HIGH STATISTICS STUDY OF THE REDUCED, QUENCHED LARGE N GAUGE THEORY
IN FOUR DIMENSIONS

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A B S T R A C T

We study the reduced, quenched $d = 4$ large N gauge theory for $N = 8, 10, 12$ with very high statistics. The development of the first-order transition expected in this model at $N = \infty$ is clearly seen in our simulation as a mismatch between strong and weak coupling curves for the plaquette energy. Locating the "critical" values of β as the intersection of the strong and weak coupling data and extrapolating to $N = \infty$ using a linear fit gives the critical point in the $N = \infty$ theory at $\beta_c = 0.66 \pm 0.02$. We also discuss the difficulties of measuring Wilson^C loops in the reduced model and give an estimate of the Λ parameter for the Wilson action at $N = \beta$.

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It has now become clear that the reduced⁽¹⁾, quenched⁽²⁾ $N = \infty$ theory is equivalent to the infinite volume gauge theory at $N = \infty$ for all values of the coupling^(3, 4). One must now determine whether anything has been gained by quenching. The initial hope would be that the reduction in the number of variables might lead both to new analytic insight and also ease in numerical computation.

To study the latter issue, we report in this paper on a high statistics simulation of the $d = 4$, $N = 8, 10, 12$ reduced quenched model.

The normalization we choose for the coupling β is defined by stating the action which is given by,

$$S = \frac{\beta N}{2} \sum_{\mu, \nu=1}^4 (\text{tr } U_{\mu} U_{\nu} U_{\mu}^{\dagger} U_{\nu}^{\dagger}) \quad (1)$$

where space-time consists of one site plus 4 links and periodic boundary conditions and U_{μ} is the link variable, an element of $SU(N)$, in the μ direction. This defines the reduced model.

Quenching consists in writing,

$$U_{\mu} = V_{\mu} D_{\mu} V_{\mu}^{\dagger} \quad (2)$$

where D_{μ} is the diagonal matrix of eigenvalues of U_{μ} and V_{μ} is the unitary matrix consisting of columns of the eigenvectors of U_{μ} . The quenched free energy is defined by restricting the integration over U_{μ} to one over V_{μ} alone and then averaging over the D_{μ} 's. Thus

$$F_Q = \int \prod_{\mu} dV_{\mu} \ln \left\{ \int \prod_{\mu} dD_{\mu} \left(\prod_{\mu} V_{\mu}^{\dagger} D_{\mu} V_{\mu} \right) e^{S(V_{\mu} D_{\mu})} \right\}. \quad (3)$$

In a Monte Carlo simulation, quenching is easy to enforce. One starts with a set of N diagonal matrices with elements of the form $e^{i\theta_\mu^i}$, $i = 1, 2, \dots, N$ where the θ 's are chosen uniformly in the interval $[0, 2\pi]$. These are the D_μ matrices and form the initial set of U_μ 's for the simulation. Now one goes through the standard Metropolis algorithm. This consists of generating a sequence of new matrices U'_μ as trial matrices to replace the old ones U_μ , comparing at each step in the sequence the new action with the old and using the Metropolis algorithm to decide whether or not to accept the change. These new matrices must have the same eigenvalues as the old ones to enforce the quenching prescription. This is guaranteed by generating the new U'_μ 's by conjugation. Thus

$$U'_\mu = V U_\mu V^\dagger \quad (4)$$

where V is an $SU(N)$ matrix.

We found that it is easy to get the optimum acceptance rate of 50% if V is chosen such that it equals the identity except for a randomly chosen $SU(2)$ embedding. Thus V has the form

$$V = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & a & & b \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \\ & & & & & & & -b^* & & \\ & & & & & & & & a^* & \\ & & & & & & & & & 1 \\ & & & & & & & & & & 1 \\ & & & & & & & & & & \ddots \end{bmatrix}, \quad |a|^2 + |b|^2 = 1. \quad (5)$$

The order parameter we measure is the average plaquette E defined by,

$$E = \frac{1}{2Nd(d-1)} \left\langle \sum_{\mu > \nu} \text{tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \text{h.c.}) \right\rangle_Q \quad . \quad (6)$$

The correct quenching procedure⁽²⁾ would be to measure order parameters for a given distribution of eigenvalues D_μ and then average over many such distributions. For numerical simulations, this is very time consuming. Instead, we choose the most symmetric distribution of the eigenvalues and measure only in this single configuration of eigenvalues. It has been argued that this procedure is correct at $N = \infty$ ⁽³⁾. For finite N , it introduces an error of $O(1/N)$ which in principle could be large. We find that our procedure is adequate to measure E but may not work for loops of larger sizes. For large loops it may be necessary, in simulations at finite N , to average over various eigenvalue distributions. Alternatively stated, the values of N that might be necessary to measure large loops may be so large that the simulation is impossible. One argument that suggests that this might indeed be the case is the following:

One knows from the usual Monte Carlo work on $SU(N)$ for $N \sim 2$ to 6 that to measure $L \times L$ loops one needs lattices of $(2L)^d$ or more sites. In the quenched model, one is working not on a spatial lattice but rather on a lattice in momentum space (recall that in the weak coupling limit, the θ_μ^i 's are like momenta in the planar perturbation theory⁽³⁾). The number of points in the quenched

model in momentum space for a given N is dN in d dimensions. Thus, to measure $L \times L$ loops, one would expect that one needs to have

$$dN \sim (2L)^d. \quad (7)$$

For $L = 2$, $d = 4$, this gives $N \sim 64$. In our simulation, the iteration time for $N = 10, 12$ was approximately 1.6 seconds and 2.4 seconds respectively on a CDC 7600. The simulation time scales like N^2 in the quenched model. For $N = 64$, an iteration would take approximately 65 seconds. To measure 3×3 loops, the N value which is necessary is about 324 and one iteration would take 1680 seconds.

It is possible that these estimates are unduly pessimistic and the real situation is much more benign. We point out these issues just to show that even if finite N simulations do not give the loop expectation values compatible with what one expects at $N = \infty$, there is a simple explanation for why this happens. These issues are currently under investigation.

The distribution we choose for the D_μ^i 's is defined by picking the θ_μ^i 's ($D_\mu^i = \exp i\theta_\mu^i$) equally spaced in $[0, 2\pi]$ for a given μ . When all the θ_μ^i 's for $\mu = 1, 2, 3, 4$ are put on the same interval $[0, 2\pi]$, they are also equally spaced. This is clearly the most symmetric distribution for our momentum space lattice and corresponds in some sense to the hyper-cubic lattices with direction independent lattice spacing used in the usual simulations.

In the figure, we show our data for E vs β . For $SU(10)$,

the points in the interval $\beta = 0.5$ to $\beta = 0.7$ are averages over every 5th of 3000 iterations after discarding the first 100 interactions. The data for all the other points is from a sample of 1,000 iterations, averaging over every fifth iteration. The lattice was initialized in a "mixed" configuration; with 2 link elements equal to the diagonal D_μ 's and 2 links equal to D_μ 's conjugated with a random unitary V matrix (see Eqn.4). The curves in the figure are separate polynomial fits to power series in β in the interval $[0, 0.4]$ and $[0.56, 1.2]$. The intersection of these curves for a given value of N gives the position of the point of crossover from strong to weak coupling. This crossover is expected to become a first order transition point at $N = \infty$. In our finite N simulation, the first order transition is seen as a kink in the data. The values we get for the critical β for $N = 8, 10$ and 12 are $0.46, 0.50$ and 0.53 . When these are combined with Okawa's⁽⁴⁾ value of 0.58 as the "critical" point for $N = 20$ and extrapolated to $N = \infty$ using a linear fit in $1/N$, one gets the critical value of β at $N = \infty$. In this way, we find,

$$\beta_c^{N=\infty} = 0.66 \pm 0.02 \quad . \quad (8)$$

This is in very good agreement with the extrapolated critical point for $N = \infty$ obtained by simulations of $SU(4)$, $SU(5)$ and $SU(6)$ ⁽⁵⁾ gauge theories.

One can also estimate the ratio σ/Λ^2 at $N = \infty$ merely from a knowledge of $\beta_c^{N=\infty}$. This is done as follows:

In weak coupling, the dimensionless string tension σa^2 (a is the lattice spacing) is given by

$$\sigma a^2 = \frac{\sigma}{\Lambda^2} \left(\frac{24\pi^2\beta}{11} \right)^{102/121} \exp\left(- \frac{24\pi^2}{11} \beta \right) \quad (9)$$

where β is defined in Eqn.1 and is related to the bare charge $g_0^2 N$ according to

$$g_0^2 N = \frac{2}{\beta} \quad (10)$$

In simulations of the SU(2), SU(3) gauge theory, the value of σa^2 at the crossover is around (0.5 - 1.0). For β larger than the crossover value, the scaling behaviour of Eqn. 9 is obtained. An estimate of $\Lambda/\sqrt{\sigma}$ for $N = \infty$ is then obtained merely by substituting the value $\beta_c^{N=\infty} = 0.66$ we have determined into Eqn.9 and estimating σa^2 to be between 0.5 and 1.0 at this value of β . This gives the result

$$\Lambda_{N=\infty}^{\text{Wilson}} = (0.002 \pm .001)\sqrt{\sigma} \quad (11)$$

It has been pointed out that higher loop effects in perturbation theory affect the Λ estimates of Monte Carlo simulations for SU(2) gauge theory quite dramatically with various actions⁽⁶⁾. But it has also been shown as a numerical fact that the Wilson action, which is used in the present paper, is somewhat favoured in the sense that these higher order effects are the smallest for it⁽⁷⁾. If this fact persists up to $N = \infty$, our estimate of $\Lambda^{N=\infty}/\sqrt{\sigma}$ should be quite reliable.

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FIGURE CAPTION

Monte Carlo data for $1-E$ vs β for $N = 8, 10$ and 12 .
The curves are least square polynomial fits in β in weak
and strong coupling.