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LOCAL HAMILTONIAN MONTE CARLO STUDY OF THE MASSIVE SCHWINGER MODEL
IN AN EXTERNAL BACKGROUND FIELD

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A B S T R A C T

Using a local Hamiltonian Monte Carlo approach the massive Schwinger model in an external background field is studied on the lattice. A non-vanishing quark condensate is found for zero-fermion masses. The dependence of condensate and string tension on the background field shows the expected continuum behaviour. The approach can be used to extrapolate between weak and strong coupling regions. A first-order phase transition for a non-small mass/quark charge ratio is examined for a special background field value.

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Monte Carlo calculations on the lattice allow us to study non-perturbative properties of gauge theories. The inclusion of fermions in the studies remains one of the most important problems. Various methods ¹⁾⁻⁴⁾ have been proposed to handle the fermionic degrees of freedom. A recently proposed local Monte Carlo algorithm for theories formulated in Hamiltonian language ⁵⁾ seems to be an elegant way in two dimensions to include the fermions into the theory, avoiding too large a consumption of computer time.

So far this method has been successfully applied to non-relativistic models of electrons in 1+1 dimensions ⁵⁾, as well as to the massless Schwinger model ⁶⁾. In this paper we extend this Monte Carlo approach to study the massive Schwinger model in an external constant background field produced by test charges.

The well-known Schwinger model ^{7),8)} is a good laboratory to study the usefulness of this local Monte Carlo algorithm. Some of the main properties of this model will be repeated briefly here.

- 1) Chiral symmetry breaking in case of vanishing quark masses connected with a non-zero quark condensate.
- 2) Screening of all external test charges q in case of zero-quark masses corresponding to a vanishing string tension defined as increase in the ground state energy per unit length due to external sources.
- 3) Screening of the test charges in the massive Schwinger model if q is an integer multiple of the quark charge g . For any other value of q/g the test charges are confined and a non-zero string tension occurs.
- 4) First-order phase transition above some critical m/g value (m being the quark mass) at background field $q/g = \frac{1}{2}$ between two degenerate vacuum states.

Using finite lattice scaling methods, Hamer et al. ⁹⁾ have recently reproduced these main properties of the model. To our knowledge, up to now only special features have been studied using Monte Carlo methods, partly due to particular difficulties ^{2),10)}. The aim of this paper is rather to show that in the local Hamiltonian approach all these described features can be reproduced than to try to reach a good accuracy in the Monte Carlo data. In most of the calculations we use relative large couplings g to reach more rapidly the equilibrium state. In the algorithm used the properties of the system are given by averages of certain operators O

$$\langle 0 \rangle = \frac{\langle 0 | e^{-HT} \rangle}{\langle e^{-HT} \rangle} \quad (1)$$

with T being the inverse temperature. The partition function can be approximately calculated introducing a space-time lattice with $\Delta\tau = T/N_T$ and splitting the Hamiltonian H into two pieces H_1 and H_2 .

$$\begin{aligned} \text{Tr}(e^{-HT}) &= \sum_{i_1, \dots, i_{2N_T}} \langle i_1 | e^{-\Delta\tau H_1} | i_2 \rangle \langle i_2 | e^{-\Delta\tau H_2} | i_3 \rangle \dots \\ &\dots \langle i_{2N_T-1} | e^{-\Delta\tau H_1} | i_{2N_T} \rangle \langle i_{2N_T} | e^{-\Delta\tau H_2} | i_1 \rangle + O(\Delta\tau^3). \end{aligned} \quad (2)$$

The pieces H_1 and H_2 couple only with the nearest neighbour sites and consist of a sum of commuting terms. The sum in the partition function runs over complete sets of intermediate states. Due to the properties of H_1 and H_2 any of the above matrix elements becomes a product of local matrix elements easily calculable. Therefore, the intermediate states can be efficiently sampled by Monte Carlo.

The Hamiltonian formulation of the massive Schwinger has been extensively discussed in the literature ^{11),12)}. So we start immediately with the lattice Hamiltonian. To avoid fermion doubling we use the method of Susskind ¹³⁾ setting the fermion spinor components on neighbouring sites. Denoting the space-like lattice spacing by a and the boson-fermion coupling with a dimension of a mass by g we use the following form of the Hamiltonian (in temporal gauge)

$$H = \frac{1}{2} g^2 a \sum_n (L_n - \alpha)^2 + m \sum_n (-1)^n \chi_n^\dagger \chi_n + \frac{1}{2a} \sum_n (\chi_n^\dagger e^{i\theta_n} \chi_{n+1} + \text{h.c.}) \quad (3)$$

The sum runs over all sites at a given time. The fermion operators with only one degree of freedom per site satisfy

$$\{\chi_n^\dagger \chi_m\} = \delta_{nm}, \quad \{\chi_n \chi_m\} = 0. \quad (4)$$

The gauge fields are defined on links between sites

$$U_{n,n+1} = e^{i\theta_n}. \quad (5)$$

The operator L_n which generates rotation in θ_n is just a multiple of the electric field, $L_n = 1/g E_n$. In the boson number representation

$$e^{i\theta_n} = a_n^+,$$

$$[L_n a_m^+] = \delta_{nm} a_m^+, \quad [L_n a_m] = -\delta_{nm} a_m. \quad (6)$$

$\alpha = q/g$ is the constant background field caused by the external test charges. The physics of the model in the continuum is a periodic function of α with $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. We will study the interval $0 \leq \alpha \leq 1$ in order to find whether this periodicity is reproduced in this lattice formulation.

The boson and fermion operators satisfy the usual commutation relations. Therefore both fermions and bosons are described by occupation numbers. The site n can be either empty or full, whereas the link between site n and $n+1$ is occupied by positive or negative integers for the bosons. In order to use the local Hamiltonian method we split H into terms acting on variables attached to even (odd) links. Each term in $H_{1,2}$ only changes the quantum numbers on a link and the adjoint sites. As discussed by Martin and Otto⁶⁾ the inclusion of only gauge invariant states in the trace (2) is guaranteed if one starts the calculation with the correct ground state which is gauge invariant.

We randomly generate new possible configurations, and accept or reject them according to a heat bath algorithm. Taking into account conservation laws associated with the Hamiltonian, only six non-vanishing matrix elements within a two-site block remain analogous to the calculation in the massless Schwinger model. Periodic boundary conditions have been chosen in the space direction. Due to the trace the periodicity in time is obvious.

Our Monte Carlo calculations have been performed on ESER and IBM computers. We have used a space-time lattice of size 40×100 and 20×50 and fixed the asymmetry $\Delta\tau/a = 0.2$ to minimize the neglected contributions in the partition sum (2). We have checked that a change to $\Delta\tau/a = 0.1$ does not influence our results within statistical errors.

As the initial zero temperature lattice, we have used a ground state configuration for which the external background field vanishes and the quark masses are equal to zero. The particle configuration is characterized by a zero-boson occupation number for the links whereas the fermion occupation numbers 0 and 1 are equally spaced. To obtain the correct ground state as a massless limit from the massive Schwinger model, the odd sites have to be occupied.

To decrease correlation effects, measurements have been done always only after five sweeps through the lattice. All the errors given in the Figures are statistical ones. They may underestimate the real errors since our lattice size is rather small and we have heated the lattice at every value of m and α for the most part only in 200 iterations before starting measurements.

We have roughly reproduced the results for the massless model ⁶⁾, however, without the attempt to reach the accuracy reported there. In what follows the lattice spacing a is set equal to one. In Figs. 1 and 2 we represent our results for the ground state energy normalized to the strong coupling limit

$$E_{\text{norm}} = E - m \langle \bar{\Psi} \Psi \rangle . \quad (7)$$

For different couplings, masses and background field values, typically 200 iterations per point have been used on the 40×100 lattice. The continuum limit for vanishing coupling g is the energy of the X-Y antiferromagnetic spin model ¹⁴⁾ independent of m and α , $E_{\text{cont}} = -1/\pi$. There is hope from Fig. 1 that this limit is reached for all masses and background field values considered.

Of special interest is the study of the quark condensate. Due to chiral symmetry breaking one expects for $m = 0$ in the continuum limit

$$\langle \bar{\Psi} \Psi \rangle / g = -\frac{1}{2\pi^{3/2}} \exp(\gamma) \cos(2\pi\alpha) , \quad \gamma = 0.5774... \quad (8)$$

In Fig. 3, we report our results for the quark condensate as a function over the m/g ratio at zero background field. Clearly, a non-zero condensate has been found consistent with (8). The dependence of $\langle \bar{\Psi} \Psi \rangle$ on the background field is studied in Fig. 4 at zero mass using a coupling $g = 0.4$. We find the expected sign change, especially also for $\alpha > \frac{1}{2}$. The normalization is in rough agreement with the continuum expectation. We have checked that with increasing mass this sign changing effect disappears and the condensate becomes almost independent of α .

Now let us study the string tension defined as

$$\sigma(\alpha) = (E(\alpha) - E(0)) / 2L \quad (9)$$

where L is the distance between the external charges. For small masses its behaviour in the continuum limit is expected to be

$$\zeta(\alpha) = \frac{1}{4\pi^{3/2}} \exp(\gamma) g m (1 - \cos(2\pi\alpha)) \quad (10)$$

Studying in our Monte Carlo approach the ground state energy E as a function of the background for different masses, we should find clear evidence for the non-vanishing string tension. For this analysis we have used a 20×50 lattice.

First we convince ourselves that the string tension vanishes in the massless model for any value of α . As can be seen from the Table the ground state energy is nearly independent of α . A coupling $g = 0.4$ and 600 iterations per Monte Carlo data point have been used.

Now we study E vs. α in the massive Schwinger model. Fixing the coupling to $g = 0.7$ and the mass to $m = 0.5$ we obtain the results presented in Fig. 5. Starting first with $\alpha = 0$ we run to $\alpha = 1$ and then come back to zero. 600 (1200) iterations have been used to measure every data point in the region $0 \leq \alpha \leq \frac{1}{2}$ ($\frac{1}{2} < \alpha \leq 1$) and $1 > \alpha \geq \frac{1}{2}$ ($\frac{1}{2} > \alpha \geq 0$). Strong evidence for an increasing string tension starting from $\alpha = 0$ or $\alpha = 1$ as a function over the background field is found. Crossing $\alpha = \frac{1}{2}$ the system first runs into an energetically unfavoured metastable state followed by a hopping into the stable ground state phase. The system undergoes a first-order phase transition with the typical hysteresis behaviour for the energy as a function on α . Below $\alpha = \frac{1}{2}$ in our boson number representation the true ground state is characterized by a configuration where most of the links are unoccupied, above $\alpha = \frac{1}{2}$ the energetically favoured state is that of an average occupation number one accompanied by some fluctuations. We have checked that the system, iterated for sufficiently long will be in the correct ground state.

The maximal stable string tension can be extracted at $\alpha = \frac{1}{2}$. We obtain for the given m and g values of Fig. 5

$$\zeta\left(\frac{1}{2}\right) \approx 0.032 \pm 0.006 \quad (11)$$

This result can be compared with the upper limit on the continuum string tension in the non-relativistic limit ($g/m \rightarrow \infty$)

$$\zeta_{\text{non-rel}}(\alpha) = \frac{g^2}{4} \alpha \quad |\alpha| \leq \frac{1}{2} \quad (12)$$

giving $\sigma_{\text{non-rel}}\left(\frac{1}{2}\right) = 0.0306$.

We note the non-vanishing slope of the "stable" string tension at $\alpha = \frac{1}{2}$. Coleman ⁸⁾ has first expected that here a first-order phase transition appears between two degenerate vacuum states breaking the charge conjugation symmetry spontaneously. This phenomenon has to appear during the transition from the strong coupling region ($m \ll g$) where the slope of the string tension at $\alpha = \frac{1}{2}$ vanishes to the weak coupling region. Hamer et al. ⁹⁾ have estimated the critical ratio

$$\left(\frac{m}{g}\right)_c = 0.325 \pm 0.02 \quad (13)$$

above which this transition should start.

We have also tried to find this critical point in our approach. Our results (Fig. 6) are obtained on a 20×50 lattice with 600 iterations per point. In spite of errors in the energy caused by the large fluctuations in the boson representation number near $\alpha = \frac{1}{2}$ we only report a region for this critical point

$$0.30 < \left(\frac{m}{g}\right)_c < 0.40 \quad (14)$$

consistent with Eq. (13).

In conclusion, we have applied a local Monte Carlo approach which allows us to include fermions in two dimensions in a local way in calculations on the lattice to the study of the massive Schwinger model in an external background field. The main properties of the model mentioned at the beginning and the periodicity in the background field can be well reproduced by this lattice version. To increase accuracy, larger lattices and more iterations per point are necessary. This seems to be non-problematic since the algorithm used works relatively fast.

At present, it is unclear how to extend this method successfully to larger dimensions.

A more detailed version of this paper will be presented elsewhere.

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α	0.0	0.1	0.2	0.3	0.4	0.5
E	-0.297 ± 0.003	-0.298 ± 0.003	-0.298 ± 0.003	-0.290 ± 0.003	-0.293 ± 0.003	-0.287 ± 0.003
α	0.6	0.7	0.8	0.9	1.0	
E	-0.298 ± 0.003	-0.291 ± 0.003	-0.294 ± 0.003	-0.291 ± 0.003	-0.254 ± 0.003	

TABLE - Ground state energy versus background field at $m = 0$
and $g = 0.4$

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FIGURE CAPTIONS

- Figure 1 Ground state energy normalized to the strong coupling limit as function of the coupling constant for different mass and background field values.
- Figure 2 Ground state energy normalized to the strong coupling limit as a function of the fermion mass for two coupling constants and zero background field.
- Figure 3 Quark condensate over coupling ratio as a function of mass over coupling ratio for different coupling constants at zero background field.
- Figure 4 Quark condensate at $m = 0$ and $g = 0.4$ versus background field compared to the expected continuum behaviour of Eq. (8).
- Figure 5 Ground state energy at $m = 0.5$ and $g = 0.7$ as a function over increasing and decreasing background
- Figure 6 Ground state energy at $g = 0.7$ versus background field near $\alpha = \frac{1}{2}$ for different mass over coupling ratios.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6