A MODEL REGULARIZING GAUGE FIELDS ON THE STRONG COUPLING REGION

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A B S T R A C T

We propose a model using an effective Lagrangian for bilocal gauge extended fields, looking somehow as an average over all open strings with fixed end points. The locality of the action is retrieved as a range parameter, which acts as a regulator, shrinks, and the Yang-Mills theory is recovered in this limit. The system is invariant by rotation and translation, generalizes lattice gauge theories and allows regularized strong coupling expansions for Yang-Mills field theory.

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1. Introduction

Lattice gauge theory [1] has been introduced as a regularization for the Yang-Mills field theory which is particularly adapted to the strong coupling limit. This technique is promising and has already brought a lot of interesting results: evidence of the existence of different regimes as the coupling constants vary and, in particular, a confined phase; possibility of expansions in the confined region; numerical calculations using Monte-Carlo techniques; ...

However the problem of removing the regularization remains in a quite unsatisfactory state. It was first thought that a second order transition point exists somewhere; physical correlation lengths become infinite with respect to the lattice spacing at such a point and the continuum limit is described in the neighbourhood of this kind of transition. But this has not yet been observed. There is nevertheless a region where numerical results agree with the prediction of asymptotic freedom [2]. Using numerical measurements in this region with a continuation based on renormalization group equations would hence allow the approach of the continuum limit. However this numerical analysis is rather delicate. From the analytical point of view, the situation is certainly worse. Strong coupling series are difficult to analyze due to complex singularities or structures which are probably artifacts of the lattice; a supplementary problem comes from the unclear role of the roughening transition [3] and from the (related?) restoration of the rotational invariance [4].

It may be thought that most of these troubles are an artifact of the introduced lattice, and in particular a consequence of the lack of translation and rotation invariance. However, very few cures have been proposed up to now (as for instance the use of random lattices [5]). We propose in this paper a model which hopefully overcome these problems. It is built directly in the continuum space and uses a delocalization as regularization. We particularly focus on two points in its construction. The first one is to remove the regularization easily and to keep a control on the approach of the exact quantum gauge field theory. The second one is to generate rather easily strong coupling expansions at high orders for interesting physical quantities (as mass spectra). This paper is essentially a presentation of the model and we particularly focus on motivations and physical interpretations rather than on the rigour of the derivations. We should therefore apologize for a lot of weak points and conjectures (particularly about the treatment of double limits, often inverted without care) which will, of course, need deeper study in the future. Also we only consider here the pure gauge case and postpone the (although quite easy)

introduction of matter fields to further works.

As in lattice gauge theory, we never use in our model the field derivatives; they are replaced instead by difference of fields at different locations. Therefore we deal with non-local Lagrangians. This procedure induces the definition of gauge fields as bilocal objects depending on a couple of sites. We work here in a continuous Euclidean space and introduce gauge fields $\zeta(x,y)$. Physically, they can be imagined as an open string starting from x and ending at y , regardless of the path joining the two points x and y . Due to this fact, $\zeta(x,y)$ is considered as an average (over all possible paths) of gauge group elements, and is therefore no longer an element of the group, but belongs now to the matrix algebra constructed from the fundamental representation of the gauge group. This extension is one of the main keys of the model and will allow the regularization to work properly (a similar trick is used in the non-linear σ -model while relating it with its linearized version). Of course, an infinite number of supplementary degrees of freedom for the fields $\zeta(x,y)$ with respect to the traditional local gauge field $A_\mu(x)$ have appeared. They play here the role of auxiliary regularizing fields for the standard Yang-Mills theory and are expected to disappear in the local limit, for which $\zeta(x,y)$ reduces to an extended object constructed from only the usual fundamental field A_μ . Our model is an effective Lagrangian (rotationally and translationally invariant, gauge invariant, but non-local) for these extended fields ζ . The action involves different space points within a finite range b , and the local limit $b \rightarrow 0$ is hoped to reduce to the usual local Yang-Mills theory.

This model can also be viewed as an intermediate stage in a real space renormalization group procedure going from the usual lattice gauge theory to the continuum limit. The decoupling procedure is supposed to be performed in two successive steps, first restoring the rotation invariance at medium distance and then analyzing the long distance behaviour. In the first step, one considers the set of generalized actions with all possible closed loops with fixed upper bound for their length. As the lattice spacing shrinks, such loops will involve more and more links and such models become rapidly impossible. Nevertheless, a simplification occurs with the following supplementary restriction. The action must be re-expressed in terms of a few numbers of composite bilocal fields $\zeta^{(i,j)}$, namely the (suitably weighted) sum of products of lattice gauge fields along all paths joining i to j with a maximum given length b . Our model may be viewed as the limit of this hypothetical process when the original lattice spacing goes to zero at fixed range b , in the rotationally invariant phase. This is not yet the continuum limit, since the range of the interaction has been kept fixed; the second step will be the analysis of large (with respect to

b) distance behaviour, or in other words the local limit $b \rightarrow 0$. We are not interested here in the detailed study of such a decimalization procedure, which is reported only as a pictorial and intuitive presentation of our goals.

It is also possible to introduce the model as a generalized mean field analysis for lattice gauge theory. We first introduced generalized fields $U^{(i,j)} = \sum U_{ik} U_{kl} \dots U_{nj}$, summing over all paths from i to j , with at most K steps; they are added as new independent fields on which one integrates. Their effect is nullified by adding suitable δ functions in the measure. Now mean fields $\zeta^{(i,j)}$ are associated to each one of these $U^{(i,j)}$ fields, using the integral method described in reference [6]. The result will be a smooth effective action for the mean fields $\zeta^{(i,j)}$ (because integral transform on the suitably introduced δ functions will produce a smooth functional integrand for $\zeta^{(i,j)}$). Then the mean field solution is searched for in the region where rotation invariance is retrieved as a continuous function $\zeta(x,y)$; the limit of zero spacing (at fixed interaction range $b=Ka$) is then taken without any problem. This procedure is unfortunately almost impossible to perform explicitly; our model may be viewed as an analysis of the possible answers of this process.

In Section 2, the Lagrangian of the model is discussed and constructed in detail. Section 3 deals with the classical limit and the relation with a perturbative regularization of the usual Yang-Mills field theory. Section 4 outlines the possibility of strong coupling expansions for this model.

2. Formulation of the model.

2.1. Fields

The fields $\zeta(x,y)$ depend on two points x and y of the d dimensional Euclidean space and take their values in the real algebra constructed from the fundamental matrix representation of the chosen gauge group (i.e. the set containing all matrices of the fundamental representation and closed both by addition and multiplication). They are to be considered intuitively as an average over the path-ordered exponential integral of the usual Yang-Mills gauge field

$$\zeta(x,y) \approx \int_{\text{all paths}} dC_{xy} \text{Pexp} \int_C A \cdot ds \quad (2.1)$$

This formula is only a heuristic support for physical intuition and must not be considered as a rigorous introduction of our fields. In particular,

the exact meaning of the measure over the paths xy is not precised. The further discussion on the classical limit will enlighten its exact signification.

The choice of the gauge group does not matter at this stage. However, to be more explicit and to write explicit formulae, we restrict ourselves to $U(N)$ and $SU(N)$. In these cases, ζ may be any $N \times N$ complex matrix of $GL(N, \mathbb{C})$, except for $SU(2)$ where some peculiarities described in appendix A occur. Following the image of eq.(2.1), we impose the constraint

$$\zeta(x, y) = \zeta^\dagger(y, x) \quad (2.2)$$

and the transformation law under a local gauge transformation

$$\zeta(x, y) \rightarrow G(x) \zeta(x, y) G^{-1}(y) , \text{ with } G(x) \in U(N) \text{ or } SU(N) . \quad (2.3)$$

Any $N \times N$ matrix can be decomposed in a unique way (up to inessential discrete transformations; some exceptional cases give a degeneracy, however) as the product of a unitary unimodular matrix $U \in SU(N)$, a hermitian unimodular matrix H and a complex number. Thus we write

$$\zeta(x, y) = \phi(x, y) e^{i\theta(x, y)} U(x, y) H(x, y), \quad (2.4)$$

with ϕ and θ real. All the gauge degrees of freedom are borne by $e^{i\theta}U$ (or just by U for the $SU(N)$ gauge group), and this part moreover transforms as indicated in (2.3) under a gauge transformation. Note also that $GL(N, \mathbb{C})$ (resp. $SL(N, \mathbb{C})$) is the complex extension of $U(N)$ (resp. $SU(N)$). Therefore the part ϕH belongs to the coset $GL(N, \mathbb{C})/U(N)$ of the group $GL(N, \mathbb{C})$ with respect to its subgroup $U(N)$; hence a natural parametrization for ϕH is to use the Weyl unitary trick, i.e. to change the real parameters defining the unitary matrices to pure imaginary parameters. This will be done in the analysis of the classical limit. Note also that $e^{i\theta}U$ reduces to 1 due to the constraint (2.2) whenever $x \equiv y$.

Before writing the action, it is necessary to precise the exact measure over the fields $\zeta(x, y)$. This measure $d\mu(\zeta)$ must be invariant under the local transformations (2.3). This condition is, however, insufficient to determine completely the measure on the whole algebra. In any case, a change of this measure can be absorbed in a redefinition of the action through the addition of a gauge invariant term and is, at this stage, unimportant. It may seem for instance natural to use the invariant measure over $GL(N, \mathbb{C})$, which is in particular invariant under the $U(N)$ gauge group. We rather use independant flat integrations over the complex matrix elements

of the fields, which differ by the invariant factor $|\det \zeta|^N$ and which will allow explicit calculations. We use all through this paper a functional integral formalism; the functional integration needs some regularization scheme to be completely defined. We proceed as follows.

- i) The space is divided into cells of equal volume a^d . This seems to reintroduce a lattice; however, our results will be independent (at least at the leading order in a when $a \rightarrow 0$) of the chosen division into cells.
- ii) To each pair (i,j) of cells is associated a $N \times N$ complex matrix $\zeta^{(i,j)} \equiv \zeta^{(j,i)\dagger}$; the integrand is computed with fields so that $\zeta(x,y) = \zeta^{(i,j)}$ if x is within cell i and y within cell j . As our fields are not derived, this step definition does not lead to any problem.
- iii) Integration is then performed using the chosen measure $\prod d\mu(\zeta^{(i,j)})$ over all pairs (i,j) of cells.
- iv) The limit $a \rightarrow 0$ is taken. All through this paper, this limit will be taken as soon as possible and we shall immediately write continuous integrals rather than sums over cells, which lead to the same result up to $O(a)$ corrections.

The generating functional will be written for instance as

$$Z(J) = \lim_{a \rightarrow 0} \int e^{-S + \text{source terms}} \prod_{(i,j)} \prod_{\alpha\beta} d\zeta_{\alpha\beta}^{(i,j)} d\zeta_{\alpha\beta}^{(i,j)\dagger} \quad (2.5)$$

The source term uses a complex matricial field $J(x,y) = J^\dagger(y,x)$ and reads

$$\iint (\text{Tr} [\zeta(x,y) J^T(x,y)] + \text{h.c.}) dx dy \quad (2.6)$$

We have now to choose a form for the action S .

2.2. Action.

With only the constraint to be gauge invariant and to have no field derivatives, there is of course a large arbitrariness for the bare actions concerning the fields $\zeta(x,y)$. It is in particular possible to write such actions which lead to the usual lattice gauge theory (as will be shown later). However, we are not interested by this case here; we impose from now on the translation and rotation invariances; moreover we use a spatial cut-off b restraining all the space points involved in an interaction term to be separated at most by the distance b , or more generally, the interaction is constrained to have a fast decrease (at least exponential) beyond this distance. The local Yang-Mills theory must be reached in the limit

$b \rightarrow 0$; this limit will be reached hopefully with only the simplest and the more practical (from a computational point of view) terms if universality arguments can be applied.

We classify terms following the number of space points used. In particular, the "free part" S_0 involving only two space points x and y depends on only one field $\zeta(x,y)$ and represents a theory of decoupled fields. Conversely, we will call "interaction terms" terms involving more space points.

The more general term constructed from one field $\zeta(x,y)$ is

$$S_0 = \iint f(\zeta(x,y); x, y) dx dy \quad (2.7)$$

As one can find a local gauge transformation (2.3) which suppresses U , diagonalizes H in (2.4) and, for $U(N)$ gauge groups, suppresses θ , f must be a symmetric function of the (real) eigenvalues of ϕH ; it depends on θ only in the $SU(N)$ case. Finally, the rotation and translation invariances demand that f depends on x and y only by the distance $r = |x-y|$.

We have noted that the gauge degrees of freedom (contained in U and θ) disappear in (2.7) term. However we insist on the fact that this term is essential, first to suppress the auxiliary degrees of freedom contained in H and ϕ in the local limit, secondly to provide a basis for strong coupling expansions.

With only this term, the theory is exactly soluble; introducing the function

$$u(J; r) = \frac{1}{2a^{2d}} \ln \int e^{-2a^{2d}[f(\zeta; r) - 2 \operatorname{ReTr}(\zeta J^T)]} d\mu(\zeta) - (\text{value at } J = 0) \quad (2.8)$$

the generating functional $Z_0(J)$ is given by

$$\ln Z_0(J) = \text{Cst} + \iint u(J(x,y); |x-y|) dx dy \quad (2.9)$$

The role of the cut-off a is apparent in (2.8), although the dependence of u with respect to this regularizator has not been explicitly indicated. Equation (2.9) is only written at the lowest order in a (sum over cells replaced by integral). Of course, this theory of non-interacting fields is trivial.

The invariant functions f and u can be expanded in terms of Casimir invariants; the lowest degree in the matrix elements uses only

$\text{Tr}[\zeta(x,y)\zeta(y,x)]$ and hence

$$S_0 = \iint K_2(|x-y|) \text{Tr}[\zeta(x,y)\zeta(y,x)] dx dy + \dots, \quad (2.10)$$

$$u(J;r) = F_2(r) \text{Tr}(JJ^\dagger) + \dots \quad (2.11)$$

As a generalization, f can also be chosen as a distribution, for instance forcing ζ to belong to either $U(N)$ or $SU(N)$. This choice is quite comparable to the non-linear σ -model, where the constraint is entered as a δ function, while linearization is done by taking just quadratic and quartic terms, i.e. a smooth function f . Of course, the expansion (2.10) is no longer valid, but in any case u remains smooth as a function of J and (2.11) is always valid.

If one restricts to the first term in the r.h.s. of (2.10), S_0 is quadratic in the field matrix elements and the integral (2.8) can be exactly performed, yielding again a quadratic form for u with

$$F_2(r) = K_2^{-1}(r) \quad (2.12)$$

However, further terms, as e.g.

$$\iint K_4(|x-y|) \text{Tr}^2(\zeta(x,y)\zeta(y,x)) dx dy,$$

will have to be introduced beyond this approximation; the interaction term proposed below is indeed cubic, and the absence of such terms would lead to unbounded actions and associated problems (as in the ϕ^3 field theory).

For $SU(N)$ rather than $U(N)$ gauge theory, other types of terms must be added in order to break $U(N)$ to $SU(N)$. They involve determinants such as

$$\iint K_D(|x-y|) \det \zeta(x,y) dx dy \quad (2.13)$$

The function f will play a role in the classical limit studied in the next section, although the function u is used in the strong coupling expansions. These functions will be more or less restricted in this paper to their first approximations in (2.10) and (2.11), which are hopefully sufficient to show the main features of our model.

We turn now to the interaction terms $S_{\text{int}} = S - S_0$. The simplest and lowest degree invariant one is the plaquette term $\text{Tr}[\zeta(x,y)\zeta(y,z)\zeta(z,x)]$, which now depends on the gauge degrees of freedom contained in the part U of ζ . We associate it to a form factor $K_3(x,y,z)$ with a range of order b , invariant under space translations and rotations (i.e. a symmetric function

of the triangle side lengths $|x-y|$, $|y-z|$ and $|z-x|$)

$$S_{\text{int}} = \iiint K_3(x,y,z) \text{Tr}[\zeta(x,y)\zeta(y,z)\zeta(z,x)] dx dy dz \quad (2.14)$$

We do not discuss further terms, such as a four point term which will be nevertheless very useful in further developments of the model.

We assume that, in any case, the minimal number of terms in the action will be sufficient, from universality arguments, to generate a non-trivial system with all the desired features. We conclude this subsection by summarizing these necessary terms which are: (2.14) as it involves the gauge degrees of freedom; and (2.10) up to quartic terms in ζ (with K_2 , K_4 and K_D kernels) for the free part.

2.3. Choice of kernels

The bare form of the kernels is thought to be unessential in the local limit. Note, however, that the kernels can be redefined using the transformation

$$\zeta(x,y) \rightarrow \lambda(|x-y|)\zeta(x,y) , J(x,y) \rightarrow \lambda^{-1}(|x-y|)J(x,y) \quad (2.15a)$$

where λ is any real function. This induces a redefinition of the kernels

$$\begin{aligned} K_3(x,y,z) &\rightarrow K_3(x,y,z) \lambda(|x-y|) \lambda(|y-z|) \lambda(|z-x|), \\ f(\zeta(x,y);x,y) &\rightarrow f(\lambda(x,y)\zeta(x,y);x,y) \\ F_2(x,y) &\rightarrow F_2(x,y) \lambda^{-2}(|x-y|) , \dots \end{aligned} \quad (2.15b)$$

so that the theory obtained with the transformed kernels will be identical to the original one. This transformation can be interpreted as a change on the measure used in (2.1) for the paths xy . Note, however, that the physical gauge degrees of freedom are not affected by this transformation.

The final choice for the kernels is governed only by considerations such as easy computations of expansions at a large order, or rapid convergence for these series. In this paper, when a choice is necessary, we use for the following combination invariant through the (2.15) transformation

$$\begin{aligned} K_3(x,y,z) &\sqrt{F_2(|x-y|)F_2(|y-z|)F_2(|z-x|)} \\ &= \beta \exp\{ - [(x-y)^2 + (y-z)^2 + (z-x)^2]/b^2 \} \end{aligned} \quad (2.16)$$

which leads to an easy evaluation of strong coupling diagrams.

Let us also indicate a particular non-rotational invariant choice which

leads to the standard lattice theory, with spacing b . The kernels must be chosen as distributions rather than functions. We recall first that any lattice action can be reduced to an action involving only triangular plaquettes by dividing arbitrarily the plaquettes into triangles by addition of auxiliary links. Then

$$K_3(x,y,z) = \sum \delta(x-i)\delta(y-j)\delta(z-k) \quad (2.17)$$

where the r.h.s. sum runs over all triangular plaquettes (ijk) , leads to the lattice action; $\exp(f(\zeta))$ must be chosen as the δ -distribution for $SU(N)$ considered as a submanifold of $GL(N,C)$ in order to recover the usual lattice gauge fields.

3. Classical limit and quantum fluctuations.

3.1. Retrieving $SU(N)$ Yang-Mills theory.

As we already pointed out, we are interested by the double limit, first $a \rightarrow 0$, and then $b \rightarrow 0$. In order to obtain non-trivial results in these limits, one may be obliged to modify the magnitude and eventually the shape of the kernels initially chosen for the bare action. Due to these "renormalization effects", we must consider the effective action as unknown at this stage; it is therefore not possible to extract explicit results from this section. Nevertheless, some results are available and in particular we are able to identify the local field theory governing the quantum fluctuations around the classical solution.

Varying the field $\zeta(x,y)$ leads immediately to the classical field equation

$$\frac{\partial f(\zeta_{cl}(x,y);x,y)}{\partial \zeta_{cl}(x,y)} + 3 \int K_3(x,y,z) \zeta_{cl}(x,z) \zeta_{cl}(z,y) dz = 0 \quad (3.1)$$

If f is quadratic in ζ , this equation is coherent with the image (2.1) of a string xy averaged over all possible paths. Indeed this formula states that a path from x to y can be divided into two paths xz and zy , with all possible z . Furthermore, this formula can be considered as the rigorous version of (2.1), and as a rigorous starting point for defining correctly the "measure over all paths xy ". A more general f involves non-linear terms in $\partial f / \partial \zeta$ and adds to the image (2.1) the possibility for the string

of splittings and bifurcations.

Here we analyze only the implications of a solution of the form

$$\zeta_{c1}(x,y) = \phi_{c1}(x,y) G(x).G^{-1}(y) \quad (3.2)$$

where G is any local gauge transformation and $\phi_{c1}(x,y) \neq 0$ is a real scalar function. Other possibilities will be briefly discussed in the next subsection. Due to the transformation (2.15), the exact form of the solution ϕ_{c1} cannot be directly physically interpreted. We suppose that ϕ_{c1} remains a smooth function (rather than a distribution) while the effective range of the action terms (f and K_3) shrinks with b (or that a transformation of type (2.15) leads to an equivalent action verifying this assumption). Explicit examples of such a behaviour can be easily constructed; for instance, suitable gaussian kernels K_2 , K_D , K_4 and K_3 lead to a gaussian shape for ϕ_{c1} with a fixed finite range.

Using the decomposition (2.4), the fluctuations around the classical solution can be parametrized as

$$\begin{aligned} \zeta(x,y) = & \phi_{c1}(x,y) (1 + \Phi(x,y)) e^{ig_0 \int \Theta(x,y)} \\ & \times e^{ig_0 \int A_\alpha(x,y) t_\alpha/2} e^{\int B_\alpha(x,y) t^\alpha/2} \end{aligned} \quad (3.3)$$

where t^α are the generators of the $SU(N)$ Lie algebra, normalized according to $\text{Tr } t^\alpha t^\beta = 2 \delta^{\alpha\beta}$. The real fields Φ , Θ , A_α , B_α are then expanded with respect to $y-x$ for fixed $x+y$, in terms of (now local) fields located at the point $(x+y)/2$. Using the constraint (2.2), this leads to

$$\begin{aligned} \Phi(x,y) &= \varphi + \mathcal{O}((x-y)^2) \\ \Theta(x,y) &= (x-y)_\mu \mathfrak{J}^\mu + \mathcal{O}((x-y)^3) \\ A_\alpha(x,y) &= (x-y)_\mu A_\alpha^\mu + \mathcal{O}((x-y)^3) \\ B_\alpha(x,y) &= \mathfrak{B}_\alpha + \mathcal{O}((x-y)^2) \end{aligned} \quad (3.4)$$

The action part S_0 does not depend on A_α^μ (nor on $\mathfrak{J}^\mu$ in $U(N)$ case) and does not involve field derivatives, since all these local fields are evaluated at the same point. This is not the case for the interaction part S_{int} . One performs for this term a Taylor expansion around the centre of the triangle $(x+y+z)/3$; we do not reproduce the quite straightforward, although tedious calculations (very similar to those done in lattice gauge theory) and we quote only the obtained results.

The term containing A_α^μ comes just from the interaction part S_{int} and yields

$$-1/4 \int (F_{\mu\nu}^\alpha(x))^2 dx,$$

with $F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + i g_0 f_{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma$

(i.e. the well known Yang-Mills Lagrangian for the SU(N) gauge field A_μ^α) provided the coupling g_0 introduced in the parametrization (3.3) verifies

$$\frac{1}{g_0^2} = \frac{1}{d(d-1)} \iint K_3(x,y,0) \phi_{cl}(x,y) \phi_{cl}(y,0) \phi_{cl}(0,x) S^2 dx dy \quad (3.5)$$

where $S^2 = [x^2 y^2 - (x \cdot y)^2]/4$ is the squared area of the triangle $(x,y,0)$.

The region of the small interaction term (K_3 small) thus corresponds from this formula to the strong coupling region, around a classical solution satisfying $\partial f / \partial \zeta = 0$.

For the fields φ and $\mathcal{B}_\alpha$, the situation is different. Due to the finite range b of the interaction, each term containing a derivative is of order b with respect to the same term without derivative. Hence the kinetic part is of order b^2 with respect to the mass term which comes from S_0 and does not vanish identically for these fields (as can be easily checked). The mass thus behaves as b^{-1} as b vanishes and hence the corresponding fields can be considered as auxiliary fields (as in the Pauli-Villars regularization); they insure a correct regularization, but do not propagate and disappear in the local limit $b \rightarrow 0$.

Finally, the field $\mathfrak{V}_\mu$ yields the U(1) Lagrangian of electrodynamics concerning the interaction part of the action, with an inverse squared coupling constant $1/g_0'$ again given by (3.5). If we deal with a U(N) gauge theory, the free part S_0 does not depend on $\mathfrak{V}_\mu$, which is the U(1) part of the U(N) Yang-Mills Lagrangian. (It is possible to get different coupling constants g_0 and g_0' for the SU(N) and U(1) parts in variant actions, for instance by adding a term as $\det(\zeta(x,y)\zeta(y,z)\zeta(z,x))$). If we only desire a SU(N) theory, S_0 does depend on $\mathfrak{V}_\mu$ (e.g. through the above-quoted possible term (2.13)); again a mass of order b^{-1} is generated and the (here unwanted) field disappears in the local limit $b \rightarrow 0$.

3.2. Other possible classical solutions.

Solutions different from the assumed form (3.2) can also be considered. The possible solution $\zeta_{c1}(x,y) \equiv 0$ cannot be studied as before since the parametrization (3.3) is singular at this zero point. One uses directly the complex matrix elements of ζ for parametrizing the fluctuations. The study is then quite similar and shows that there is a mass term, but no kinetic term for the quadratic part at the tree level. Kinetic terms only appear at the one loop level; they are again (for the same reason as before) of order b^2 with respect to the mass term and this is the indication for a trivial theory, with only non-propagating fields (of infinite mass) in the limit $b \rightarrow 0$.

The two regimes $\zeta_{c1} \equiv 0$ and $\zeta_{c1} \sim 1$ correspond to the two possible phases, confined and unconfined. This analysis is very parallel to the mean field method [6] in lattice gauge theories, with ζ_{c1} playing the role of the mean field. It is in particular possible to observe a transition between these two regimes. However a supplementary phenomenon may be obtained in our model. Let us consider the possibility in (3.2) of $\phi_{c1}(x,y) = 0$ for $|x-y|$ greater than some fixed distance R , and different from 0 for smaller distances (we have underlined in the preceding subsection a similar possibility on an explicit construction of $\phi_{c1}(x,y)$ with a gaussian shape). If such a classical solution occurs as a self-consistently determined static gauge field for a system of interacting matter fields, this looks very much like the bag model [7], with perturbative interactions within the radius R and confinement without propagation beyond this distance. We do not study this possibility in more depth.

To get more general solutions of (3.1), we use the decomposition (2.4) for the classical fields ζ_{c1} . As quoted before, the gauge independent degrees of freedom are just ϕ_{c1} , the set of eigenvalues of the unimodular matrix H_{c1} and (for $SU(N)$ only) θ_{c1} . The form (3.2) assumed that all eigenvalues of H_{c1} were equal, i.e. $H_{c1} \equiv 1$. It is quite easy to generalize to other cases, which correspond to a spontaneous breaking of $SU(N)$ into $SU(p_1) \otimes SU(p_2) \otimes \dots$ (where the eigenvalues have been separated into different sets of $p_1, p_2, \dots$ equal values). It is however far beyond the scope of this paper to specify the conditions in which such a breaking will occur.

4. Strong coupling expansions

4.1. Regularized expansions.

This subsection is rather technical. We show that any physical quantity for the regularized system can be expanded as double series in two reduced variables and we display the diagrammatic rules for the computation of the coefficients.

The action has already been divided as a sum $S_0 + S_{\text{int}}$ of a free part and an interaction part. According to the results of the preceding section, the strong coupling expansion is obtained by expanding with respect to S_{int} . This is done formally on the expression

$$Z(J) = \exp(-S_{\text{int}}(\partial/\partial J)) Z_0(J). \quad (4.1)$$

However, it is necessary to use the regularization procedure described in subsection 2.1 for the functional integration over fields in order to get finite results. With this regularization, the interaction term reads

$$S_{\text{int}}(\zeta^{(i,j)}) = \sum_{(ijk)} \sum_{\alpha\beta\gamma} 3 a^{3d} K_3(i,j,k) \zeta^{(i,j)}_{\alpha\beta} \zeta^{(j,k)}_{\beta\gamma} \zeta^{(k,i)}_{\gamma\alpha} \quad (4.2)$$

and involves all oriented triangles (ijk) (i.e. all sets of three different cells i, j, k defined up to a circular permutation), while the unperturbed generating functional, defined in subsection 2.1 is

$$\begin{aligned} Z_0(J) &= \exp \left\{ \sum_{(i,j)} 2 a^{2d} u(J^{(i,j)}; |i-j|) \right\} \\ &= \exp \left\{ \sum_{(i,j)} \sum_{\alpha,\beta} 2 a^{2d} F_2(|i-j|) |J^{(i,j)}_{\alpha\beta}|^2 + \dots \right\} \end{aligned} \quad (4.3)$$

Now the usual diagrammatic techniques will work properly. Let us summarize the corresponding rules.

A diagram appears as a set of oriented triangles (each one contributing a factor $3 a^{3d} K_3$ from (4.2)) so that each side of a given triangle is shared by other triangles. This means that there are either zero or at least two triangle sides joining a given pair of cells. In particular, if a side is shared by exactly two triangles, these triangles must be coherently oriented. This follows immediately from the term indicated in (4.3) for the expansion of u which yields

$$\langle \zeta_{\alpha\beta}^{(i,j)} \zeta_{\gamma\epsilon}^{(k,l)} \rangle_0 = \frac{1}{2 a^{2d}} F_2(|i-j|) \delta_{i1} \delta_{kj} \delta_{\alpha\epsilon} \delta_{\gamma\beta} \quad (4.4)$$

We restrict ourselves to only the quadratic term for the function $u(J;r)$ and postpone to further study the possibility for a side to be shared by more than two triangles. With this assumption, the diagram has the topology of a two-dimensional orientable surface, a triangulation of which it realizes.

Concerning the gauge group indices, the inspection of Kronecker symbols in (4.2) and (4.4) shows that a unique index must be assigned to each vertex of the diagram. As the other parts of the various contributions do not depend on these group indices, the summation over them can be carried out and yields a factor N per vertex.

Finally, we have to divide the contribution by the order of symmetry of the diagram and to sum over all the positions of the vertices. We do not care about possible problems which may occur as a goes to zero (non-uniform limits) and replace this last sum by an integral. In view of preparing the limit $b \rightarrow 0$, a dilatation of ratio b^{-1} is simultaneously done, so that this replacement reads

$$\sum_i \rightarrow (b/a)^d \int d^d X$$

for a summation on a cell i containing the point bX . In the integrand of this integral, the coordinate dependent part is constructed by assigning to each triangle (bX, bY, bZ) the form factor

$$\beta G(X, Y, Z) = K_3(bX, bY, bZ) \sqrt{F_2(b|X-Y|) F_2(b|Y-Z|) F_2(b|Z-X|)} \quad (4.5)$$

which has a definite, non-trivial functional limit as b vanishes. For instance, the choice made in (2.16) leads to a function G which depends no more on b

$$G(X, Y, Z) = \exp \{ -(X-Y)^2 - (Y-Z)^2 - (Z-X)^2 \} \quad (4.6)$$

It is clear that, if a diagram has several disconnected pieces, its contribution will be the product of the contributions of its connected parts. Hence the general theorems on diagrammatic expansions apply. First, the generating functional is the exponentiated sum of contributions of all connected diagrams. Average values can be decomposed as usual in terms of

"connected" averages $\langle \rangle_c$ involving only connected diagrams, e.g.

$$\langle X Y \rangle = \langle X Y \rangle_c + \langle X \rangle_c \langle Y \rangle_c$$

We shall now recast the various rules for these connected diagrams in a very simple and compact form, using the topological relations

$$3 n_2 = 2 n_1 + p \quad (4.7)$$

$$n_2 - n_1 + n_0 = 2 - g - r \quad (\text{Euler relation})$$

which hold for a connected diagram with a boundary made of r closed loops along which there are p links (and therefore also p vertices), n_0 internal (i.e. not on boundaries) vertices, n_1 internal links, n_2 triangles, and which has the topology of a surface with genus g ($g \geq 0$ and even for orientable surfaces; $g/2$ is the number of "handles").

We evaluate now the gauge invariant quantity connecting r given closed loops

$$X_r = \langle \prod_{\lambda=1}^r \text{Tr} \{ \xi_{12}^{(\lambda)} \xi_{23}^{(\lambda)} \dots \xi_{p_\lambda 1}^{(\lambda)} \} \rangle_c \quad (4.8)$$

involving the suitably normalized and dilatized fields

$$\xi(X,Y) = \frac{4}{3} \frac{a^d}{\beta N} F_2^{-1/2}(b|X-Y|) \zeta(bX,bY) \quad (4.9)$$

The use of (4.7) allows us to recast all factors in a single formula as

$$X_r = w^{r-2} \sum_I v^{n_0} w^g, \quad (4.10)$$

where the sum runs over connected diagrams, $n_0=0, 1, 2, \dots$ and $g=0, 2, 4, \dots$, with

$$v = 9/8 \beta^2 N (b/a)^d, \quad (4.11)$$

$$w = 9/8 \beta^2, \quad (4.12)$$

$$I = \int \dots \int [\prod G(X,Y,Z)] [\prod d^d X], \quad (4.13)$$

that is double series in the reduced variables v and w , with numerical coefficients I depending on the external coordinates $X_i(\lambda)$ and on the chosen bare form G for the action, but in any case independent on the regularizers a , b and on the coupling strength β .

Some care must, however, be taken with the free energy which corresponds to the case $r = 0$. First, the order of symmetry S is not trivially one and must divide the contribution (4.13) of each diagram. Secondly, the integral is divergent and must be computed as a finite part I' while fixing the position of one vertex; the remaining integration on this fixed point yields a supplementary term $V b^{-d}$, V being the space volume (the free energy is indeed an extensive quantity) and b^{-d} corresponding to the dilatation made in the integration. Hence

$$\ln Z = V b^{-d} \sum 1/S I' v^{n_0} w^{g-2}, \quad (4.14)$$

where the sum runs over all connected diagrams ($n_0=3, 4, \dots$).

4.2. Removal of the regularization.

We give here some considerations about what may happen as the regularization is removed. This is, however, speculations just supported by some preliminary calculations (summarized in appendix B) and which must be taken with a pinch of salt at the present stage. It is in particular unclear whether the regularization can be removed to give a non-trivial system or not, as long as we restrict the function $u(J;r)$ to its first quadratic term in (4.3) and the interaction term to a triangular part (hence the action is not upper bounded). In any case, it is remarkable that this approximation leads to a result independent of the chosen gauge group.

As the regularization is removed, the three remaining variables b , v and w are expected to approach some limits 0 , v_c , w_c so that the physical quantities will have finite non-trivial limits.

On sight of (4.11), v goes to $v_c = \infty$ in the first step of the removal of regularizers ($a \rightarrow 0$ at fixed b). This statement can, however, be modified in practical calculations. It is indeed useful to perform a resummation of

sub-classes of diagrams; this is equivalent to considering G as a formal series in v and w (exactly as one adds counterterms in usual field theory) and this process induces a redefinition of v and w . We do not consider this possibility further in the present paper. While evaluating the series at $v = \infty$, some singularities may be encountered. The numerical analysis reported in Appendix B for the free energy shows that this is indeed the case. In the (v, w^2) plane, there is evidence of a swallow-tail shaped singularity, corresponding to a first order transition line ending on this singularity, and which is reminiscent of the often-encountered transition in lattice theories. This line can be turned out, but we will have to pay the price of a careful numerical analysis in order to obtain significant results in the limit $v \rightarrow \infty$.

After this first limit, each physical quantity is now a function of w . We note the nice interpretation of the corresponding series as a topological expansion, w being the conjugate variable of the genus (number of handles) of the diagrams.

We now turn toward the local limit $b \rightarrow 0$, which also governs the approach of w_c . In practical calculations, this will be achieved by the following process. From the two-loop correlation function X_2 in (4.8), one can extract a mass spectrum, and in particular e.g. the spin 0 lowest excitation mass. Due to the dilatation used in the normalization (4.9), this mass is measured in $1/b$ units and appears also as a double series in (4.10). After taking the limit $v \rightarrow \infty$, one finds

$$m = b^{-1} h(w) \quad (4.15)$$

This formula taken at fixed m will govern the approach of w to its limit.

A little more can be said if asymptotic freedom is fulfilled. We assume that renormalization group equations can be written for our (non-local) regularization of Yang-Mills theory; thus b is related to the bare coupling g_0 in four dimensions by

$$\frac{1}{g_0^2} = - \frac{11 N}{24 \pi^2} \ln(\Lambda b) + \mathcal{O}(\ln|\ln \Lambda b|) \quad (4.16)$$

and hence to w by (3.5) and (4.12)

If we restrict ourselves to our approximation (quadratic form for u , only triangles in S_{int} , gaussian shape for G), the classical equation (3.1) is solved and leads to

$$\phi_{cl}(x,y) = - \frac{2}{3\beta} \left(\frac{8}{\pi b^2} \right)^{d/2} \exp\left\{ - \frac{3}{b^2} (x-y)^2 \right\} \quad (4.17)$$

$$\text{and} \quad g_0^2 = - \frac{2187 \pi^2}{8d(d-1)} \left(\frac{32}{3 \pi b^2} \right)^{(d-4)/2} \beta^2 \quad (4.18)$$

Hence, in four dimensions,

$$w = \frac{32}{297 N \ln(\Lambda b)} \quad (4.19)$$

However these last three results must not be taken too seriously, due to the insufficiency of the approximation for constructing a coherent theory (and in particular no upper bounded action). The minus signs, for instance, are cured in a more realistic model. The fact that w vanishes in (4.19) is also not a general fact; nevertheless, the N factor in the denominator is always present. This means that, if the limit $N \rightarrow \infty$ is taken at fixed b , the system reduces to the planar approximation $w=0$. We emphasize, however, that such an inversion of the limits $b \rightarrow 0$ and $N \rightarrow \infty$ may lead to erroneous results in our model if w_c is not 0.

4.3. Further generalizations.

If one does not restrict oneself to the first quadratic term in the expansion (2.11) of the u function, sides of triangles in the diagrammatic expansion can be grouped by more than two. Hence more complex topologies (well-known in the strong coupling expansions of lattice gauge theories) will arise. Topologically, the diagrams are 2-dimensional manifolds with singular lines along which three (or more) regular surfaces join together.

In the interaction part, we can also introduce, beyond the triangular term, plaquettes with four or more sides. The technique used in subsection 4.1 is valid, with corresponding complications, while treating all these extensions.

5. Conclusion.

We have proposed a model for gauge fields in a continuum Euclidean space which may be viewed as a regularization of the Yang-Mills field theory by mean of a delocalization parameter b . As very little is known about non local theories, we have been obliged to present the hopefully expected properties of this model as speculations, which nevertheless seem quite reasonable. This model has essentially the same features as lattice gauge theories; it avoids the problem of restoration of rotation invariance and seems to provide a better control on the local limit.

Further developments will include a careful study on the influence of various terms in the action, computation of glueball mass spectra, introduction of matter fields, ...

Appendix A

SU(2) peculiarities

The matrix algebra generated by 2×2 unitary matrices is not $GL(2, \mathbb{C})$, but $R \otimes SU(2) \simeq E_4$, i.e. the set of matrices of the form $a_0 + i a_i \sigma_i$ (σ_i being the Pauli matrices and the a_i 's being real). The invariant measure on this set is $\prod_i da_i$ and reduces to the $SU(2)$ Haar measure on the sphere $a_0^2 + a^2 = 1$.

In the decomposition (2.4) of the fields, there are neither the hermitian part H , nor the phase θ . Action terms using determinants such as (2.13) are now useless since they coincide with trace terms such as (2.10). The discrimination between $U(2)$ and $SU(2)$ gauge groups is indeed not due to terms such as (2.13) in the action, but to the set onto which the field ζ varies ($GL(2, \mathbb{C})$ for $U(2)$, $R \otimes SU(2)$ for $SU(2)$).

In the diagrammatic expansion, non-orientable diagrams are allowed since the fundamental representation of $SU(2)$ is equivalent to its conjugate. Several minor changes occur in the formula of section 4 to take into account this property. Triangles are no more oriented in the diagrams; g now runs over all non-negative integers; changes appear in the following formulae

$$S_{\text{int}}(\zeta^{(i,j)}) = \sum_{(ijk)} \sum_{\alpha\beta\gamma} 6 a^{3d} K_3(i,j,k) \zeta^{(i,j)}_{\alpha\beta} \zeta^{(j,k)}_{\beta\gamma} \zeta^{(k,i)}_{\gamma\alpha} \quad (4.2')$$

$$\langle \zeta^{(i,j)}_{\alpha\beta} \zeta^{(k,l)}_{\gamma\epsilon} \rangle_0 = \frac{1}{2 a^{2d}} F_2(|i-j|) \{ \delta_{i1} \delta_{kj} \delta_{\alpha\epsilon} \delta_{\gamma\beta} + \delta_{ik} \delta_{jl} (\delta_{\alpha\beta} \delta_{\gamma\epsilon} - \delta_{\alpha\epsilon} \delta_{\beta\gamma}) \} \quad (4.4')$$

The supplementary term in this formula comes from the fact that the fundamental representation is equivalent to its adjoint; it plays a role while evaluating the non-orientable diagrams for each disorienting sequence. It will be responsible for the minus sign in (4.12').

$$\xi(X,Y) = \frac{1}{3} \frac{a^d}{\beta} F_2^{-1/2}(b|X-Y|) \zeta(bX, bY) \quad (4.9')$$

$$v = 9 \beta^2 (b/a)^d \tag{4.11'}$$

$$w = - 9/2 \beta^2 \tag{4.12'}$$

Note that it is also possible to construct another theory for SU(2) with a non-minimal set for the fields ζ , using (as for any other SU(N) model) the U(2) model with a supplementary term (2.13).

Appendix B

Preliminary numerical analysis

We have generated on the computer all closed diagrams up to 12 triangles for the approximation described in subsection 4.1 (u quadratic, G gaussian) for the free energy (4.14). The integrals (4.13) are gaussian and hence can be evaluated exactly. The corresponding coefficients C_{ng} of the series

$$w^2 b^d \ln Z / V = \sum C_{ng} v^n w^g$$

are displayed in Table 1.

It can be seen that the number of diagrams increases drastically with the order. Nevertheless, most of them can be generated from a few "irreducible" diagrams using techniques which will be described in further works.

The resulting series have been analyzed using the technique of Dlog Pade approximants in the variable t along the lines ($v = t \cos \theta$, $w = t \sin \theta$) and ($v = t \cos \theta$, $w = i t \sin \theta$) for fixed θ . We find evidence of a singularity line in the (v, w^2) plane which stops at some point $w^2 < 0$. Beyond this point, the singularity becomes imaginary and disappears. This analysis is illustrated on figure 1 by the dashed line. Due to the small number of terms in the series, the errors of the Pade analysis are expected to be rather large, and the figure is somewhat idealized and qualitative. Nevertheless, it clearly suggests the behaviour of a first order transition line, with an end point (plain line on the figure); with this interpretation, the found dashed line would represent the end of the metastable phase encountered when one crosses the transition line while coming from the origin.

To verify this hypothesis, we have computed the partial differential approximants [8] of the series; this technique consists to approximate the series by the solution $h(v, w^2)$ of the differential equation

$$P \partial h / \partial v + Q \partial h / \partial w^2 + R h = 0$$

where we choose P , Q and R as polynomials in v and w^2 (here of degrees 1, 1 and 0). The solution h is singular at the point $v = 1.52$, $w^2 = -1.04$. It scales with a power law $X^{-\chi}$ ($\chi \approx 0.42$) along an axis with slope -3.37. This analysis thus confirms the precendently postulated behaviour.

Of course, the small number of terms in the actual series forbids a definitive answer. In any case, we found evidence of singularities in the

(v, w^2) plane and therefore the limit $v \rightarrow \infty$ must be estimated carefully. There is, however, a path avoiding the singularities, as indicated in the figure by an arrow. This analysis is only a preliminary one; further calculations are in progress and involve in particular a more realistic action.

	n=3	n=4	n=5	n=6	n=7	n=8
q=0	.0563710017558 (1 diagram)	.0154846146291 (2 diagrams)	.0104630107206 (4 diagrams)	.0102486408102 (14 diagrams)	.0123073347207 (54 diagrams)	.0168110235370 (291 diagrams)
q=2	.0002319794311 (1 diagram)	.0017132388116 (7 diagrams)	.0076947639879 (51 diagrams)	.0278975032485 (596 diagrams)		
q=4	.0001443097645 (2 diagrams)	.0043344874819 (7 diagrams)				

Table 1
Coefficients C_{nq} of the series for $\ln Z$

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Table Captions

Table 1 : Coefficients C_{ng} of the series for $\ln Z$

Figure Captions

Figure 1 : Singularities in the (v, w^2) plane. This figure is semi-quantitative. The arrow indicates the free singularity way to reach the $v \rightarrow \infty$ limit.