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NON-ABELIAN PLANE-WAVES IN THE HIGGS MODEL

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ABSTRACT

We establish the equations of motion for the Higgs model by using an ansatz due to Teh and Oh for the gauge potentials. We present two types of plane-waves solutions; these are non-Abelian, real and analytic over the Minkowski space-time. Their velocity of energy transport is shown to be less than the speed of light in the vacuum.

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Very recently, Teh and Oh obtained interesting plane-wave solutions¹⁾ for a pure Yang-Mills theory in Minkowskian space-time. Their solutions are non-Abelian (in the sense that the commutator $[A_\mu, A_\nu]$ is not zero in general) and are formulated in terms of the Jacobi elliptic functions which are periodic ones.

In this note, considering an $SU(2)$ Yang-Mills theory minimally coupled to a triplet of Higgs fields Φ and performing an ansatz inspired from Ref. 1), we also obtain non-Abelian plane-waves solutions. Our solutions appear in two classes formulated respectively in term of the Jacobi elliptic functions and of trigonometric functions; in this respect they are periodic; they occur only for particular values of the Higgs potential. Some properties of these solutions are studied, namely their velocity of energy transport is shown to be smaller than the velocity of light in the vacuum.

Let us consider the so-called Higgs model for a triplet of scalar fields ϕ_a minimally coupled to the $SU(2)$ Yang-Mills potentials A_μ [we use the matrix notation $\Phi = \lambda^a \phi_a$ etc., and the metric $(+, -, -, -)$]

$$\mathcal{L} = -\frac{1}{4\gamma} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\gamma} \text{tr} D_\mu \Phi D^\mu \Phi - V(\Phi) \quad (1)$$

Where γ is the normalization of the $SU(2)$ generators λ^a and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu] \quad (2)$$

$$D_\mu \Phi = \partial_\mu \Phi + ig [A_\mu, \Phi] \quad (3)$$

$$V(\Phi) = \lambda \left(\frac{\text{tr} \Phi^2}{\gamma} - \frac{a^2}{\lambda} \right)^2 \quad (4)$$

To solve the equations of motion of this theory²⁾, we will use the ansatz below, largely inspired from the reference 1):

$$A_\mu = r_\mu \alpha(\mu_1, \mu_2) G + q_\mu \beta(\mu_1, \mu_2) H \quad (5a)$$

$$\Phi = \varphi(\mu_1, \mu_2) F \quad (5b)$$

In (5), F, G, H are constant generators of $SU(2)$ while $\mu_A = s_A^u x_\mu$; the four vectors p^u, q^u, s_1^u, s_2^u being orthogonal to each other.

Without losing generality, we will suppose in the following that the generators F, G, H are normalized to $\sqrt{1/2}$ (i.e., $\text{tr} G^2 = 1/2$, etc.) and that the four-vectors are normalized to 1 or -1 or 0 in space time. Then, when the matrices F, G, H form a basis of the algebra of the gauge group, the equations of motions of (1) reduce to the system below where we have put $\epsilon_1 = -p^u p_u$ and $\epsilon_2 = -q^u q_u$

$$\square \alpha + g^2 \alpha (\epsilon_1 \beta^2 + \varphi^2) = 0 \quad (6a)$$

$$\square \beta + g^2 \beta (\epsilon_1 \alpha^2 + \varphi^2) = 0 \quad (6b)$$

$$\square \varphi + g^2 \varphi (\epsilon_1 \alpha^2 + \epsilon_2 \beta^2 + \frac{\lambda}{g^2} \varphi^2 - \frac{a^2}{g^2}) = 0 \quad (6c)$$

In the following, we propose two types of solutions obtained for

$$\delta_1^\mu = \delta_2^\mu = \delta^\mu, \quad \delta^\mu \delta_\mu = 1, \quad \mu = \delta^\mu x_\mu, \quad \epsilon_1 = \epsilon_2 = 1 \quad (7)$$

i) let $\lambda = 2g^2$; then a solution of (6) is

$$\alpha = \pm \beta = \frac{a}{g} \cos(a\mu) \quad (8)$$

$$\varphi = \frac{a}{g} \sin(a\mu) \quad (9)$$

This solution is manifestly periodic and non-singular; it depends crucially on the presence of the Higgs field and potential. Note that the Higgs field self-coupling constant is fixed in terms of the gauge coupling constant ($\lambda = 2g^2$); the parameter a being related to the common period of α, β , and φ .

ii) let $a = 0$ and try a solution like $\alpha/m = \beta/n = \varphi$; then, as long as $g^2 > \lambda$, a real solution exists provided that

$$m^2 = n^2 = 1 - \frac{\lambda}{g^2} \quad (10)$$

In this case, the system (6) reduces to the equation $\square \varphi + (2g^2 - \lambda)\varphi^3 = 0$. Up to a trivial scale factor, it is the basic equation obtained in Ref. 1) where it has been largely studied. We will retain only two bounded solutions formulated in terms of Jacobi elliptic functions:

$$\varphi_1(\mu) = \frac{\gamma}{\sqrt{2g^2 - \lambda}} \operatorname{cn}(\gamma\mu, \frac{1}{2}) \quad (11)$$

$$\varphi_2(\mu) = \frac{\gamma}{\sqrt{2(2g^2 - \lambda)}} \operatorname{sd}(\gamma\mu, \frac{1}{2}) \quad (12)$$

In (11) and (12), γ is an arbitrary parameter determining the period and amplitude of the solutions.

So far, we have computed the field strength $F_{\mu\nu}$ the energy and momentum densities Θ_{00} , Θ_{0i} [see Ref. 2] associated with the solutions (i) and (ii). For the solution (i), it appears that

$$F_{\mu\nu} = -\frac{a^2}{g} \left\{ \sin(a\mu) (p_\mu s_\nu - p_\nu s_\mu) G + \sin(a\mu) (q_\mu s_\nu - q_\nu s_\mu) H + \cos^2(a\mu) (p_\mu q_\nu - p_\nu q_\mu) F \right\} \quad (13)$$

The field strength (13) sustains the whole algebra of $SU(2)$. Taking $p_\mu = \delta_{2\mu}$, $q_\mu = \delta_{3\mu}$, and $s_\mu = (\cosh \theta, 0, 0, \sinh \theta)$, we have obtained

$$\theta^{00} = \frac{a^4}{g^2} \cdot (2 \cosh^2 \theta \cdot (1 + \sin^2(a\mu)) - \frac{3}{4}) \quad (14)$$

$$\theta^{0i} = \delta_{3i} \cdot \frac{a^4}{g^2} \cdot \cosh \theta \cdot \sinh \theta \cdot (1 + \cos^2(a\mu)) \quad (15)$$

From these formulas, it is apparent that the energy density is positive definite bounded over space-time and attains its extrema on spacelike hyperplanes

$$\mu = x_0 \cosh \theta + x_3 \sinh \theta = \frac{k\pi}{2a}, \quad k = 0, \pm 1, \pm 2, \dots \quad (16)$$

the direction of propagation is constant and the phase velocity ($= \coth \theta$) is bigger than one for any values of θ . However, we have also studied the velocity of energy transport which is a more relevant physical quantity. As stressed by Teh and Oh, it is defined by the relation

$$v^i = \frac{\theta^{0i}}{\theta^{00}}. \quad (17)$$

Using (14) and (15), one can see that $|v_i|$, the length of v_i , is always less than one, the velocity of light. Indeed, $|v_i|$ is close to zero for infinitesimal values of θ and, increasing monotonically, becomes asymptotic to one for large values of θ .

The solutions (ii) can be shown to have very similar properties. We do not write here their energy and momentum densities because of their complicated form; however, we have verified that the velocity of energy transport is also smaller than one.

Consequently the solutions (i) and (ii) can be considered as infinitely long plane-waves propagating through a medium with phase velocity greater than one. These plane-waves occur only for restricted values of the coupling constants ratio λ/g^2 :

$$\begin{aligned} \lambda/g^2 &= 2 \quad \text{for solutions (i)} \\ 0 \leq \lambda/g^2 &\leq 1 \quad \text{for solutions (ii)} \end{aligned}$$

We have looked without success for other non-singular and bounded solutions to the equations (6); particularly for solutions with $\alpha \neq \beta$ and $\alpha \neq -\beta$. It would be interesting to discover solutions for the other values of the ratio λ/g^2 , perhaps by adding spinor fields and (or) new scalar fields in the Lagrangian density (1). At least, it would also be nice to find any relation between the solution (i) and some supersymmetric theories which describe particles with different spins with a unique coupling constant.

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REFERENCES

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- 2) See e.g. in P. Goddard and D.I. Olive, Rep. Prog. Phys. 41 (1978) 1357.