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VACUUM POLARIZATION EFFECTS IN THE $(\mu^{-4}\text{He})^{+}$ ATOM
AND THE BORN-INFELD ELECTROMAGNETIC THEORY

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ABSTRACT

We show that the Born-Infeld non-linear electromagnetic theory predicts the $n = 2$ fine-structure differences in the $(\mu^{-4}\text{He})^{+}$ system in fair agreement with the experimental results.

A discussion on the birefringence induced in vacuum by a static magnetic field within the frame of classical Born-Infeld type theories is also presented.

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1. INTRODUCTION

The experimental results for the transition energies $\Delta E_1 = 2S_{1/2} - 2P_{1/2}$ and $\Delta E_2 = 2S_{1/2} - 2P_{3/2}$ of the muonic helium ion $(\mu^{-4}\text{He})^+$ agree very well with the theoretical values obtained using the theory of quantum electrodynamics (QED).

The latest experimental results^{1,2)} are

$$\Delta E_1^{\text{exp}} = 1381.3 \pm 0.5 \text{ MeV} , \quad (1)$$

$$\Delta E_2^{\text{exp}} = 1527.5 \pm 0.3 \text{ MeV} . \quad (2)$$

The corresponding updated theoretical values³⁾ are given in Table I, where we have used, for the helium nucleus form factor, the newly determined value⁴⁾

$$\langle r^2 \rangle^{1/2} = 1.676 \pm 0.008 \text{ fm}. \quad (3)$$

The comparison between values (1), (2), and Table I shows one of the most stringent tests (1.5%) of the vacuum-polarization corrections predicted by QED. A similar sort of agreement has also been achieved recently in medium Z muonic-atom experiments⁵⁾.

Since in muonic atoms the electric field intensities explored by the muon around the nucleus are rather intense ($E \approx 2 \times 10^{16} \text{ V/m}$), these systems are very suitable for testing the limits of validity of the classical non-linear electromagnetic theory of Born-Infeld, which, in the recent past, has gained some popularity⁶⁻¹⁰⁾.

In this theory the self-energy of a point-like charge is always finite and its value is related to the Born-Infeld parameter b , which measures the maximum field intensity allowed. Born and Infeld^{11,12)} fix the parameter $b = b_0$ to have the rest mass of the electron equal to its electrostatic self-energy (see Appendix A).

On a general basis, it is expected¹³⁾ that in a non-linear classical electromagnetic theory one could have effects similar to those generally described by the QED vacuum polarization phenomena. The muonic atoms are particularly interesting for such a comparison since, owing to the high mass of the muon compared

with that of the electron, vacuum-polarization corrections to the Coulomb potential are dominant over corrections due to radiation reaction.

We have used the Born-Infeld theory, in its electrostatic version, to calculate the energy differences ΔE_1 and ΔE_2 (which are very sensitive to possible departures from the Coulomb potential), and we have compared the results with the experimental values (1) and (2). The calculations have been performed by leaving free the parameter b and by using the approximate procedure suggested elsewhere^{7,14}). A fair agreement between theory and experiment has been obtained for a value of the parameter b very near to b_0 .

Another effect which, owing to its non-linear character, can in principle be described by the Born-Infeld theory is the photon-photon interaction. Such an effect is also predicted by QED^{15,16}) and one of its consequences is the birefringence of vacuum induced by a static magnetic field¹⁷). This effect has never been observed directly; however, since there is an experimental program to detect such a space birefringence by optical means^{18,19}), we have investigated the possibility of describing this birefringence within the frame of the Born-Infeld theory.

In Appendix B we show that, as it stands, this theory is not able to predict the birefringence obtained in the standard QED and this result has nothing to do with the actual value of the parameter b . We show also that there is, however, a generalization of the Born-Infeld electromagnetic Lagrangian which can reproduce both the muonic-helium data and the space birefringence expected in QED, at least in the weak-field approximation.

Finally, we want to point out the methodological interest of the non-linear theories of the Born-Infeld type in QCD²⁰) and in problems dealing with solitons¹⁰).

2. TWO-PARTICLE INTERACTION

The electrostatic energy $W(\vec{D})$ of a charge distribution $\rho(x)$ is given, according to eq. (A.12), by

$$W(\vec{D}) = \int d^3x u(x) = \frac{b^2}{4\pi} \int d^3x \left\{ \left[1 + \left(\frac{\vec{D}}{b} \right)^2 \right]^{1/2} - 1 \right\} \quad (4)$$

and the auxiliary vector field $\vec{D}$ satisfies the linear differential equation

$$\text{div } \vec{D} = 4\pi\rho . \quad (5)$$

Moreover, from eq. (A.9) and $\text{curl } \vec{E} = 0$, one obtains

$$\text{curl } \vec{D} = \frac{\vec{D} \times \vec{\nabla}(D^2)}{2(b^2 + D^2)} , \quad (6)$$

which shows the non-linear character of the theory.

It is because of eq. (6) that the field $\vec{D}$ does not superimpose linearly (as in the Maxwell theory); i.e. if $\vec{D}_1$ is the solution for the charge distribution ρ_1 , and $\vec{D}_2$ for ρ_2 , the solution for the charge distribution $(\rho_1 + \rho_2)$ is not the sum $\vec{D}_1 + \vec{D}_2$.

To our knowledge, no exact solutions of eqs. (5) and (6) exist, apart from the case in which the charge distribution is spherically^{11,12)} symmetric. In this case $\text{curl } \vec{D} = 0$ for symmetry reasons; therefore we are brought to the usual electrostatic Maxwell solutions.

A way of obtaining an approximate solution of eqs. (5) and (6) has been suggested in ref. 14, and it consists in dividing the field $\vec{D}$ into a "Coulomb" part $\vec{D}_c$ plus a rotational part $\vec{D}'$; i.e.

$$\vec{D} = \vec{D}_c + \vec{D}' . \quad (7)$$

Since, by definition

$$\text{div } \vec{D}_c = 4\pi\rho, \quad \text{curl } \vec{D}_c = 0 , \quad (8)$$

the field $\vec{D}'$ is divergence free and satisfies the non-linear differential equation

$$\text{curl } \vec{D}' = \frac{\vec{D} \times \vec{\nabla}(D^2)}{2(b^2 + D^2)} \quad (\vec{D} = \vec{D}_c + \vec{D}') . \quad (9)$$

In an attempt to solve eq. (9), one can try to use the following iterative procedure

$$\vec{D} = \lim_{n \rightarrow \infty} \vec{D}^{(n)} \quad (10)$$

$$\vec{D}^{(n)} = \vec{D}_c + \vec{D}'^{(n)} ,$$

where

$$\begin{aligned} \operatorname{div} \vec{D}'^{(n+1)} &= 0 \\ \operatorname{curl} \vec{D}'^{(n+1)} &= \frac{\vec{D}^{(n)} \times \vec{\nabla} [\vec{D}^{(n)}]^2}{2\{b^2 + [\vec{D}^{(n)}]^2\}}. \end{aligned} \quad (11)$$

The convergence of this scheme has been considered in ref. 14 up to $\vec{D}'^{(1)}$. The authors show by direct computation that, for a system of two elementary charges, the correction $\vec{D}'^{(1)}$ is less than a few per cent of $\vec{D}_c$.

Moreover, by substituting eq. (7) in eq. (4) it can be shown that, at least in the case of two equal charges of the same or opposite sign, the following result holds:

$$W(\vec{D}) = W(\vec{D}_c) \left\{ 1 + O[(D'/D_c)^2] \right\}. \quad (12)$$

Hence the field $\vec{D}'^{(1)}$ does not contribute to the electrostatic energy $W(\vec{D})$ of two equal interacting charges at least up to the second order in the ratio (D'/D_c) . Therefore, insofar as the energy W is concerned, a good approximation of the field $\vec{D}$ should indeed be represented, in general, by the Coulomb solution $\vec{D}_c$, i.e. $W(\vec{D}) \approx W(\vec{D}_c)$.

Similarly to ref. 8, we have assumed that the interaction energy $V(d)$ between two charge distributions ρ_1 and ρ_2 , placed at a distance d from one another, is given by

$$V(d) = W(\vec{D}_{c_1} + \vec{D}_{c_2}) - W(\vec{D}_{c_1}) - W(\vec{D}_{c_2}), \quad (13)$$

where $\vec{D}_{c_1}$ and $\vec{D}_{c_2}$ are the Coulomb solutions associated, respectively, with the charge distributions ρ_1 and ρ_2 .

Let $U(d)$ be the interaction energy between the two charges, as obtained in the frame of the Maxwell theory, in general different from $V(d)$. It is not difficult to show that, asymptotically in the charge distance d , the following relation holds

$$\operatorname{sign} [V(d) - U(d)] = \operatorname{sign} (q_1 q_2), \quad (14)$$

where

$$q_i = \int d^3x \rho_i(x), \quad i = 1, 2. \quad (15)$$

As a consequence, at large distances, two opposite charges appear to be more bounded in the Born-Infeld theory than in the Maxwell theory.

3. RESULTS FOR THE $(\mu^{-4}\text{He})^{+}$

In order to calculate the energy differences ΔE_1 and ΔE_2 in the $(\mu^{-4}\text{He})^{+}$ system, obtained according to the Born-Infeld electromagnetic theory, we have started by computing numerically the interaction energy $V(d)$ defined by eq. (13) on the hypothesis that the helium nuclear charge distribution is Gaussian, with a r.m.s. given by eq. (3), and that the μ^{-} charge distribution is point-like.

Figure 1 shows the results obtained for the ratio

$$S(d) = V(d)/U_p(d) \quad (16)$$

for different values of the parameter b , as a function of the distance d between the two charges. The normalizing factor $U_p(d)$ is here the pure Coulomb interaction energy between the two charges both assumed point-like, i.e.

$$U_p(d) = -2e^2/d \quad (17)$$

To obtain ΔE_1 and ΔE_2 , we have used the energy difference

$$V'(d) = V(d) - U_p(d) = U_p(d) [S(d) - 1] \quad (18)$$

as a perturbation of the $n = 2$ hydrogen-like levels of the $(\mu^{-4}\text{He})^{+}$ system.

The energy splitting²¹⁾

$$\Delta E = \int_0^{\infty} r^2 dr (R_{21}^2 - R_{20}^2) V'(r) \quad (19)$$

is shown in fig. 2 as a function of the parameter b of the Born-Infeld theory.

A fair agreement between the experimental value (1) and the calculation is reached at the peak of the curve, for a value of b given by

$$b_{\text{helium}} = 0.382 b_0 \quad (20)$$

where b_0 is the value fixed by Born and Infeld. By taking care of the fine-structure interaction, in the usual way (see Table I), the energy splitting ΔE_2 is also reproduced by the same value of the parameter b as given by eq. (20).

Using the same procedure, we have in addition computed the energy splitting $DE_3 = 3S_{1/2} - 3P_{1/2}$ and we have compared the result with the sum of the Uhling + form-factor corrections for the $(\mu^{-4}\text{He})^+$ $n = 3$ level. In Table II we summarize the results obtained for the energy differences DE_1 and DE_3 within the frame of QED (leaving out fine, hyperfine, and other minor corrections) and of the Born-Infeld theory with $b = b_{\text{helium}}$.

4. CONCLUSIONS

The experimental values of the energy splittings ΔE_1 and ΔE_2 in muonic helium can be reproduced with fair accuracy within the frame of the Born-Infeld non-linear electromagnetic theory. This theory is indeed able to reproduce some of the effects described in QED through the vacuum-polarization phenomenon.

It is interesting to note that the value of the parameter b_{helium} necessary to fit the above-mentioned data is not far from the original Born-Infeld value b_0 .

We have also made investigations on the Born-Infeld prediction for the Lamb shift in the helium ion $(e^{-4}\text{He})^+$. We have found that, to the first order in the perturbation expansion, the energy of any S state with principal quantum number n is changed by the following amount

$$\Delta E_n = -8\pi e^2 |\psi_n(0)|^2 Y(b), \quad (21)$$

where the function $Y(b)$, plotted in fig. 3, is defined as

$$Y(b) = \int_0^\infty dr \, r [S(r) - 1]. \quad (22)$$

For $b = b_{\text{helium}}$ the above correction ΔE shifts the $n = 2$ state by approximately 200 MHz.

We have also studied in the same approximation the (μp) atom. We have found that the energy-splitting $2S_{1/2} - 2P_{1/2}$ (apart from the hyperfine contribution), calculated according to the Born-Infeld theory, agrees with the QED prediction^{22,23)} (Uhling's part + form factor), provided one takes

$$b_{\text{hydrogen}} = 0.6 \, b_{\text{helium}}. \quad (23)$$

Another effect that we have considered, within the frame of the Born-Infeld theory, is the birefringence of vacuum^{18,19}). This phenomenon is predicted by QED^{16,17}) and, according to eqs. (B.24) and (B.25), one expects a magnetic birefringence of vacuum equal to

$$\Delta n = n_p - n_t = 3RB_0^2 \sin^2 \theta , \quad (24)$$

where B_0 is the magnetic field intensity, θ is the angle between the magnetic field direction and the light-propagation axis, and R is given by eq. (B.21) (its value being $1.33 \times 10^{-33} \text{ cm}^3/\text{erg}$).

The Born-Infeld classical theory, according to eq. (B.9), predicts no magnetic induced vacuum birefringence, whatever the value of the parameter b ¹⁰).

We show in Appendix B that there exists, however, a generalization (B.10) of the Born-Infeld Lagrangian (A.4), which is able to predict (in the weak-field approximation) a vacuum magnetic induced birefringence.

In order that the indices of refraction given in eqs. (B.18) and (B.19) be equal to those in eqs. (B.24) and (B.25) one must have

$$2\tau = 7(1 - p) \quad (25)$$

$$1 - p = 4Rb^2 . \quad (26)$$

If one wants to maintain $p = 1/2$, then

$$\tau \approx 1.75 , \quad b = \bar{b} \approx 0.76 b_0 . \quad (27)$$

Since the energy splitting DE , as given by eq. (19), does not depend on the value of the parameter τ , its value for $b = \bar{b}$ can be inferred from fig. 1.

On the other hand, if the condition $\tau = 1$ is imposed, then from eqs. (25) and (26) one has

$$p \approx 0.71 , \quad b \approx 0.59 b_0 . \quad (28)$$

In conclusion, as far as the Born-Infeld non-linear electromagnetic theory is concerned, we can say the following:

- i) The agreement with the μ -helium experimental data (see fig. 2) can be made quite good, by using the approximate scheme suggested in ref. 14.
- ii) It is possible to generalize the Born-Infeld theory to describe also the magnetic induced birefringence of vacuum, and this can be done without spoiling too much the agreement with the μ -helium experimental data.

We close by saying that we are aware of the fact that the Born-Infeld theory is a rather primitive theory compared to QED. However, we have estimated that it is worth while to show the possibility of explaining directly, with a classical non-linear theory, effects otherwise predicted through second-quantization mechanisms.

APPENDIX A

In the Born-Infeld theory the electric and magnetic fields $\vec{E}$ and $\vec{B}$ are obtained from a relativistic covariant four-potential A_μ in the same way as in the Maxwell theory, i.e.

$$E_i = F_{0i} , \quad i = 1, 3 \quad (A.1)$$

$$B_i = -\frac{1}{2}\epsilon_{ijk} F_{jk} , \quad i, j, k = 1, 3 \quad (A.2)$$

where $F_{\mu\nu}$ is the usual electromagnetic antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad \mu, \nu = 0, 3 . \quad (A.3)$$

The equations of motion for the fields $\vec{E}$ and $\vec{B}$ are obtained, via the minimum action principle, from the free Lagrangian density

$$\mathcal{L} = \frac{b^2}{4\pi} \left[1 - \left(1 - C - G^2 \right)^{1/2} \right] , \quad (A.4)$$

plus the interaction term $J^\mu A_\mu$. C and G are, respectively,

$$C = \frac{E^2 - B^2}{b^2} = -\frac{1}{2b^2} F_{\mu\nu} F^{\mu\nu} \quad (A.5)$$

$$G = \frac{\vec{E} \cdot \vec{B}}{b^2} = \frac{1}{4b^2} \epsilon_{\mu\nu\sigma\rho} F^{\mu\nu} F^{\sigma\rho} , \quad (A.6)$$

and b is the parameter of the theory, which has the dimensions of a field intensity (in the limit of b going to infinity, one comes back to the Maxwell Lagrangian).

If we introduce, in the standard way, the two auxiliary fields

$$D_i = 4\pi \frac{\partial \mathcal{L}}{\partial E_i} = \frac{E_i + GB_i}{[1 - C - G^2]^{1/2}} \quad (A.7)$$

$$H_i = -4\pi \frac{\partial \mathcal{L}}{\partial B_i} = \frac{B_i - GE_i}{[1 - C - G^2]^{1/2}} , \quad (A.8)$$

then the field equations read

$$\begin{aligned} \text{curl } \vec{H} &= \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{J}, \quad \text{curl } \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \\ \text{div } \vec{D} &= 4\pi\rho, \quad \text{div } \vec{B} = 0. \end{aligned} \quad (\text{A.9})$$

One of the most interesting features characterizing the Born-Infeld theory is that the electrostatic self-energy associated with a point-like charge is finite¹¹⁾.

Let us consider the energy-momentum density tensor defined by the Born-Infeld Lagrangian (A.4): one has

$$T_{\mu\nu} = F_{\mu\rho} \frac{\partial \mathcal{L}}{\partial F^{\nu\rho}} - g_{\mu\nu} \mathcal{L}, \quad (\text{A.10})$$

from which one obtains for the energy density the expression

$$u = T_{00} = \frac{\vec{E} \cdot \vec{D}}{4\pi} - \mathcal{L}, \quad (\text{A.11})$$

and this becomes, in the electrostatic situation ($\vec{B} = 0$),

$$\begin{aligned} u &= \frac{b^2}{4\pi[1 - (\vec{E}/b)^2]^{1/2}} \left\{ 1 - [1 - (\vec{E}/b)^2]^{1/2} \right\} = \\ &= \frac{b^2}{4\pi} \left\{ [1 + (\vec{D}/b)^2]^{1/2} - 1 \right\}. \end{aligned} \quad (\text{A.12})$$

For a point-like charge q , the Gauss theorem and rotation symmetry require that

$$\vec{D} = \frac{q}{r^3} \vec{r}, \quad (\text{A.13})$$

which means, for the electrostatic energy W ,

$$W = b^2 \int_0^\infty dr r^2 \left\{ [1 + (\frac{q}{br^2})^2]^{1/2} - 1 \right\} = 1.2361 q^{3/2} b^{1/2}. \quad (\text{A.14})$$

Born and Infeld fixed the parameter $b = b_0$ in order to have the electrostatic self-energy W of the electron equal to its rest mass. In this case b_0 is equal to

$$b_0 = \frac{m_e^2 c^4}{e^3 (1.2361)^2} \simeq 1.2 \times 10^{20} \text{ V/m}. \quad (\text{A.15})$$

APPENDIX B

In the weak-field approximation ($|C| \ll 1$; $|G| \ll 1$) the Born-Infeld free Lagrangian (A.4) becomes

$$\mathcal{L} = \frac{1}{8\pi} \left\{ (E^2 - B^2) + \frac{1}{4b^2} [(E^2 - B^2)^2 + 4(\vec{E} \cdot \vec{B})^2] \right\}. \quad (B.1)$$

The auxiliary fields $\vec{D}$ and $\vec{H}$, from eqs. (A.7) and (A.8), read

$$D_i = E_i + \frac{1}{2b^2} [(E^2 - B^2)E_i + 2(\vec{E} \cdot \vec{B})B_i] \quad (B.2)$$

$$H_i = B_i + \frac{1}{2b^2} [(E^2 - B^2)B_i - 2(\vec{E} \cdot \vec{B})E_i]. \quad (B.3)$$

If we introduce now the electric and magnetic permeability tensors ϵ and μ in the usual way:

$$D_i = \epsilon(\vec{E}, \vec{B})_{ij} E_j \quad (B.4)$$

$$B_i = \mu(\vec{E}, \vec{B})_{ij} H_j, \quad (B.5)$$

we obtain

$$\epsilon(\vec{E}, \vec{B})_{ij} = \delta_{ij} + \frac{1}{2b^2} [(E^2 - B^2)\delta_{ij} + 2B_i B_j] \quad (B.6)$$

$$\mu(\vec{E}, \vec{B})_{ij} = \delta_{ij} - \frac{1}{2b^2} [(E^2 - B^2)\delta_{ij} - 2E_i E_j]. \quad (B.7)$$

The fact that ϵ and μ are different from unity means that, because of the presence of the fields, the vacuum behaves as a material medium and gets polarized.

In the presence of a static magnetic field $\vec{B}$ (the situation also considered in ref. 18 to test the QED magnetic birefringence of vacuum), substituting eqs. (B.2) and (B.3) in eq. (A.9) one obtains plane wave solutions which propagate in the direction k with a phase velocity

$$v = c/n \quad (B.8)$$

$$n = 1 + \frac{B_0^2}{2b^2} \sin^2 \theta, \quad (B.9)$$

where θ is the angle between the vectors $\vec{B}_0$ and $\vec{k}$. Since the phase velocity is independent of the polarization state, no vacuum birefringence can be generated in the canonical Born-Infeld theory whatever the value of b .

Let us consider now the following relativistic invariant Lagrangian:

$$\mathcal{L} = \frac{b^2}{8\pi p} \left\{ 1 - [1 - C - \tau G^2]^p \right\}, \quad (B.10)$$

which reproduces the Born-Infeld theory for $p = 1/2$, $\tau = 1$ and the Maxwell theory for $p = 1$, $\tau = 0$. According to eq. (A.11), in the electrostatic situation ($\vec{B} = 0$), the energy density associated with such a Lagrangian reads

$$u = \frac{b^2}{8\pi p} \left\{ \left[1 + (2p - 1) \frac{E^2}{b^2} \right] \left[1 - \frac{E^2}{b^2} \right]^{p-1} - 1 \right\} \quad (B.11)$$

and the self-energy W of a point-like charge q becomes

$$W = \frac{(q^3 b)^{1/2}}{4p} \int_0^1 d\xi \xi^{-5/2} (1 - \xi^2)^{(1-3p)/2} \times [\xi^2(2p-1) - 1] \left\{ (1 - \xi^2)^{p-1} [1 + \xi^2(2p-1)] - 1 \right\}, \quad (B.12)$$

which is independent of τ and always finite, provided $p < 1$.

In the weak-field approximation ($|C| \ll 1$, $|G| \ll 1$), the Lagrangian (B.10) gives

$$\mathcal{L} = \frac{1}{8\pi} \left\{ (E^2 - B^2) + \frac{1}{2b^2} [(1-p)(E^2 - B^2)^2 + 2\tau(\vec{E} \cdot \vec{B})^2] \right\}. \quad (B.13)$$

According to the definitions (B.4) and (B.5), the electric and magnetic permeability tensors now become

$$\epsilon(\vec{E}, \vec{B})_{ij} = \delta_{ij} + \frac{1}{b^2} [(1-p)(E^2 - B^2)\delta_{ij} + \tau B_i B_j] \quad (B.14)$$

$$\mu(\vec{E}, \vec{B})_{ij} = \delta_{ij} - \frac{1}{b^2} [(1-p)(E^2 - B^2)\delta_{ij} - \tau E_i E_j] \quad (B.15)$$

and, in the particular case in which $\vec{E} = 0$, $\vec{B} = \vec{B}_0$, they read

$$\epsilon_{ij} = \left[1 - (1-p) \frac{B_0^2}{b^2} \right] \delta_{ij} + \tau \frac{B_i B_j}{b^2} \quad (B.16)$$

$$\mu_{ij} = \left[1 + (1 - p) \frac{B_0^2}{b^2} \right] \delta_{ij} . \quad (B.17)$$

With these permeability tensors, the vacuum becomes (in general) birefringent. As a matter of fact, it can be shown that the indices of refraction n_t and n_p for light propagating in the $\vec{k}$ direction, linearly polarized respectively in the $(\vec{k}, \vec{B}_0)$ plane and orthogonal to it, are

$$n_t = 1 + (1 - p) \frac{B_0^2}{b^2} \sin^2 \theta \quad (B.18)$$

$$n_p = 1 + \frac{1}{2} \frac{B_0^2}{b^2} \sin^2 \theta , \quad (B.19)$$

where θ is the angle between the two vectors $\vec{k}$ and $\vec{B}_0$.

For the sake of completeness, we point out that the QED Lagrangian, which describes the photon-photon interaction to zero order in the radiative corrections is the Euler-Heisenberg-Weisskopf Lagrangian^{15,16)} and, in the weak-field approximation, this reads

$$\mathcal{L} = \frac{1}{8\pi} \left\{ (E^2 - B^2) + 2R[(E^2 - B^2)^2 + 7(\vec{E} \cdot \vec{B})^2] \right\} , \quad (B.20)$$

where the constant R is related to the fine-structure constant α , to the rest mass of the electron m , and to its Compton wavelength λ_e in the following way:

$$R = \frac{1}{90\pi} \alpha^2 \frac{\lambda_e^3}{mc^2} . \quad (B.21)$$

With this Lagrangian, the electric and magnetic permeability tensors for the vacuum are

$$\epsilon_{ij} = \delta_{ij} + 2R[2(E^2 - B^2)\delta_{ij} + 7B_i B_j] \quad (B.22)$$

$$\mu_{ij} = \delta_{ij} - 2R[2(E^2 - B^2)\delta_{ij} - 7E_i E_j] \quad (B.23)$$

and the two indices of refraction n_t and n_p , in the particular case in which $\vec{E} = 0$, $\vec{B} = \vec{B}_0$, become¹⁷⁾

$$n_t = 1 + 4RB_0^2 \sin^2 \theta \quad (B.24)$$

$$n_p = 1 + 7RB_0^2 \sin^2 \theta . \quad (B.25)$$

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Table I

Various contributions to the $n = 2$ energy splittings in
the $(\mu^{-4}\text{He})^{+}$ system

Contributions	Transitions energies (meV)	
	$2S_{1/2} - 2P_{3/2}$	$2S_{1/2} - 2P_{1/2}$
Dirac contribution with Coulomb potential and point-like charges	145.70	0
Nuclear polarizability	3.1 ± 0.6	3.1 ± 0.6
Finite size ⁴⁾	-289.5 ± 2.8	-289.5 ± 2.8
Electronic vacuum polarization:		
Uehling term: first iteration	1664.44	1664.17
higher "	1.70	1.70
Kallen-Sabry term ($\alpha^2 Z\alpha$)	11.55	11.55
$\alpha(Z\alpha)^n$, $n > 3$	-0.02	-0.02
$\alpha^2(Z\alpha)^2$	0.02	0.02
muon vacuum polarization	0.33	0.33
μ -e vacuum polarization	0.02	0.02
hadron vacuum polarization	0.15	0.15
Vertex corrections and $(g - 2)$:		
$\alpha(Z\alpha)$	-10.52	-10.85
$\alpha(Z\alpha)^n$, $n > 1$	-0.16	-0.16
$\alpha^2 Z\alpha$	-0.03	-0.03
Recoil terms:		
Breit	0.28	0.28
Two photons	-0.44	-0.44
Weak contribution	0.00002	0.00002
Sum theory	1526.0 ± 2.8	1380.3 ± 2.8
Experiment	1527.5 ± 0.3	1381.3 ± 0.5

Table II

Energy differences (in meV)

Splitting	Born-Infeld from eq. (19)	QED Uhling + helium form factor
DE ₁	1302	1380
DE ₃	376	398

Figure captions

Fig. 1 : Behaviour of the function $S(d)$ defined in the text by eq. (16), for the following values of the ratio $y = b/b_0$: $a-h = 2.5, 1.5, 1.0, 0.6, 0.38, 0.25, 0.15, 0.1$. The dashed line corresponds to the case $b = b_{\text{helium}}$.

Fig. 2 : Energy splitting DE calculated for the $(\mu^{-4}\text{He})^{+}$ system according to the Born-Infeld theory [see eq. (19)] as a function of the parameter $y = b/b_0$. The dashed line represents the experimental value DE_1 as given by eq. (1), whereas the dash-dotted line represents the form-factor contribution.

Fig. 3 : $Y(b)$ [see eq. (22)] as a function of the ratio $y = b/b_0$: for $b = b_{\text{helium}}$ the Born-Infeld correction on the 2S level amounts to ~ 200 MHz. The dotted line represents the Uhling correction.

Fig. 1

Fig. 2

Fig. 3