2)] = -i\hbar (k v^{-1})_{ij}(t_1 - t_2)$. It is then possible to show that

$$\mathcal{I}_+^{(t_1, -t_2, \dots, \varepsilon_s t_s)} = g_0^{(\varepsilon_s)}(T_s) \mathcal{I}_+ (g_0^{(\varepsilon_s)}(T_s))^{-1}, \quad (4.8)$$

where

$$g_0^{(+1)}(T) \equiv g_0(T), g_0^{(-1)}(T) \equiv \exp[-(\hbar k^{-1} \hat{A})T + i\hbar/2(\hbar k^{-1} \hat{v}^{-1})T],$$

$$\hat{A}_j \equiv \hat{A}_j + i\hbar v_j^{-1}.$$

Analogous formulae for $J_{-}^{(t_1, -t_2, \dots, \varepsilon_s t_s)}$ follow from the previous ones by applying Hermitian conjugation (+) and taking into account the relations $\mathcal{X}_{+j}^{\dagger} = \mathcal{X}_{-j}$, $h_j^{\dagger} = h_j$. (Remember that in the case under consideration $J_{\pm} = \sum_j \delta_j^{\frac{1}{2}} \mathcal{X}_{\pm j}$.) The proof of (4.8) can be obtained using a reduction procedure and direct calculation for $s=2$. In fact

$$\begin{aligned} \mathcal{I}_+^{(t_1, -t_2)} &= g_0(t_1) \left[\sum_j \delta_j^{1/2} \exp(k\hat{q}(t_2))_j \mathcal{X}_j \right] g_0^{-1}(t_1) = \sum_j \delta_j^{1/2} [g_0(t_1) \times \\ &\times \exp(k\hat{q}(t_2))_j g_0^{-1}(t_1)] [\mathcal{X}_j g_0^{-1}(t_1)] = \sum_j \delta_j^{1/2} \{ \exp[i\hbar(\hbar v^{-1})_j T_2] \times \\ &\times \exp(k\hat{q}(t_2))_j \} \{ \exp[-(k\hat{q}(t_1))_j] \mathcal{X}_j \} = \\ &= \sum_j \delta_j^{1/2} \exp[i\hbar(\hbar v^{-1})_j T_2] \exp\{(\kappa[\hat{q}(t_2) - \hat{q}(t_1)])_j - i\hbar v_j^{-1} T_2\} \mathcal{X}_j = \\ &\equiv g_0^{(-1)}(T_2) \mathcal{I}_+ (g_0^{(-1)}(T_2))^{-1}. \end{aligned}$$

Substitution of these relations for $J_{\pm}^{(t_1, -t_2, \dots, e_s t_s)}$ into the corresponding formulae for $\hat{u}_m(t)$ leads, in the same way as in Section 3, to the following final expression for the Heisenberg operators $\exp(-\hat{x}_i) \equiv \prod_j \delta_j^{-k_{ij}} \langle i | g(t) | i \rangle$,

$$\exp(-\hat{x}_i(t)) = \langle i | \hat{M}(t) \exp(-\hat{q}_i(t)) \hat{M}^\dagger(t) | i \rangle, \quad (4.9)$$

where

$$\hat{M}(t) \equiv \sum_{m=0}^{\infty} \lambda^{m/2} \sum_{j_1 \dots j_m} R_{j_1 \dots j_m} \hat{x}_{j_1} \dots \hat{x}_{j_m},$$

$$R_{j_1 \dots j_m} \equiv \prod_{s=1}^m (\exp f_{j_s}/2) \left[\sum_{\ell=1}^m \hat{A}_{j_\ell} \right]^{-1},$$

which coincides with those of Ref. 14). This formula, after using relation (3.11), can be rewritten as

$$\exp(-\hat{x}_i(t)) = \sum_{m=0}^{\infty} \lambda^m \exp[-\hat{q}_i(t)] \sum_{j_1 \dots j_m} \prod_{s=1}^m (\exp f_{j_s}(t)) \times$$

$$\times P_{j_1 \dots j_m}^{(1)} \sum_{\omega(j_1 \dots j_m)} D_{i j \omega} P_{j_1 \dots j_m}^{(2)}, \quad (4.10)$$

where

$$P_{j_1 \dots j_m}^{(\varepsilon)} \equiv \prod_{\ell=1}^m \left(\sum_{s=\ell}^m a_{j_s}^{(\varepsilon)} \right)^{-1}, \quad \varepsilon = 1, 2;$$

$$a_{j_s}^{(1)} \equiv \hat{A}_{j_s} - i\hbar/2 \hat{k}_{j_s j_s} - i\hbar \sum_{\ell=s+1}^m \hat{k}_{j_s j_\ell},$$

$$a_{j_s}^{(2)} \equiv \hat{A}_{j_s} - i\hbar/2 \hat{k}_{j_s j_s} + i\hbar \sum_{\ell=1}^{s-1} \hat{k}_{j_\ell j_s} - i\hbar \sum_{\ell=1}^{s-1} \hat{k}_{j_\ell j_s}^1; \quad k \equiv k v^{-1}.$$

Transition to the classical limit ($\hbar \rightarrow 0$) is nothing but a trivial change $a_i^{(\varepsilon)} \rightarrow \hat{A}_i = -(kp)_i$, $\hat{B}_i = -(kq)_i$ and gives an equivalent form of expression (3.12).

The main steps of the given calculations apply also to the general case of the system (1.2), but with very tedious efforts connected with the diagonalization of the operator matrix $\exp f$, $\hat{f}_{\alpha\gamma} \equiv \sum_{\beta} c_{\beta\alpha}^{\gamma} \hat{q}_{\beta}$.

ACKNOWLEDGEMENTS

We would like to thank B.A. Arbuzov, O.A. Khrustalev, A.A. Kirillov, Yu.I. Manin and M.A. Mestvirishvili for many useful discussions. One of us (M.S.) also wishes to thank P. Goddard, D. Olive and L. Palla for helpful comments concerning the developments in the domain of monopoles, S. Ferrara for his kind help and the CERN Theoretical Physics Division (where this paper was completed) for its hospitality.

R E F E R E N C E S

- 1) A.N. Leznov and M.V. Saveliev, Comm.Math.Phys. 74 (1980) 111;
A.N. Leznov, M.V. Saveliev and V.G. Smirnov, Sov.J.Theor.Math.Phys.
48 (1981) 3.
- 2) A.N. Leznov and M.V. Saveliev, preprint IHEP82-113, Serpukhov (1982).
- 3) A.N. Leznov and M.V. Saveliev, Lett.Math.Phys. 3 (1979) 207.
- 4) N. Gaioulis, P. Goddard and D. Olive, Nuclear Phys. B205 (1982) 601.
- 5) A.T. Filippov, Sov.J.Part.Nucl. 11 (1980) 735;
A.N. Leznov and M.V. Saveliev, ibid 12 (1981) 126.
- 6) I.B. Frenkel and V.G. Kac, Inv.Math. 62 (1980) 23.
- 7) A.N. Leznov and M.V. Saveliev, Lett.Math.Phys. 6 (1982)
- 8) B.M. Barbashov, A.M. Chervyakov and V.V. Nesterenko, Comm.Math.Phys.
84 (1982) 471.
- 9) F. Lund and T. Regge, Phys.Rev. D14 (1976) 1524.
- 10) M. Toda, Physics Reports 180 (1975) 1;
O.I. Bogoyavlensky, Comm.Math.Phys. 51 (1976) 201;
J. Moser, Adv.Math. 16 (1975) 197;
B. Kostant, Adv.Math. 34 (1979) 195;
A.N. Leznov and M.V. Saveliev, Sov.J.Funct.Anal.Appl. 14 (1980) 87;
Physica 3D (1981) 62.
- 11) A.N. Leznov and M.V. Saveliev, Sov.J.Funct.Anal.Appl. 8 (1974) 87;
Sov.J.Part.Nucl. 7 (1976) 55.
- 12) B. Kostant, Lecture Notes in Math. 70 (1970) 80; Lecture Notes Series,
London Math.Soc. 34 (1979) 287.
- 13) A.G. Reyman and M.A. Semenov-Tian-Shansky, Inv.Math. 54 (1979) 81;
M. Adler, Inv.Math. 50 (1979) 219.
- 14) I.A. Fedoseev, A.N. Leznov and M.V. Saveliev, Phys.Letters 116B (1982)
49.