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QUANTUM PROPERTIES OF HIGHER DIMENSIONAL AND DIMENSIONALLY
REDUCED SUPERSYMMETRIC THEORIES

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ABSTRACT

We discuss the relation of quantum properties of d dimensional and dimensionally reduced theories with a special emphasis on the use of a regularisation accounting for power divergences. The main examples are $N = 1, d = 10$ super Yang-Mills (SYM_{10}^1) and $N = 1, d = 11$ supergravity (SG_{11}^1) and their reductions. In particular, we understand the vanishing of one-loop β -function in SYM_4^4 (zero conformal anomalies in SG_4^8) as a consequence of the absence of $L^6(L^2)$ power divergences in $SYM_n^1(SG_n^1)$. The heat kernel expansion coefficients b_{2k} , $k = 0, \dots, 3$ are found to be zeroes for SG_{11}^1 (as well as for SYM_{10}^1 and SG_{10}^1) and thus the one-loop finiteness of maximal SG 's in $d \leq 7$ is explicitly demonstrated. We also present the expressions for one-loop constant gauge field effective lagrangian and scalar effective potential in the SYM_4^4 theory and analyse the problem of $N = 4$ supersymmetry breaking.

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1. INTRODUCTION

Recently it was understood that a number of interesting four dimensional theories may be obtained by a reduction of higher dimensional ones, most notably, from $N = 1$, $d=10$ super Yang-Mills (SYM_{10}^1) [1] and $N = 1$, $d = 11$ supergravity (SG_{11}^1) [2] (see e.g. [3-8]). Two points of view are possible on the relation of d -dim and 4-dim theories: (A) our world is described by a complete d -dim theory, while its apparent four dimensionality is only a low energy phenomenon and can be attributed to a particular structure of d -dim vacuum space (e.g., $M^4 \times N^{d-4}$ with N being a compact manifold of a "scale" R , for example, a group G [9] or a coset G/K space [10, 11]); (B) dimensional reduction is a technical tool in analysis of 4-dim theories (useful, e.g., in constructing various supergravities [3, 12]). Believing in (A), one can classically consider the observed light particles in M as massless modes of "Fourier" expansion in internal coordinates. However, at the quantum level it is necessary to include the infinity of massive modes ($M_n \sim 1/R$). The reason is that the equality of d -dim (Γ_d) and 4-dim (Γ_4) quantum effective actions (the latter defined to be that of d -dim theory rewritten on terms of 4-dim fields) holds only when all modes are taken into account^{a)}. Being

^{a)} It is worth stressing that $\Gamma_d = \Gamma_4$ is valid for infinite as well as for finite parts; to avoid possible contradictions (cf. [6]) one is to observe that $(\Gamma_4)_\infty = (\text{divergences for every mode}) + \text{additional divergences which are due to infinite sums of finite parts of partial 4-dim effective actions for separate modes.}$

interested only in low energy processes we may try to define some effective light particles theory, integrating over massive modes. The resulting (non-local) effective lagrangian $\mathcal{L}_{light}$ does not coincide with the classical zero mode lagrangian $\mathcal{L}^{(0)}$, following by reduction from d-dim classical action. Naturally, it is tempting to connect both expressions in some approximation. This program may seem analogous to constructing "effective field theories" (see e.g. [13]): if the external momenta $p_i \ll \min \{M_n\}$, then $\mathcal{L}_{light}$ is given by the infinite sum of local terms, with those of $dim > 4$ being suppressed by $1/M$ -factors. However, there is one essential difference: d-dim theories are generally non-renormalizable while the analysis in [13] assumes renormalizability of complete theory (evident for a finite number of heavy states). That is why, it seems impossible to describe consistently the low energy sector of quantized d-dim theory with the help of some quantized 4-dim one. Thus one can a priori criticize the attempts [14, 7, 8] (if viewed from the point of (A)) to construct a unified theory starting with power counting non-renormalizable d-dim gauge theories. However, two improvements are possible: (A') it may happen that d-dim theory is finite on shell (either in each order of loop expansion or after its summation, cf. [15]); (A'') one may pass to a more sophisticated picture where d-dim theory is only an intermediate stage, being a low energy ($\alpha' \rightarrow 0$) limit of some finite or renormalizable theory of (super) strings in d dimensions (see e.g. [16] and refs. therein). Disregarding (A'') in this paper, let us point out that existence of finite $d > 4$ -dim theories is doubtful at present. Namely, according to superstring (supergraph) power counting rules [17, 16] ([18]) we expect UV finiteness of maximally

extended SYM and SG d-dim theories (following by reduction from SYM_{10}^1 and SG_{11}^1) for the following number of loops

$$SYM : \quad l < \frac{4}{d-4} \quad (l < \frac{6}{d-4}), \quad (1.1)$$

$$SG : \quad l < \frac{6}{d-2} \quad (l < \frac{14}{d-2}),$$

i.e. SYM_{10}^1 is infinite already in the first loop (while SG_{11}^1 is infinite for $l \geq 1$ ($l \geq 2$)). We will confirm this conclusion by demonstrating the presence of one-loop on shell quadratic divergence in SYM_{10}^1 . Once again we see that the approaches [7,8], starting with SYM_{10}^1 are to be considered as unsatisfactory at the quantum level. The following remark may be useful concerning the reasoning in ref. [8]. Here the probable finiteness of SYM_{10}^1 (i.e. the simplest reduction of SYM_{10}^1) was implicitly treated as an indication of some good quantum properties of SYM_{10}^1 itself. The latter theory was then used as a starting point for a different (coset) reduction. Finally, it was conjectured that the resulting "realistic" d=4 theory may thus be distinguished from the point of divergences. It should, however, be stressed that different reductions a priori have different quantum behaviour^{*/} and it is the simplest supersymmetry preserving one which is singled out by

^{*/} for example, the coset reduction generally changes the number of degrees of freedom and in this respect is analogous to truncation, which is known to modify quantum results (e.g. $\beta = 0$ in SYM_{10}^1 but $\beta_{1-loop} \neq 0$ in SYM_{10}^1).

its UV properties. Different reductions *do* have the same UV limit but only when all (massive) modes are taken into account. ^{E/} But then we again confront the problem of infinities of complete d-dim theory.

Therefore let us now turn to the second point of view (B). Here we are to quantize a 4-dim theory marked by the fact that its classical action can be obtained by a reduction of some d-dim one. Then a natural question is about information, concerning the quantum theory, which can be gained from this circumstance. To provide an answer we are to relate the effective action $\Gamma_4^{(\tau)}$ of our 4-dim theory to that (Γ_d) of initial d-dim one. In view of the above discussion, the latter is defined only for a fixed UV cut-off L . Then it is clear that shrinking the radii of compact dimensions to zero we get $\lim_{R \rightarrow 0} (\Gamma_d)_{\text{fixed } L} = \Gamma_4^{(\tau)}$ (after proper rescalings of couplings and wave functions to absorb all R^{d-4} -factors). Generally, this implies that the study of infinities of Γ_d may tell us something about those of $\Gamma_4^{(\tau)}$. However, a care is needed in interpreting the above equality. A subtle point is that one should use different cut-offs for four (L_4) and $d-4$ (L_{d-4}) dimensions, relating the limits $L_{d-4} \rightarrow \infty$ and $R \rightarrow 0$ e.g. by $L_{d-4} \sim \frac{1}{R}$ (because $L_{d-4} \rightarrow \infty$ after $R \rightarrow 0$ is senseless). Now it is evident that in order to relate the

E/ One should also bear in mind that initial gauge group of d-dim theory (e.g. E_8 in [7,8]) will be restored in the high energy limit of the final d=4 theory only after summing contributions of all modes.

infinities of Γ_d and $\Gamma_d^{(2)}$ it is necessary to use a power divergences preserving regularization ^{*/} (i.e. one cannot choose the standard dimensionnal or ζ -function regularizations in d dimensions). In this connection let us comment on the proposal [22] to understand the finiteness of SYM_4^y by relating the absence of certain $(F \square^3 F)$ logarithmic counter-terms for SYM_{10}^4 to that of $"F^2"$ ones for SYM_4^y . From our point of view this relation cannot be justified. In fact, the simplest counter-example is provided by YM in odd number of dimensions. This theory is free from one-loop logarithmic divergences but the reduced theory has a non-zero one-loop β -function. It turns that it is the $"L^{d-4}"$ -type d-dim counter-term which is related to the logarithmic one of 4-dim theory. Another important clarification is that ^{it} is not sufficient to study only d-dim invariant counter-terms because reduction breaks d-dim symmetry (e.g. $O(d) \rightarrow O(4) \times O(d-4)$) and thus the infinities of reduced theory cannot be expressed solely in terms of d-dim invariants. The moral is that one must be cautious in applying d-dimensional considerations in the reduced theory. But having properly understood the connection of d-dim and 4-dim results for some particular reduction one can use d-dim theory to facilitate the analysis. This appears to be especially efficient in supersymmetric theories where different background sectors are tied together by supersymmetry and hence one can do calculations in a suitable one, common for reduced and d-dim theories.

The aim of this paper (based mainly on the point of view (B))

^{*/} The use of such a regularization naturally solves possible paradoxes discussed in [6] . Note also that regularizations accounting for power divergences were recently discussed, e.g., in [19-21] .

is: (i) to study some properties of higher dimensional theories; (ii) to connect them with those of reduced theories; (iii) to study some new features of SYM_4^V , illustrating the advantages of $d=10$ -dimensional approach. In sect.2 we prove the "Lemma" establishing the connection of one-loop results in d-dim and (simply) reduced theories. Then we discuss correspondence of (one and higher loop) infinities with the special emphasis on the use of a gauge invariant, power divergences preserving regularization. Sect. 3 is devoted to the one-loop analysis of SYM_4^V using SYM_{10}^1 as a guide. First, in sect.3a we show the absence of $L^{10-2\kappa}$, $\kappa \leq 3$ one-loop infinities (i.e. the vanishing of $b_{2\kappa} \sim F^{2\kappa}$ counter-terms) in SYM_{10}^1 and relate the $b_4 = 0$ property of SYM_{10}^1 to that of zero one-loop β -function in SYM_4^V . We also observe that $b_4(YM_{26}) = 0$ and remark on connection with string theory. It turns, however, that $b_8(SYM_{10}^1) \neq 0$ (sect.3 b,c) and so there are at least L^2 (on shell) infinities in ten dimensions (implying logarithmic ones in d=8 reduction of SYM_{10}^1). Thus we explicitly demonstrate the absence of one-loop finiteness of SYM_{10}^1 . The above conclusions are in accordance with power counting results (1.1) and also with those of one-loop four-particle amplitude ^{calculations} in maximal SYM_4 theories [17] (see also [18]), which showed its UV finiteness for $d < 8$.^{*/} Next, in sect. 3g, we calculate the SYM_4^V one-loop homogeneous abelian gauge field effective action, which is distinguished

^{*/} The vanishing of one-loop SYM_4^V κ -particle ($\kappa \leq 3$) amplitude (or corresponding Green function in a suitable gauge) can be related to $b_{2\kappa} = 0$, while its non-zero value for $\kappa = 4$ indicates $b_8 \neq 0$.

by its UV finiteness (as compared to corresponding expressions for QED [23] and YM theory [24], cf. also [25, 26]). However, supersymmetry does not cure the well-known IR instability of constant background (see e.g. [27]), which is due to the negative mode of gauge field operator. In sect. 3c we analyse the one-loop effective potential for scalars. The problem appears to be closely connected with that of effective action for constant non-abelian gauge potential background ($A_i \neq 0$, $i = 5, \dots, 10$) in SYM_{10}^1 . In particular, this provides understanding of IR instability (cf. [28]) of the final (off-shell) answer. Here we also *show* the absence of perturbative SYM_4^1 supersymmetry breaking and discuss a possibility to construct a realistic model starting with SYM_4^1 with softly broken supersymmetry (this theory still has zero β -function and thus should be contrasted to those of refs. [7, 8] which do not provide soft supersymmetry breaking).

The main topic of sect. 4 is the study of one-loop divergences in higher dimensional supergravities. We start (in sect. 4a) with pure d -dim gravity and prove that it is one-loop on shell infinite for any $d > 4$ by computing the $b_{2k}(\sim R^k)$ coefficients of Z^{d-2k} divergences for $k \leq 3$. Then we comment on infinities in reduced Kaluza-Klein theories. Sect. 4b deals with gravitational sector b_{2k} -calculation for antisymmetric tensor gauge fields which are known to be present in supergravities. Fairly ^{general} results are presented, clarifying the subject of "quantum (in) equivalence" and extending the previous work [29]. Further, in sect. 4c we solve the non-trivial problem of background field method quantization of gravitino in $d > 4$ dimensions. We explicitly construct the "standart" gauge where the gravitino operator takes its simplest

form and thus, e.g., it is straightforward to compute its β_{2k} - coefficients. Finally, all is prepared for discussion of supergravities (sect. 4d). We start with the maximal one, SG_{11}^4 , which appears to be free from $L^{11}, \dots, L^5$ one-loop divergences, i.e. have $\beta_{2k} = 0$, $k \leq 3$. Though the algorithm for β_8 is not presently available (cf. [30]) it seems very probable that $\beta_8(SG_{11}^4) \neq 0$, implying the absence of one-loop finiteness of this theory (and thus invalidating the conjecture about on shell finiteness of SG_{11}^4 made in [5,6]). These results for β_{2k} are in agreement with superstring counter rules (1.1) and also with one-loop four-particle amplitude calculations for maximal SG_d 's [17] ([18]), which indicate the presence of UV infinities for $d \geq 8$ ^{1/}. At the same time, we find one-loop finiteness of maximal supergravities in $d \leq 7$ dimensions. To appreciate the non-triviality of this result (based on cancellations and not merely on non-existence of possible on shell invariants) one is to recall the absence of finiteness of pure gravity in $d > 4$. For example, the SG_6^4 theory, corresponding from 4-dim point of view to SG_7^8 , plus the infinity of its "massive copies", appears to be one-loop finite and thus may serve as a basis for (at least at one loop) consistent complete Kaluza-Klein theory. Returning to four dimensions, we recognize the $\beta_\gamma = 0$ property of SG_{11}^4 as the vanishing of conformal anomalies (or topological infinities) in the version of SG_γ^8 , obtained by dimensional reduction (cf. [31-33]). Finally, we prove that $\beta_{2k} = 0$, $k \leq 3$ is valid also for SG_{10}^4 but raise doubt in suggestion [22]

^{1/} note that β_8 gives logarithmic counter-term in $d=8$.

(see also [16]) that d=4 reduction of this non-maximal supergravity (with d=4 reducible supersymmetry) may be (at least) one-loop finite.

Some speculations and concluding remarks are gathered in sect.5. Appendix A gives our notations and some useful identities while Appendix B contains γ -matrices relations in d dimensions.

2. Correspondence of divergences in d-dimensional and dimensionally reduced theories.

Let us start with the one-loop approximation which can generally be represented as

$$Z = \int d\phi e^{-\frac{1}{2} \phi \Delta_2 \phi}, \quad \Gamma = \frac{1}{2} \log \det \Delta_2, \quad (2.1)$$

$$\Delta_2 = -\frac{1}{\sqrt{g}} \partial_M g^{MN} \sqrt{g} \partial_N + X, \quad (2.2)$$

where $\partial_M = \partial_M + A_M$ and $M, N = 1, \dots, d$ (for notations see Appendix A). Using the well-known expansion [9, 34, 30]

$$\left(t_2 e^{-s \Delta_2} \right)_{s \rightarrow 0} \simeq \sum_{p=0}^d s^{\frac{p-d}{2}} \mathcal{H}_p, \quad (2.3)$$

$$\begin{aligned} \mathcal{H}_p &= B_p + C_p, \quad B_p = \int_M b_p \sqrt{g} d^d x, \quad B_{2k+1} = 0, \\ C_p &= \int_{\partial M} c_p \sqrt{g} d^{d-1} x, \end{aligned} \quad (2.4)$$

We get the infinite part of the effective action

$$\Gamma_\infty = -\frac{1}{2} \int_E \frac{ds}{s} t_2 e^{-s \Delta_2} = \sum_{p=0}^d \frac{\epsilon^{\frac{p-d}{2}}}{p-d} \mathcal{H}_p \quad (2.5)$$

More explicitly

$$\Gamma_{\infty} = - \left(\frac{1}{d} L^d A_0 + \dots + \frac{1}{d-p} L^{d-p} A_p + \dots + \frac{1}{2} A_d \log \frac{L^2}{\mu^2} \right) \quad (2.6)$$

where $L = \varepsilon^{-1/2} \rightarrow \infty$ is the "proper time" cut-off. This formula is valid for odd as well as for even dimension d . If $d = 2k+1$ we conclude that there is no volume logarithmic infinities. However, this does not imply (even if $\partial M = 0$, i.e. $A_{2k+1} = 0$) the one-loop finiteness of any theory in M^{2k+1} because of possible "power-type" divergences. The account of these (L^p) terms (in any appropriate regularization) appears necessary for establishing the relation of counter-terms in d-dimensional and reduced theories.

In this paper we shall neglect surface infinities, assuming $\partial M = 0$. As for the volume terms in (2.4), only the following four coefficients are presently explicitly known [30]

$$\begin{aligned} \bar{b}_p &\equiv (4\pi)^{d/2} b_p, \quad \bar{b}_0 = t_2 1, \\ \bar{b}_2 &= t_2 \left(1 \cdot \frac{R}{6} - X \right), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \bar{b}_4 &= t_2 \left[1 \cdot \left(\frac{1}{180} R_{MN PQ}^2 - \frac{1}{180} R_{MN}^2 + \frac{1}{72} R^2 + \right. \right. \\ &\quad \left. \left. + \frac{1}{30} \partial^2 R \right) + \frac{1}{12} F_{MN}^2 + \frac{1}{2} X^2 - \frac{R}{6} X - \frac{1}{6} \partial^2 X \right], \end{aligned} \quad (2.8)$$

$$\begin{aligned} \bar{b}_6 &= t_2 \left\{ 1 \cdot \left[\frac{1}{15120} R^{MN}_{PQ} R^{PQ}_{KS} R^{KS}_{MN} + \right. \right. \\ &\quad \left. \left. + \frac{1}{3240} (R^{MN}_{PQ} R^{PQ}_{KS} R^{KS}_{MN} - 2 R^{MN}_{PQ} R^{QS}_{NK} R^{KM}_{SP}) \right] - \right. \\ &\quad \left. - \frac{1}{120} (\partial_M F_{NK})^2 - \frac{1}{45} F_{MN} F_{NK} F_{KM} + \frac{1}{72} R_{MNKP} F_{MN} F_{KP} \right. \\ &\quad \left. + \frac{1}{12} X \partial^2 X - \frac{1}{6} X^3 - \frac{1}{12} X F_{MN}^2 - \frac{1}{180} X R_{MNKP}^2 \right\}, \end{aligned} \quad (2.9)$$

where in (2.9) we assumed $R_{MN} = 0$, omitted total derivative terms, and used relations of appendix A (for complete expression see [30]). Thus (2.7)-(2.9) give the algorithm to compute one-loop L^d , L^{d-2} , L^{d-4} and L^{d-6} divergences in any d-dimensional theory.

Now let us consider a dimensionally reduced theory, assuming the simplest reduction when $M^d = M^n \times S^1 \times \dots \times S^1$ and only "zero modes" in "internal" coordinates are retained, i.e.

$$\partial_i \phi = 0, \quad i = 1, \dots, d-n \quad (2.10)$$

is assumed in the classical action and in the path integral. This condition implies the breaking of d-dimensional general covariance group to the product of n-dimensional one and $[U(1)]_{local}^{d-n} \times SL(d-n, R)_{global}$ (and also $O(d) \rightarrow O(n) \times [U(1)]^{d-n}$), see e.g. [3]. That is why we may use the standard Kaluza-Klein parametrization of g_{MN} in terms of h-dimensional metric $g_{\mu\nu}$, vectors and scalars

$$g^{MN} \sqrt{g_{(d)}} = \left[\begin{array}{c|c} g^{\mu\nu} \sqrt{g} & -B^{i\nu} \sqrt{g} \\ \hline -B^{i\mu} \sqrt{g} & (g^{ij} + g^{\mu\nu} B_\mu^i B_\nu^j) \sqrt{g} \end{array} \right] \quad (2.11)$$

(here we used the rescaled metric:

$$g_{MN} = \left[\begin{array}{c|c} \bar{g}_{\mu\nu} + \bar{\varphi}_{ij} B_\mu^i B_\nu^j & \bar{\varphi}_{ij} B_\mu^i \\ \hline \bar{\varphi}_{ij} B_\nu^j & \bar{\varphi}_{ij} \end{array} \right], \quad g_{\mu\nu} = \lambda^{1/n-2} \bar{g}_{\mu\nu}, \quad \lambda = \det \bar{\varphi}_{ij},$$

$g = \det g_{MN}$).

The classical action in (2.1)

now takes the form:

$$\frac{1}{2} \left(\int d^{d-n} x \right) \int d^n x \sqrt{g} \phi \tilde{\Delta}_2 \phi$$

with the analog of Δ_2

(2.2) being

$$\tilde{\Delta}_2 = - \frac{1}{\sqrt{g}} \tilde{\partial}_\mu g^{\mu\nu} \sqrt{g} \tilde{\partial}_\nu + \tilde{X} \quad (2.12)$$

$$\begin{aligned}\tilde{\partial}_\mu &= \partial_\mu + \tilde{A}_\mu, & \tilde{A}_\mu &= A_\mu - B_\mu^i A_i, \\ \tilde{X} &= \lambda^{-1/n-2} X - \varphi^i A_i A_i,\end{aligned}\quad (2.13)$$

(we put $A_M = (A_\mu, A_i)$ and assumed that $\partial_i X = 0$, $\partial_i g_{MN} = 0$). As a result, the divergences of, first reduced and then quantized, theory are given by (2.5), (2.6)-(2.9) with $d \rightarrow n$, $g_{MN} \rightarrow g_{\mu\nu}$ etc.

It is easy to understand that correspondence of d-dimensional divergences is established by comparing $\mathcal{A}_p$'s for equal p ^{*/}. Thus $L^*(\log L^2)$ infinities in n -dimensions are counterparts of $L^{d-n+2}(L^{d+2})$ ^{in d dimensions} Comparing $\mathcal{C}_p(\Delta_2)$ and $\mathcal{C}_p(\tilde{\Delta}_2)$ for (2.2) and (2.12) we conclude that in general they do not coincide. It is important to observe that in view of explicit breaking of d-dimensional symmetry by the reduction (2.10), the counter-terms (i.e. $\mathcal{C}_p$'s) in reduced theory cannot be written only in terms of d-dim covariant objects. Hence the analysis of only d-covariant counter-terms in d-dimensional theory is not sufficient for obtaining information about counter-terms of reduced theory (cf. [22]). Nevertheless, it is possible to prove the equality of $\mathcal{C}_p$ in d and n dimensions under some special choice of background fields. Namely, the following "Lemma" is true: if

$$A_M = \{A_\mu, 0\}, \quad g_{MN} = \left[\begin{array}{c|c} g_{\mu\nu} & \\ \hline & \delta_{ij} \end{array} \right], \quad (2.14)$$

then $\mathcal{C}_p(\Delta_2) = \mathcal{C}_p(\tilde{\Delta}_2)$. Thus in order to compute counter-terms of reduced theory in the case of (2.14), one may first calculate

*/ The $(d-n)$ -volume can always be trivially factorized if divergences are local.

corresponding ones in d-dimensions and then substitute (2.14). This non-trivial observation is based on universality ^{no explicit} dependence on d , (cf. (2.7)-(2.9)) of $\bar{c}_p$ -coefficients [30]. It provides essential simplifications in calculations when quantum fields

have d-dimensional indices: we need not do reduction for quantum fields (and therefore may use natural d-dimensional gauges etc.).

It should be understood that this Lemma is valid only for the simplest reduction (2.10), which does not involve indices of fields

(and thus "conserves" the number of degrees of freedom). Different reductions (for example, the coset space one [10, 11, 14]) may lead to quite different quantum theories with counter-terms having no natural connection with d-dimensional ones.

Now let us make several remarks on possible higher loop generalization of the above discussion. The most adequate framework is again the background field method and coordinate space heat kernel technique (for recent progress see [35, 36]). With one-loop experience in mind, we need a (gauge -invariant) regularization preserving power-type divergences. A natural candidate is a generalization of the "proper-time" one in (2.5): given a diagram with

K internal lines we may represent all K propagators as $G(x_1, x_2) = \int ds \langle x_1 | e^{-s\Delta_1} | x_2 \rangle$, thus finally obtaining a gauge invariant expression like (cf. [36]) $\int \dots \int_{\epsilon} ds_1 \dots ds_k J(s_1, \dots, s_k / g, A)$,

where g, A are background fields and $L = \epsilon^{-\frac{1}{2}} \rightarrow \infty$. This procedure can be straightforwardly implemented for calculation of counter-terms (at least at two loop level) generalizing various earlier background field method results [37, 35, 36] on d-dimensional case. Another appropriate regularization is a modification of the standard dimensional one. The main idea is to consider d

as a parameter taking any integer value $\bar{d} = 0, 1, 2, \dots, d$ and to sum all corresponding infinities in ordinary dimensional regularization, as if applied in $0, 1, \dots, d$ dimensions. For example, $\mathcal{H}_p$ in (2.6) can be considered as a (logarithmic)

counter-term in dimensional regularization for $\bar{d} \rightarrow p$, because $(\frac{1}{\bar{d}-p} L^{\bar{d}-p})_{\bar{d} \rightarrow p} \rightarrow \frac{1}{2} \log L^2$ (cf. [20, 21]). Thus to N -loop order we shall have: $\Gamma_\infty = \sum_{\ell=0}^N \hbar^\ell \sum_{k=0}^{\ell} \sum_{p=0}^d \left[\frac{a_k^{(\ell, d)}}{(\bar{d}-p)^k} \right]_{\bar{d} \rightarrow p}$. The analogous expression with an explicit (dimension-1) cut-off L reads

$$\Gamma_\infty = \sum_{\ell=0}^N \hbar^\ell \sum_{k=1}^{\ell} \left(\sum_{p=0}^{d-1} L^{d-p} \mathcal{H}_p^{(\ell, k)} + \log L^2 \mathcal{H}_d^{(\ell, k)} \right)^k \quad (2.15)$$

where $\mathcal{H}_s$ depend on background fields and no renormalization was carried out (except neglecting non-local parts in $\mathcal{H}_s$). As a result, simply on dimensional grounds, any possible relation of infinities in d and n dimensional theories must have the form

$$\left\{ L^{d-n+k} \sum_{2 \leq s} \alpha_s (\log L^2)^s \right\}_d \rightarrow \left\{ L^k (\log L^2)^s \right\}_n \quad (2.16)$$

$k = 0, \dots, N \cdot d$.

However, it seems difficult to establish any relation between $\mathcal{H}_s$ without information about interaction terms in the action and structure of indices of quantized fields ϕ .

In the above analysis we assumed the same cut-off L for all dimensions in d -dimensional theory, which was then supposed to be related to the cut-off in reduced theory. It is possible to trace

^{N/} This should be compared with the ordinary $d=4$ dimensional regularization result

$$\Gamma_\infty = \sum_{\ell=0}^N \hbar^\ell \sum_{k=1}^{\ell} \left[\frac{a_k^{(\ell)}}{(\bar{d}-4)^k} \right]_{\bar{d} \rightarrow 4}$$

the emergence of reduced theory infinities employing different cut-offs for n and $d-n$ momenta ($|p_\mu| \leq L_n$, $\mu=1, \dots, n$, $|p_i| \leq L_{d-n}$, $i=1, \dots, d-n$). Then the structure of divergences is given by (2.15) under substitution: $L^p \rightarrow \sum_{k=0}^p c_{k,p} (L_n)^k (L_{d-n})^{p-k}$, $\log L^2 \rightarrow \log L_n^2 + \log L_{d-n}^2 + \dots$, etc. Imposing

the reduction condition (2.10) on d -dim path integral we get $\prod \delta(p_i)^{(d-n)}$ -factors in all loop integrations. As a consequence, the infinities of reduced theory as viewed from d dimensions, will take the form: $\Gamma_\infty^{(n)} \sim h (L_{d-n})^{d-n} \times [(L_n)^n + \dots + \log L_n^2] + h^2 (L_{d-n})^{2(d-n)} \times \{ [(L_n)^n + \dots + \log L_n^2]^2 + (L_n)^n + \dots \log (L_n)^2 \} + \dots$. Finally, we are to absorb all L_{d-n} -factors by the wave function and (or) coupling constant redefinitions. This remark provides explicit

illustration of correspondence (2.16) in the case of regularization, breaking initial d -dim symmetry. It seems, however, that no constructive algorithm, relating d -dim and n -dim counter-terms, exists (beyond one or at least two loop level) if we start with quantum theory having the unbroken d -dimensional symmetry.

3. Super-Yang-Mills: results for $d=4$

as they follow from $d=10$

(3a) One-loop infinities

As is well-known [1] the lagrangian of $N=4$, $d=4$ super Yang-Mills theory (containing 1 vector, 6 (pseudo) scalars, and 4 Majorana spinors, all in adjoint representation of some gauge group G) can be obtained by simple reduction (2.10) from that of SYM'_{10} in ten dimensions (we omit possible(?) auxiliary fields)

$$\mathcal{L}_{10} = \frac{1}{4g^2} (F_{\mu\nu}^a)^2 + i \bar{\psi}^a \hat{\partial} \psi^a \quad (3.1)$$

where $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f^{abc} A_M^b A_N^c$,
 $M, N = 1, \dots, 10$; $a, b = 1, \dots, \dim G$,

$\hat{\mathcal{D}} = \gamma_M \mathcal{D}_M$, γ_M - are 32×32 Dirac matrices (see Appendix B) and ψ is a Majorana-Weyl spinor, (the first term in (3.1))

After the reduction $A_M = \{A_\mu, A_i\}$ takes the form

$$\mathcal{L}_\psi^{(bose)} = \frac{1}{4g^2} (F_{\mu\nu}^a)^2 + \frac{1}{2g^2} (\mathcal{D}_\mu A_i^a)^2 + \frac{1}{4g^2} (F_{ij}^a)^2, \quad (3.2)$$

$$F_{ij}^a = f^{abc} A_i^b A_j^c.$$

It is straightforward to quantize (3.1) in one-loop approximation assuming that only A_M has a non-trivial background

$$Z = \frac{\det \Delta_0}{\sqrt{\det \Delta_1}} \cdot [\sqrt{\det \Delta_2}]^{1/4}, \quad (3.3)$$

$$\Delta_0 = -\mathcal{D}^2, \quad \Delta_{1MN} = -g_{MN} \mathcal{D}^2 - 2F_{MN},$$

$$\Delta_2 = -(\hat{\mathcal{D}})^2 = -\mathcal{D}^2 - \frac{1}{2} \gamma_{MN} F_{MN}. \quad (3.4)$$

We have chosen the background gauge $\mathcal{D}_M A_M^{(1)} = \zeta(x)$, used matrix notations (A.2) and took into account the Majorana-Weyl constraint on ψ . It is easy now to calculate the Z^{d-2k} , $k \leq 3$ infinities in d dimensions using (2.6)-(2.9). Introducing the notations

$$\bar{b}_0 = \beta_0, \quad \bar{b}_1 = \beta_1 \frac{1}{12} t_2 F_{MN}^2,$$

$$\bar{b}_2 = -\beta_2 \frac{1}{60} t_2 (\mathcal{D}_M F_{MN})^2 - \beta_3 \frac{1}{72} t_2 (F_{MN} F_{NK} F_{KM}), \quad (3.5)$$

Note that for Majorana spinor $\int d^4x e^{\bar{\psi} \hat{\mathcal{D}} \psi} = [\sqrt{\det \hat{\mathcal{D}}^2}]^{1/2}$; note also that our gauge contains quantum scalars after the reduction:

$$\mathcal{D}_\mu A_\mu^{(1)} + [A_i^{(c)}, A_i^{(1)}] = \zeta(x).$$

and using (A.3), (A.4), (B.9), we get $\beta_2 = 0$ and $(\gamma = 2^{\lfloor d/2 \rfloor})$

Δ_i	β_0	β_1	β_2	β_3
Δ_0	1	1	1	1
Δ_1	d	$d-2\gamma$	$d-4\gamma$	d
$\Delta_{\gamma/2}$	$-\gamma$	-2γ	-4γ	$-\gamma$

(3.6)

Let us now establish total results for a d-dimensional system of one gauge vector, N_0 scalars and $N_{\gamma/2}$ spinors (all in adjoint representation)

$$\bar{\ell}_p = \bar{\ell}_p^{(1)} + N_0 \bar{\ell}_p(\Delta_0) - \frac{1}{\gamma} N_{\gamma/2} \bar{\ell}_p(\Delta_{\gamma/2}), \quad \bar{\ell}_p^{(1)} = \bar{\ell}_p(\Delta_1) - 2\bar{\ell}_p(\Delta_2), \quad (3.7)$$

where $\gamma = 1, 2, 4$ for Dirac, Majorana and Majorana-Weyl spinors correspondingly. Imposing the condition

$$\bar{\ell}_0 = \bar{\ell}_2 = \bar{\ell}_\gamma = \bar{\ell}_6 = 0, \quad (3.8)$$

we get three equations

$$\begin{aligned} \beta_0 &= \beta_3 = (d-2) + N_0 - \frac{\gamma}{8} N_{\gamma/2} = 0, \\ \beta_1 &= (d-2\gamma) + N_0 + 2\frac{\gamma}{8} N_{\gamma/2} = 0, \\ \beta_2 &= (d-4\gamma) + N_0 + 4\frac{\gamma}{8} N_{\gamma/2} = 0, \end{aligned} \quad (3.9)$$

with a unique solution $d + N_0 = 10$, $\frac{\gamma}{8} N_{\gamma/2} = 8$, corresponding to SYM'_{10} ($N_0 = 0$, $d = 10$, $N_{\gamma/2} = 1$, $\gamma = 4$) and all its reductions. Thus the condition of finiteness in $d=4$ uniquely fixes SYM'_4 . Eq. (3.8) implies the absence of leading $L'^0, \dots, L'^\gamma$ one-loop divergences in ten dimensions. Applying the Lemma (2.14), we now understand the known one-loop finiteness ^{*/} of SYM'_4 as a

^{*/} Here we get finiteness in the gauge field sector, which, however, is sufficient to conclude about finiteness for a general background in view of irreducible supersymmetry.

consequence of some property ($\ell_4 = 0$) of SYM_{10}^1 . Moreover, we get $\ell_6 (SYM_4^1) = 0$, what is connected with vanishing of one-loop corrections in 3-point function in SYM_4^1 (in a proper supersymmetric background gauge [38]).

Eq. (3.8) holds also for another possible reduction, SYM_6^2 , implying its off shell one-loop finiteness in six dimensions (cf. (2.6)). This (power counting non-renormalizable) theory provides an example of off-shell cancellations due to supersymmetry. When treated from four dimensional point of view, it contains SYM_4^1 as well as the infinite number of its "massive analogs". Interestingly enough, we see that the inclusion of all massive states does not disturb the one-loop finiteness of SYM_4^1 .

One more observation following from (3.6), (3.9) is the absence of L_{22} -divergences in pure Yang-Mills theory in $d=26$ ($\ell_4 = 0$), establishing, through the Lemma, the vanishing of one-loop gauge coupling β -function in the corresponding $d=4$ reduced theory (which contains $YM_4 + 22$ scalars in adjoint representation). This "zero" is of non-supersymmetrical nature and can be understood exploiting the connection with the open sector of the bose string model (see e.g. [16] and refs. there). Namely, YM_{26} coincides with the $\alpha' \rightarrow 0$ limit of this model in $d = 26$ [39, 16]. By this statement we express a possibility to describe tree string amplitudes by the gauge field "effective" lagrangian ($a_1 = \text{const}$)

*/ as a by-product of (3.6) and (3.9) we get $\ell_6 = 0$ on shell ($\partial_m F_{mn} = 0$) in any supersymmetric theory (notice that β_1 in (3.6) is equal to the number of degrees of freedom β_0). This does not, however, mean its on shell finiteness in $d=6$ because of possible quadratic divergences (if $\ell_4 \neq 0$).

$$\mathcal{L} = t_2 \left\{ a_1 F_{MN}^2 + \alpha' \left[a_2 (\partial_M F_{MN})^2 + \right. \right. \\ \left. \left. + a_3 F_{MN} F_{NK} F_{KM} \right] + O(\alpha'^2) \right\} \quad (3.10)$$

Quantizing this theory and utilizing the fact that all non-trivial divergences of one-loop open string amplitudes can be absorbed solely by α' renormalization, we conclude that q_1 (and also q_2, q_3) must be finite to one-loop order. However, it seems impossible to generalize this reasoning to higher loops because of apparent difficulties in the quantum bose string model (cf. [16])^{*}. Turning to the supersymmetric open string theory, which is probably a renormalizable one [16] and has SYM_{10}^4 as its $\alpha' \rightarrow 0$ limit [1,17], we recognize $\beta_{\psi}(SYM_{10}^4) = 0$ as its one-loop consequence and may also conjecture about the absence of F_{MN}^2 -type infinities in SYM_{10}^4 to any loop order. It is this latter property (and not that of some class of logarithmic counter-terms in $d=10$, cf. [22]) that implies the vanishing of $F_{\mu\nu}^2$ -infinities (and thus zero β -function) in the reduced SYM_4^4 theory (cf. (2.16) and discussion in Sect.2).

The examples of YM_{26} and SYM_{10}^4 are useful to illustrate the following general observation concerning interplay of dimensional reduction and supersymmetry. While supersymmetry "glues" together all sectors of the theory, different fields initially related by d -dim symmetry are completely independent after reduction. It is only in supersymmetric dimensionally reduced theory where all

^{*} This is in agreement with a non-zero value of two-loop gauge field β -function in the $(YM_4 + 22 \text{ adjoint scalars})$ theory (see e.g. [40]).

fields are mutually connected. This can be expressed by the following diagram (we assume that supersymmetry is irreducible)

where arrows stand for supersymmetry, relating bose and fermi fields and hence bose (and fermi) fields among themselves. As a consequence, no initial relations (e.g. between couplings) following from d-dim lagrangian, survive quantization if it is not for supersymmetry. For example, one can readily check that there is one-loop renormalization of scalar potential in the reduced analog of $YM_{2c}^{\mathbb{N}/}$.

We conclude this section with several remarks.

The results (3.8) for $SYM_V^{\mathbb{N}/}$ are in agreement with existence of $N=4$ sum rules [41, 42]: $b_{2k} \sim \sum_{\lambda} (-1)^{2\lambda} d(\lambda) 1^k = 0$, $k \leq 4$. The absence of $k=4$ sum rule gives a hint for $b_8 \neq 0$ to be confirmed in the next section. Next, let us note that the formal use of d-dimensional notations was already appreciated in 3-loop calculation of β -function in $SYM_V^{\mathbb{N}/}$ in [43]. Their one-loop result $\beta_1 \sim (d-10)$ clearly coincides with ours (cf. (3.9)). Finally, we once more want to stress that $d=10$ results (3.8) imply the corresponding ones for $d=4$ only for the simplest supersymmetry preserving reduction (2.0). Quantum properties of differently reduced theories (cf. [7,8]) are to be studied independently in four dimensions and are expected to be worse than to those of $SYM_V^{\mathbb{N}/}$.

$\mathbb{N}/O(26)$, which connected A_m and A_c , is broken by reduction. Quantum corrections respect only $O(22)$ symmetry and thus induce $(A_c^2)^2$ invariant in addition to $(F_V^a)^2$ -potential in (3.2).

(3b) Constant gauge field effective lagrangian

Here we are going to consider the effective action, corresponding to (3.3), supposing the following background for d=10 potential ($\mu, \nu = 1, \dots, 4$)

$$A_\mu^a = -\frac{1}{2} F_{\mu\nu} \chi_\nu h^a, (h^a)^2 = 1, A_i = 0, \quad (3.12)$$

and taking for simplicity the gauge group to be SU_2 . The one-loop effective action for a homogeneous background is given by (see e.g. [23, 44], cf. (2.5))

$$\Gamma^{(1)} = -\frac{\hbar}{2} \int_0^\infty \frac{ds}{s} \sum_k \text{tr} e^{-A_2^{(k)} s} = -\frac{\hbar}{2} \frac{V_d}{(m)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \Phi^{(1)}(s), \quad (3.13)$$

where V_d is the d-dimensional volume and Φ depends on background fields. It is easy to understand (cf. the Lemma and (2.14)) that for (3.12) Φ 's are the same in d-dimensional and reduced theories. Thus the only difference in Γ 's is in s-integration measure

$$\mathcal{L}_{10}^{(1)} = -\frac{\hbar}{2(m)^5} \int_0^\infty \frac{ds}{s^6} \Phi(s), \quad \mathcal{L}_4^{(1)} = -\frac{\hbar}{2(m)^2} \int_0^\infty \frac{ds}{s^3} \Phi(s). \quad (3.14)$$

Therefore we may use d=10 theory to calculate Φ .

Let us introduce the following notations (our signature is euclidean)

$$J_1 = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad J_2 = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}^*, \quad F_{\mu\nu}^* = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}^{}, \quad (3.15)$$

$$F_{1,2}^2 = J_1 \pm \sqrt{J_1^2 - J_2^2} \quad (3.16)$$

✓ We note in passing that Φ is gauge independent ((3.12) satisfies the classical field equations).

Note that $J_1 = \frac{1}{2} (\vec{E}^2 + \vec{H}^2) \geq J_2 = (\vec{E} \vec{H})$ and thus the eigen-values $\{\pm F_1, \pm F_2\}$ of $F_{\mu\nu}$ are real (the transition to pseudo-euclidean notations is given by $E \rightarrow iE$). The total expression for Φ follows from (3.3)

$$\Phi = \Phi_1 - \frac{1}{4} \Phi(\Delta_{1/2}), \quad \Phi_1 \equiv \Phi(\Delta_1) - 2\Phi(\Delta_0). \quad (3.17)$$

After some standard calculations (cf. [23-24]) we have

$$\Phi_0 \equiv \Phi(\Delta_0) = c \frac{sF_1}{sh sF_1} \cdot \frac{sF_2}{sh sF_2}, \quad c(SU_2) = 2, \quad (3.18)$$

$$\Phi(\Delta_{1/2}) = \nu \, ch sF_1 \cdot ch sF_2 \cdot \Phi_0, \quad \nu = 2^{[d/2]}, \quad (3.19)$$

$$\Phi_1 = (d-2)\Phi_0 + 4(s h^2 sF_1 + s h^2 sF_2)\Phi_0. \quad (3.20)$$

Here we used that $\Phi_0 = t_2 [\det \frac{sF}{sh sF}]^{1/2}$, $\Phi(\Delta_{1/2}) = t_2 e^{\pm \frac{1}{2} F_{\mu\nu} F_{\mu\nu} s} \Phi_0$, $\Phi_1 = [t_2 e^{2 F_{\mu\nu} s} - 2] \Phi_0$ with $F_{\mu\nu} = \{F_{\mu\nu}, 0\}$, and took $(\gamma_M)_{M=1,2,3,4} = (\gamma_\mu)_4 \otimes 1_8$. The final result reads

$$\Phi(s) = 4 (ch sF_1 - ch sF_2)^2 \Phi_0 \quad (3.21)$$

and thus, e.g., in four dimensions,

$$\mathcal{L}_4^{(1)} = -\frac{\hbar}{2} \cdot \frac{4c}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} \cdot \frac{sF_1}{sh sF_1} \cdot \frac{sF_2}{sh sF_2} (ch sF_1 - ch sF_2)^2 \quad (3.22)$$

This expression is remarkable by its ultra-violet (UV) finiteness, evident from small- s expansion of (3.21)

$$\Phi(s) \Big|_{s \rightarrow 0} \simeq 4c [s'(J_1^2 - J_2^2) - \frac{1}{90} s^4 (J_1^2 - J_2^2)(3J_1^2 + J_2^2) + O(s^6)] \quad (3.23)$$

This expansion gives also information about total $\bar{b}_p$ -coefficients, calculated on the background (3.12). In view of the obvious formula

(of. (2.3), (2.5), (3.13))

$$(\Phi(s))_{s \rightarrow 0} \simeq \sum_{k=0}^{\infty} s^k \bar{b}_{2k}, \quad (3.24)$$

and noticing that $\Phi(s)$ is even in S , we get

$$\bar{b}_p = 0, \quad p \leq 6, \quad \bar{b}_8 = \frac{4C}{16} [(F_{\mu\nu} F_{\mu\nu})^2 - (F_{\mu\nu} F_{\mu\nu}^*)^2], \quad (3.25)$$

$$\bar{b}_{12} = 0, \quad \bar{b}_{\nu\kappa+2} = 0, \quad \bar{b}_{\nu\kappa} \neq 0, \quad \kappa \geq 4.$$

The $\bar{b}_8 \neq 0$ result implies the quadratic divergence of $d=10$ effective lagrangian (3.14) (see also (2.6)). This explicitly demonstrates that SYM_{10}^1 is not finite in one loop. Note that the absence of logarithmic divergence of $\mathcal{L}_{10}^{(1)} (\bar{b}_0 = 0)$ is probably an artifact of our choice of background (3.12) (recall that (1.1) predicts one-loop logarithmic infinities in $d=10$).

Let us now discuss several properties of our effective lagrangian (3.22). As is evident from (3.15), (3.16), it vanishes if $F_{\mu\nu} = \pm F_{\mu\nu}^*$ ^{x/}. This fact is a particular case of the absence of one-loop radiative corrections on the (non-abelian) self-dual gauge field background in any supersymmetric gauge theory (which is due to supersymmetry relations between non-zero eigenvalues of scalar, spinor and vector operators, cf. [45, 25]) ^{**/}. Next, let us comment on $S \rightarrow \infty$ infra-red (IR) behaviour of (3.22). Expanding it in powers of F we get a series of divergent terms $\sum_{k=1}^{\infty} \mu^{-4k} F^{2k+2}$, where μ is IR cut-off (note, that the first F^4 -term is related to the four-particle amplitude also found to be IR divergent [17, 18]). We conclude that supersymmetry, though providing UV finiteness, does not improve singular IR behaviour, characteristic to all massless theories, except for the fact, that it cancels logarithmic IR divergences. Given a particular constant background problem, one could, of ^{x/} it is also invariant under duality transformations $F \rightarrow \pm F^*$, what is connected with helicity conservation in corresponding amplitudes (cf. [25]).

^{**/} zero modes also cancel in $N=4$ SYM theory [45]

course, consider the above power-type IR divergences as artificial ones, absent after summation of the series. This is really true for the partial scalar (3.18) and spinor (3.19) contributions in effective action. However, the integral over S for the gauge field contribution is divergent for $S \rightarrow \infty$ if $F \neq \pm F^*$. This is a manifestation of the gauge field IR instability of the background (3.12) (cf. [24, 27]), which should not, of course, be confused with "ordinary" IR divergences. This instability originates from the "anomalous magnetic moment" term ($-2F_{\mu\nu}$) in Δ_1 in (3.4) (or the second term in (3.20)) and may be attributed to the negative mode of Δ_1 . Thus we are to introduce some IR cut-off or to rotate the contour of integration ($S \rightarrow iS$) for the divergent part of the integrand in (3.22). The second recipe leads to an imaginary part in the effective lagrangian, implying the decay of the "vacuum" (3.12) by gauge particles pair creation (cf. [27]). As a result, supersymmetry does not also solve the problem of IR instability of constant abelian gauge field configurations. This could be expected from the observation that it is the total (regularized) number of one-loop fluctuation modes $B_4 = N_+ + N_- + N_0$ which is equated to zero by supersymmetry, and thus it is still possible to have a non-zero number of negative modes. The above conclusions may be considered as arguments in favour of the analogy in IR behaviour of YM_4 and SYM_4 , e.g., implying the confinement (in spite of zero β -function) in the second theory.

To illustrate the discussion let us now calculate (3.22) explicitly, assuming, e.g., that $\vec{E} \neq 0$, $\vec{H} = 0$. Then the total one-loop corrected lagrangian is

$$\mathcal{L}_1 = \frac{1}{2g^2} E^2 - \frac{\hbar g}{4\pi^2} E^2 \equiv \frac{1}{2g_{eff}^2} E^2, \quad (3.26)$$

where $a = I_1 + 2I_0 - 2I_{1/2}$, $I_{1,0,1/2} = \int_0^\infty \frac{dx}{x^2} \{ shx; \frac{1}{shx}; ethx \}$, where I_1 is to be defined by $x \rightarrow ix$. In view of UV finiteness of (3.22) there is no usual $E^2 \log E$ terms in (3.26) and thus there is no regularization ambiguity in the constant a . Calculation of the integrals yields (cf. [46])

$$\frac{1}{g_{eff}^2} = \frac{1}{g^2} + \frac{\hbar}{8\pi^2} [2\eta'(-1) - 8\eta(-1)\ln 2 + i\pi] \quad (3.27)$$

Note that if we have employed the IR cut-off by inserting $e^{-\mu^2}$ factor in (3.22), then the imaginary part of (3.27) would appear through $\lim_{\mu \rightarrow 0} E^2 \log \frac{\mu^2 - E}{\mu^2 + E}$. It should, of course, be understood that the simple structure of (3.26) is due to the condition $H=0$, while in general (3.22) will non-trivially depend on dimensionless combination E/H .

(3c) Effective potential for scalars

Our aim in this subsection is to consider the effective action for a constant scalar background

$$A_i = \text{const}, \quad A_\mu = 0 \quad (3.28)$$

illustrating the efficiency of d=10 approach and pointing some general facts about (super) symmetry breaking in SYM_4^* theory. According to (3.2) the classical scalar potential and its absolute minima are given by

$$V_0 = \frac{1}{4g^2} F_{ij}^a F_{ij}^a, \quad F_{ij} = [A_i, A_j] = 0 \quad (3.29)$$

(we assume the gauge group G to be compact and six internal dimensions to be space-like so that all scalars are physical and

$V_0 \geq 0$).^{*/} Note that in our case $F_{ij} = 0$ does not imply

^{*/} The classical equations, corresponding to $V_0, [A_i, [A_i, A_j]] = 0$, coincide with the Yang-Mills equations in (d-4)-dimensional gauge theory in the case of constant gauge potentials. They can have
(see next page)

"triviality" of A_i because after the reduction A_i 's transform homogeneously under the gauge transformations. All vacua (3.29) do not break supersymmetry ($V_0 = 0$) ^{*/} but may spontaneously break the gauge symmetry, giving masses to some fields, grouped in N=4 multiplets. For example, if we take $A_i = h_i A$, $h_i^2 = 1$, then for $G = SU_5$ and a suitable matrix A we get the standard $SU_3 \times SU_2 \times U_1$ symmetry breaking as (with the help of one adjoint Higgs multiplet) in the Georgi-Glashow model [48]. The mass matrix, corresponding to $F_{ij} = 0$, is the same for scalars, vectors and spinors

$$M^{ab} = f^{abc} A_i^b A_i^c \quad (3.30)$$

One can show that the minimal number of massless N=4 multiplets which survive the tree level gauge symmetry breaking is equal to the rank τ of G ($\tau = n$ for SU_n , SO_{2n} , E_n etc.). Therefore the potential (3.29) admits a "realistic" gauge symmetry breaking but still there are the problems of spin degeneracy (unbroken supersymmetry), of energy degeneracy ($V_0 = 0$) and also of scale degeneracy of vacua ($F_{ij} = 0$ does not fix the scale

"non-trivial" ($F_{ij} \neq 0$) real solutions only if G is non-compact and (or) the (d-4)-space signature is pseudoeuclidean (cf. [47]).

^{*/} in view of the form of supersymmetry transformations

$$\delta\psi = -\frac{1}{4\sqrt{2}} F_{MN} \gamma_{MN} \epsilon \quad \text{it is obvious that } F_{ij} \neq 0$$

is a necessary condition for a supersymmetry breaking (for appearance of goldstone fermion).

of $\pi/$ terms/logarithms of dimensionless variables. For example, if $A_i = m \bar{A}_i$ for dimensionless $\bar{A}_i$, then the total potential has the form

$$V = m^4 \left\{ \frac{1}{4g^2} \bar{F}_{ij}^2 + \frac{\pi}{6m^2} (\bar{M}_0^4 \log \bar{M}_0^2 + \dots) \right\} \quad (\text{the proof is based on (3.35)}).$$

Hence the potential calculated on the extrema defined by $\frac{\partial V}{\partial m} = 0$, $\frac{\partial V}{\partial \bar{A}_i} = 0$, is always equal to zero (and thus there is no supersymmetry breaking). This simple argument can be generalized to all loops, assuming that zero β -function property [38] (and thus scale invariance) of SYM_4^4 persists to more than three loops. This connection of ^{the} absence of supersymmetry breaking and scale invariance can be seen also from the following reasoning: if $\beta = 0$ then $T_\mu^\mu = 0$ but $\langle T^{\mu\nu} \rangle = V g^{\mu\nu}$ and thus $V = 0$, which implies the absence of supersymmetry breaking (cf. [50]). Turning this argument around we may say that a non-perturbative dynamical breaking of SYM_4^4 supersymmetry (if possible at all) must be accompanied by a breaking of scale invariance of this theory. (b) One can easily check that the classical minima (3.19) are also the solutions of effective equations, i.e. $V(F_{ij} = 0) = 0$ $\pi/$ (this is the analog of $N = 1$ supersymmetry result, saying that if the classical potential is zero in some point then the effective potential is also zero in this point [49]). (c) Combining the observations made $\pi/$

Let us also mention in passing that the effective potential (3.34) is gauge independent (independent of parameter of covariant gauge); this can be understood, e.g., as a consequence of the absence of the gauge coupling and wave function renormalization. $\pi/$ this follows from (3.31), (3.32) and the zero total number of degrees of freedom in the theory, or from the invariance of (3.34) under $F_{ij} \rightarrow F_{ij}$. The latter property implies that only $\phi_{\mu\nu}$ coefficients ($k \geq 3$) $\pi/$ are non-zero for background (3.28) (cf. (3.25)).

in (a) and (b) we conclude that all effective minima of $V = V_0 + V_1$ are exhausted by the classical ones (3.29). Really, suppose that $V(A)$ is a regular function of A_i^a , which is known to have two properties: (i) if $(\frac{\partial V}{\partial A})_{\tilde{A}} = 0$, then $V(\tilde{A}) = 0$, and (ii) there exists A_0 such that $V(A_0) = 0$. Then a simple theorem of analysis states that $\tilde{A}$ coincides with A_0 . This argument is valid, of course, to any order in loop expansion. Thus all possible

patterns of gauge symmetry breaking are only the classical ones (with $F_{ij} = 0$) discussed above. (d) Our next remark is about the IR instability of the background (3.28), i.e. the negative modes of M_0^2 (for $F \neq 0$), giving the imaginary part $\sim t_2 [\pi M_0^2 \cdot \theta(-M_0^2)]$ to (3.34). These negative modes can be proved to be present always when $F_{ij} \neq 0$, but the simplest way to predict their appearance is to recall the analogous instability (negative modes of the gauge field Δ_1 -operator) of the constant non-abelian gauge potential background in YM_4 [27, 28]. Indeed, the scalar operator in (3.32) appears as a part of d=10 gauge field one and thus the origin of instability is the same: the "anomalous magnetic moment" term $-2F_{MN}$ (see also sect. (3b)). This instability in our case is, however, a pure off-shell effect, because we already know that $F_{ij} = 0$ for all solutions of effective equations.

Let us now suppose that we added scalar and spinor mass terms in the classical lagrangian (3.2)

$$\Delta \mathcal{L} = \frac{1}{2g^2} M_0^2 \delta A_i^a A_i^a + M_{\frac{1}{2}} \bar{\psi} \psi \quad (3.36)$$

(here g^2 is needed in view of scalar kinetic term normalization in (3.2)). They produce soft supersymmetry breaking, i.e. do not spoil finiteness of the theory in the sense, that gauge coupling β -function is still zero and no field-dependent quadratic divergences are induced (cf. [54]). Moreover, it is possible to cancel logarithmic "mass" renormalizations properly adjusting M_0 and $M_{1/2}$. A remarkable fact is that now we can solve the previous problems of masses and vacua degeneracies and thus (in principle) construct a realistic unified model starting with $SYM_4^{N=1}$. The important consequence of the presence of bare scalar mass terms is a possibility to avoid negative modes of the scalar one-loop operator and hence to obtain a well-defined (real) effective potential which is now not forbidden to have non-trivial effective minima. Then we are in position to study the question of dynamical gauge symmetry breaking and to search for ^asolution of gauge hierarchy problem, generating some well separated effective scales, e.g. $A_1^2 \sim \mu^2 \exp(\pm C/g^2)$ or $A_1/A_2 \sim \exp(\pm \frac{C}{g^2})$ (for a recent proposal of its solution in the framework of unified models based on $N=1$ softly broken supersymmetry [55], see ref. [56]).

It is not in the aim of this paper to present such an analysis here. That is why we shall consider only illustrative examples, calculating (3.34) explicitly for two simple scalar backgrounds. First of all, let us note that in order to get finite (up to Λ -independent divergent constant) expression for effective potential,

^{a/} This proposal seems superior to those of refs. [7,8], where a coset reduction of SYM_{10}^1 was used to generate some d=4 theory with broken supersymmetry but with (very probably) bad quantum properties.

one is to relate μ_0 and $\mu_{1/2}$ (cf. (3.35)): $\sum_i \mu_{0,i}^2 = \frac{1}{4} \sum_i \mu_{1/2,i}^2$. This condition is assumed in the following. We start with the simplest ($F_{ij}=0$) $O(6)$ -preserving background: $A_i^a = n_i A^a$, $n_i^2 = 1$, and take $G = SU_2$, $\mu_{0,i}^2 = \delta_{ij} \mu^2$, $\mu_{1/2}^2 = \frac{3}{4} \mu^2 \delta_{\rho\rho}$. Diagonalizing (3.30) $M^2 = \{0, A^2, A^2\}$, $A^2 = A_a A_a$, we get the following expression for one-loop effective potential (3.34)

$$V = \mu^4 \left\{ \frac{1}{2g^2} x + \frac{\hbar}{64\pi^2} \left[3(1+x)^2 \log(1+x) + x^2 \log x - 4\left(x + \frac{1}{4}\right)^2 \log\left(x + \frac{3}{4}\right) \right] \right\} + \text{const}, \quad (3.37)$$

where $x = A^2/\mu^2$. The $A \neq 0$ extrema of V satisfy:

$$x \log \frac{x(x+1)^3}{(x+3/4)^2} + 3 \log \frac{x+1}{x+3/4} = -\frac{4\pi^2}{xg^2}$$

and thus are absent if $x \geq 0$. Taking formally $x \leq -1$ (i.e. $\mu^2 < 0$)

it is possible to find a unique solution if g^2 is some suitable large number (which is generally considered not to be a good choice). We observe stability of $A_i = 0$ classical solution, which can be related to the fact of UV finiteness of (3.37) (i.e. to the finite $x \rightarrow \infty$ limit of the quantum term in (3.37)). Our next example (now with $F_{ij} \neq 0$) is provided by the following background ($G, \mu_0, \mu_{1/2}$ are as above)

$$A_i^a = A \left[\delta_i^a - (\xi+1) n^a n_i \right], \quad a=1,2; \quad i=1,2; \quad n_i^2=1, \quad A, \xi = \text{const} \quad (3.38)$$

with all other components being zero. Then $F_{ij}^{ab} = \epsilon^{ab} \epsilon_{ij} A^2 \xi$, $M_{ab}^2 = A^2 \left[\xi^2 \delta_{ab} - (\xi-1) h_a h_b \right]$. The eigenvalues of mass matrices in (3.31), (3.32) (where we put $\gamma_i = (\text{Pauli matrices}) \otimes \mathbf{1}_{16}$) are given by

$$M^2 = \{A^2 \xi^2, A^2, 0\}; \quad M_0^2 = \{4 \times (A^2 \xi^2 + \mu^2), 4 \times (A^2 \mu^2), 2 \times (A^2 \lambda_0^+ + \mu^2), 2 \times (A^2 \lambda_0^- + \mu^2), 6 \times \mu^2\}, \quad (3.39)$$

$$M_{1/2}^2 = \{32 \times (A^2 \lambda_{1/2}^+ + \frac{3}{4} \mu^2), 32 \times (A^2 \lambda_{1/2}^- + \frac{3}{4} \mu^2), 32 \times (\frac{3}{4} \mu^2)\},$$

where $\lambda_0^\pm = \frac{1}{2} \left((\xi^2 + 1 \pm \sqrt{(\xi^2 - 1)^2 + 16\xi^2}) \right)$, $\lambda_{1/2}^+ = 1 + \xi^2$, $\lambda_{1/2}^- = 0$. It is now

easy to establish the effective potential, using (3.34),
(3.35)

$$V = \mu^4 \left\{ \frac{1}{2g^2} (x+y+xy) + \frac{\hbar}{32\pi^2} \left[2(y+1)^2 \log(y+1) + \right. \right. \quad (3.40)$$

$$+ 2(x+1)^2 \log(x+1) + (x\lambda_0^+ + 1)^2 \log(x\lambda_0^+ + 1) + (x\lambda_0^- + 1)^2 \log(x\lambda_0^- + 1) +$$

$$+ y^2 \log y + x^2 \log x - 4(x+y + \frac{3}{4})^2 \log(x+y + \frac{3}{4}) \left. \right] \Big\} + \text{const}$$

where $x = A^2/\mu^2$, $y = x \frac{1}{\lambda^2}$. One may easily check that $\mathcal{O}(\hbar)$ part of (3.40) (as well as (3.37)) can be rewritten in terms of log's of ratios of polynomials having equal degrees and thus is bounded when $x, y \rightarrow \infty$. The $\mu \rightarrow 0$ limit of (3.40) serves as a good illustration of the statements made above concerning pure SYM_4 case. If $\mu^2 < A^2 \lambda_0^-$ we get the already discussed negative modes. That is why we are to relate x and λ properly in order to preserve the reality of (3.40). The analysis of extrema of (3.40) shows that no solutions exist for an arbitrary value of g^2 (except the classical one $x=y=0$). At the same time it is possible to find the approximate minima if g^2 is sufficiently large, while $\lambda \gg 1$, $x < \frac{1}{2}$ and $y < 1$. Thus we get dynamical SU_2 symmetry breaking and three different mass scales $\mu^2, \mu^2 x, \mu^2 y$. However, this is not an "exponential" ($\sim \exp \frac{1}{g^2}$) hierarchy, which, in fact, seems to be impossible if we demand the UV finiteness of ^{the} potential. It remains to be seen whether it can be generated, if we omit the relation between μ_0 and μ_{λ_2} and consider a realistic example of $G = SU_5$ or E_6 .

4. Supergravities in $d \leq 11$: one-loop divergences and anomalies

The subject of this section is analysis of infinities (i.e. calculation of b_p -coefficients in (2.6), (2.4)) in higher dimensional supergravity theories on a d-dimensional gravitational

background. These theories are known to contain antisymmetric tensor gauge fields and gravitinos along with graviton itself. One-loop results, therefore will be given by a sum of separate contributions of all these fields. Finally, using the Lemma of sect.2 we will be able to obtain some information concerning the corresponding reduced theories. We shall use the following notations

$$\bar{b}_0 = N, \quad \bar{b}_2 = \int R, \quad (4.1)$$

$$\bar{b}_4 = \alpha_1 R_{MN PQ}^2 + \alpha_2 R_{MN}^2 + \alpha_3 R^2 + \alpha_4 \partial^2 R, \quad (4.2)$$

$$\bar{b}_6 = G_1 I_1 + G_2 E, \quad (4.3)$$

where in (4.3) we used the definitions (A.5), (A.7), assumed that $R_{MN} = 0$ (we will compute $\bar{b}_6$ only on mass shell) and omitted all total derivative terms (as was already done in (2.9)). It should be understood that $\bar{R}R = R_{MN PQ}^2 - 4R_{MN}^2 + R^2$ (E in (A.7)) is the integrand of the Euler number (A.8) and thus may be neglected for topologically trivial backgrounds only if $d = 4$ ($d=6$). Let us also remind that only one (two) of the first three invariants in (4.3) are independent (while I_1 and E vanish on shell) when $d=2$ ($d=3$). One more remark is that $I_2 = 0$ if $d=4$ and $R_{MN} = 0$ [57] and hence $E(d=4) = I_1$.

(4a) Gravity in d dimensions

The background field method one-loop quantization of gravity in d dimensions can be done straightforwardly in the gauge

$$\partial_M (h_{MN} - \frac{1}{2} g_{MN} h) = \xi_N(x),$$

$$Z = \frac{\det \Delta_g}{\sqrt{\det \Delta_h}}, \quad \Delta_{gMN} = -g_{MN} \partial^2 - R_{MN}, \quad (4.4)$$

$$\Delta_h = -P \cdot \partial^2 + X, \quad P_{PQ}^{MN} = \delta_{(P}^{M} \delta_{Q)}^{N} - \frac{1}{2} g^{MN} g_{PQ}, \quad (4.5)$$

$$X_{PQ}^{MN} = 2R_{P(MN)Q} - 2\delta_{(P}^{(M} R_{Q)}^{N)} + g_{MN} R_{PQ} + g_{PQ} R_{MN} + P_{PQ}^{MN} R.$$

We see that eqs. (4.4)-(4.5) are universal in the chosen gauge, i.e. are the same as, e.g. in $d=4$ case. Now we are to use eqs. (2.7)-(2.9) (with $(F_{MN})_{KS}^{PQ} = 2\delta_{(K}^{(P} R_{S)MN}^{Q)}$ etc.) providing the following results for $\bar{b}_p^{(h)} = \bar{b}_p(A_h) - 2\bar{b}_p(A_3)$ (cf. (4.1)-(4.3))

$$N = \frac{1}{2}d(d-3), \quad \rho = \frac{1}{12}(-5d^2 + 9d - 48), \quad (4.5)$$

$$\alpha_1 = \frac{1}{180} \cdot \frac{d(d-3)}{2} - \frac{1}{12}(d-18), \quad \alpha_2 = \frac{1}{360}(-d^2 + 543d - 3600) \quad (4.6)$$

$$\alpha_3 = \frac{1}{144}(43d^2 - 303d + 696), \quad \alpha_4 = \frac{1}{60}(-4d^2 + 7d - 40), \quad (4.7)$$

$$\zeta_1 = \frac{1}{15120} \cdot \frac{d(d-3)}{2}, \quad \zeta_2 = \frac{1}{3240} \cdot \frac{d(d-3)}{2} - \frac{(d+30)}{180}, \quad (4.8)$$

which are in agreement with the previously known ones for $d=4$ ($b_2: [15, 19]$, $b_4: [58, 59]$) and for $d=6$ (in $b_6: [60, 59]$)^{*/}. Evidently, only N , α_1 , ζ_1 and ζ_2 are gauge independent (they contribute on shell). Recalling the meaning of b_p 's (cf. (2.6)) we immediately conclude that: (i) $d = 2, 3$ and 4 gravities are one-loop finite on-shell (the latter-up to topological divergence); (ii) all gravities in $d > 4$ dimensions have at least $\int^{d-4}$ and $\int^{d-6}(\log L^2 \text{ for } d=6)$ divergences on shell. Our results (4.5)-(4.8) give complete expressions for infinity in $d \geq 7$ theories according to (2.6) the weakest linear infinity in $d = 2k+1$ case is governed by b_{2k} . Put in this perspective the one-loop finiteness of $d=4$ gravity appears to be an accident. Thus one should strongly believe in "uniqueness" of $d=4$ number for higher loops (contradicting complete universality of one-loop expression

^{*/} it should be pointed out that $\zeta_2(d=6)$ was not computed in ref[60, 59] because their authors assumed that the integral of E is zero (cf. (A.9)). However, it is erroneous to omit E in the local expression for $d=6$ gravitational conformal anomaly $\Gamma_P^A = \bar{b}_6(x)$ (compare with $d=4$ expression $\Gamma_P^A = \bar{b}_4 = \alpha_1 R\bar{R}^{**} + \dots$).

(4.4), (4.5)) in order to start with two loop calculations.

An interesting question connected with quantization of d-dimensional gravity is about divergences in reduced Kaluza-Klein (KK) theories, following by reduction from the classical lagrangian ($\sim R$) in d-dimensions (see e.g. [9,3]). A priori one may hope that these theories have better quantum behaviour as compared to some general system of gravity, scalars and vectors (known to be one-loop infinite even on shell [58, 61]) because classical mass shell of reduced theory coincides with that of d-dimensional one ($R_{MN} = 0$). This conjecture is not however, supported by above statement on the absence of finiteness in higher dimensions. ^{M/} Strictly speaking, this result cannot be immediately applied to the reduced theory (it says only that the total KK theory, i.e. with all massive states, is infinite). Moreover, as it follows from discussion in sect.2, divergences of reduced theory generally cannot be written only in terms of d-dimensional objects. That is why one must do explicit calculation (in four dimensions) to settle the question. It seems worth presenting the outcome of this calculation of divergences in the simplest reduced KK theory, following from d=5 gravity

$$\mathcal{L} = \left(-\frac{1}{\kappa^2} R + \frac{1}{4g^2} F_{\mu\nu}^2(B) e^{2\alpha\varphi} + \frac{1}{2} g_{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \right) \sqrt{g}, \quad (4.9)$$

where $\alpha = \frac{\sqrt{3}}{2}$, $\kappa^2 = 16\pi G$. Let us consider the following background: $g_{\mu\nu} = \delta_{\mu\nu}$, $B_\mu = 0$, $\varphi = 0$. Then the quantum scalar field contribution in the one-loop infinities (which should be added to that of the Einstein-Maxwell system [61]) is given by

^{M/} the presence of R_{MNPQ}^2 - on shell infinities is a hint for e.g. $F^4(B)$ - divergences in reduced theory (for notations see (2.11)).

$$\Delta \bar{\mathcal{L}}_4 = \frac{1}{6} a^2 (\partial_\mu F_{\mu\nu})^2 - \frac{a^2}{3} T_{\mu\nu} R_{\mu\nu} + 4a^2 T_{\mu\nu}^2 + \\ + \frac{a^4}{6} (F_{\lambda\rho} F_{\mu\rho})^2 + \frac{7}{12} a^4 (F_{\mu\nu}^2)^2, \quad T_{\mu\nu} \equiv F_{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^2 \quad (4.10)$$

Thus the theory is still infinite on shell. It has even worse divergence structures (the last two terms in (4.10)) because of breaking of duality invariance by the vector-scalar coupling in (4.9). The lessons we can draw from this example are the following: (i) the fact that some theory is obtained by dimensional reduction does not by itself imply an improvement of situation with infinities; (ii) the inclusion of scalar-vector couplings does not by itself imply better one quantum behaviour. As a consequence, one-loop on shell finiteness of $\mathcal{S}G_N^N$ ($N \leq 8$) is due to (i) one-loop on shell finiteness of d=4 gravity and (ii) irreducible supersymmetry, connecting all sectors of S-matrix with finite gravitational one (as to the fact that R^2 -invariants are built up to superinvariants) and should not be specially attributed to the possibility of obtaining these theories from $\mathcal{S}G_{11}^1$ by a reduction (and truncation) ^{*/}. This conclusion is in agreement with the observation made in sect. 3a, that it is supersymmetry which provides finiteness of SYM_4^4 as compared to $(\mathcal{M}_4 + 22 \text{ scalars})$ theory, both theories being reductions of SYM_{10}^1 and YM_{26} correspondingly. To reiterate, even if $(g_{\mu\nu} - B_\mu)$ -sector of (4.10) was finite we would have troubles in the scalar one, because of breakdown of initial d-dimensional symmetry by the reduction. The following conjecture suggests itself in this connection: if it is possible to construct a theory with some symmetry connecting gravity with a finite

^{*/} in particular, it is supersymmetry that provides duality invariance of scalar-vector interactions in $\mathcal{S}G$'s.

number N of bose fields (such theories with $N = \infty$ are, of course, known being complete KK theories as seen from lower dimensions, cf. [11]) then it may have an improved one-loop on shell quantum behaviour characteristic to supergravities.

(4b) Antisymmetric tensor gauge fields

The correct recipe for quantization of antisymmetric tensor gauge field $A_{N_1 \dots N_h}$ ($h < d$) with the lagrangian $\mathcal{L} \sim (\partial_{[N} A_{N_1 \dots N_h]})^2$ on d -dimensional gravitational background is given by (cf. [62, 29])

$$Z^{(h)} = \prod_{k=0}^h [\det \Delta_{h-k}]^{-\frac{1}{2}(-1)^k(k+1)} \quad (4.11)$$

where the Hodge-DeRham operators Δ_p are defined as follows

$$(\Delta_p)_{N_1 \dots N_p}^{M_1 \dots M_p} = -\delta_{[M_1}^{N_1} \dots \delta_{M_p]}^{N_p} \partial^2 + \sum_{i=1}^p R_{[M_i}^{N_i} \delta_{M_1 \dots M_p]}^{N_1 \dots N_p} - \sum_{i>j=1}^p R_{[M_i}^{N_i} \delta_{M_j]}^{N_j} \delta_{M_1 \dots M_p}^{N_1 \dots N_p} \quad (4.12)$$

($\delta_{M_i}^{N_i}$ are omitted in sums). For example, $\Delta_0 = -\partial^2$, $\Delta_1 = -g_{MN} \partial^2 + R_{MN}$, etc. To compute the infinite part of (4.11) one must first establish $\bar{\ell}_p(\Delta_k)$ and then to find the total result, which according to (4.11), is

$$\bar{\ell}_p^{(h)} = \sum_{k=0}^h \bar{\ell}_p(\Delta_{h-k}) C_{-2}^k, \quad (4.13)$$

where we introduced binomial coefficients C_s^z obeying ($h, k = 1, 2, \dots$, $z, s = 0, \pm 1, \dots$)

$$C_z^h = \frac{z(z-1) \dots (z-h+1)}{h!}, \quad C_z^{-h} = 0, \quad C_z^0 = 1, \quad (4.14)$$

$$C_k^h = C_k^{h-k}, \quad C_h^{h+k} = 0, \quad C_{z+s}^h = \sum_{k=0}^h C_z^{h-k} C_s^k, \quad (4.15)$$

$$C_{-1}^k = (-1)^k, \quad C_{-2}^k = (-1)^k(k+1), \quad C_{d-2-p}^{h-m} = \sum_{k=0}^{h-m} C_{d-p}^{h-m-k} C_{-2}^k \quad (4.16)$$

Employing again (2.7)-(2.9), (4.1)-(4.3) we get

$$N = C_{d-2}^n, \quad \rho = \frac{1}{6} C_{d-2}^n - C_{d-4}^{n-1}, \quad (4.17)$$

$$d_2 = \frac{1}{180} C_{d-2}^n - \frac{1}{12} C_{d-4}^{n-1} + \frac{1}{2} C_{d-6}^{n-2}, \quad (4.18)$$

$$\alpha_2 = -\frac{1}{180} C_{d-2}^n + \frac{1}{2} C_{d-4}^{n-1} - 2 C_{d-6}^{n-2}, \quad (4.19)$$

$$\alpha_3 = \frac{1}{72} C_{d-2}^n - \frac{1}{6} C_{d-4}^{n-1} + \frac{1}{2} C_{d-6}^{n-2}, \quad \alpha_4 = \frac{1}{30} C_{d-2}^n - \frac{1}{6} C_{d-4}^{n-1},$$

$$G_1 = \frac{1}{15120} C_{d-2}^n, \quad G_2 = \frac{1}{3240} C_{d-2}^n - \frac{1}{180} C_{d-4}^{n-1} + \frac{1}{6} C_{d-6}^{n-2} - \frac{2}{3} C_{d-8}^{n-3} \quad (4.20)$$

If $d=4$ and $n \leq 3$ our results for $\bar{b}_p$ are in agreement with those of ref. [29]. To restore the analogous expressions for $\bar{b}_p(\Delta_n)$

($n \leq p/2$) one should simply substitute $d \rightarrow d+2$ in (4.17)-(4.20).

In view of duality invariance of Δ_n we have $\det \Delta_n = \det \Delta_{d-n}$, i.e. $\bar{b}_p(\Delta_n | d) = \bar{b}_p(\Delta_{d-n} | d)$. However, in general $Z^{(n)} \neq Z^{(d-2-n)}$. Let us make a precise statement of this "quantum inequivalence" ^{*}

$$\bar{b}_p^{(n)}(d) = \bar{b}_p^{(d-2-n)}(d), \quad p < d \quad (4.21)$$

$$\bar{b}_p^{(n)}(d) - \bar{b}_p^{(d-2-n)}(d) \begin{cases} = (-1)^n (n+1 - d/2) H_p(d), & p \geq d, \\ & d = \text{even} \\ \neq 0, & d = \text{odd} \end{cases}$$

where

$$H_p(d) \equiv \sum_{k=0}^d (-1)^k \bar{b}_p(\Delta_k | d) \begin{cases} = 0, & p < d \\ = E_d, & p = d \\ \neq 0, & p > d \end{cases} \quad (4.22)$$

for $d = \text{even}$ and $H_p(d) = 0$ for $d = \text{odd}$. E_d in (4.22) is the

^{*} comparing different terms in $\bar{b}_p$'s for $d = 2, 3$ one is to remember that not all invariants (e.g. in $\bar{b}_p$) are independent and thus these relations hold only for complete $\bar{b}_p$ -coefficients.

integrand of the Euler number (A.8). The proof of (4.21) is evident from (4.13), while (4.22) is justified in [30]. To provide understanding of these relations it is useful to consider them as consequences of the general formulas ($p=2 \cdot n$)

$$\bar{b}_p(\Delta_n | d-2) = \bar{b}_p^{(n)}(d), \quad p < d$$

$$\bar{b}_p(\Delta_n | d) = \sum_{\alpha} a_{\alpha,p} I_{\alpha,p}, \quad \bar{b}_p^{(n)}(d) = \sum_{\alpha} a_{\alpha,p}^{(n)} I_{\alpha,p}, \quad (4.23)$$

$$a_{\alpha,p} = \sum_{k=0}^{p/2} \gamma_{\alpha,p,k} \left(C_{d-2k}^{n-k} + C_{d-2k}^{d-n-k} \right), \quad (4.24)$$

$$a_{\alpha,p}^{(n)} = \sum_{k=0}^{p/2} \gamma_{\alpha,p,k}^{(n)} \cdot C_{d-2-2k}^{n-k}, \quad p \geq d,$$

where $I_{\alpha,p}$ are " $R^{p/2}$ " curvature invariants and $\gamma_{\alpha,p,k}$'s are universal numbers (of. (4.17)-(4.20)). Now it is clear that

$n \leftrightarrow d-2-n$ equivalence in (4.21) is based on the property $C_{d-2-2k}^{n-k} = C_{d-2-2k}^{d-2-n-k}$ which holds only for $d-2-2k > 0$. Therefore the $d=p$ anomaly in (4.21), (4.22) is due to the last $(C_{d-2-p}^{n-p/2})$ term in (4.24). Let us give some explicit examples of this anomaly: if $d=4$ we get the results of ref. [29] (see also [32, 5]): $\bar{b}_4^{(3)} - 0 = -2 \bar{E}_4$, $\bar{b}_4^{(2)} - \bar{b}_4^{(0)} = \bar{E}_4$;

for $d=6$ we have: $\bar{b}_6^{(5)} - 0 = -3 \bar{E}_6$, $\bar{b}_6^{(4)} - \bar{b}_6^{(0)} = \bar{E}_6$, $\bar{b}_6^{(3)} - \bar{b}_6^{(1)} = -\bar{E}_6$ (these relations can be also obtained using (4.20) (A.7), (A.9)). Next, to illustrate (4.21), (4.22) for $p > d$ we put $d=4$, $p=6$; then $\bar{b}_6^{(3)} - 0 \neq 0$, $\bar{b}_6^{(2)} - \bar{b}_6^{(0)} \neq 0$;

as follows, e.g., from (4.20). This probably implies that "quantum inequivalence" (for instance, in $d=4$) is valid /not only for conformal anomalies but also for finite parts of effective actions (which are governed by $\bar{b}_p$'s, $p > 4$).

Let us mention the following useful check of d -and n -depend-

dence in (4.24). One first observes that $A_{\mu_1, \dots, \mu_n}$ can be described as a system of fields in $m < d$ dimensions: $\{A_{\mu_1, \dots, \mu_n}; (d-m) A_{\mu_1, \dots, \mu_{n-1}}, \dots; C_{d-m}^n A\}$, $\mu = 1, \dots, m$, and thus can employ the formula

$$\bar{b}_p^{(n)}(d) = \sum_{k=0}^n C_{d-m}^k \bar{b}_p^{(n-k)}(m) \quad (4.25)$$

Then the consistency of (4.24) and (4.25) follows simply from the "sum rule" in (4.15). It is interesting also to note that using (4.25) one can compute $\bar{b}_p^{(n)}(d)$ given $\bar{b}_p^{(k)}(d)$ for all $k < n$ and $\bar{b}_p^{(n)}(m)$ for some $m < d$. Our final remark is that this "reduction" procedure works also in the calculation of gravitational contribution (sect. 4a). Indeed, $g_{\mu\nu} \rightarrow \{g_{\mu\nu}; (d-m) B_{\mu\nu}; \frac{1}{2}(d-m)(d-m+1)\varphi\}$ (cf. (2.11)) and so

$$\bar{b}_p^{(h)}(d) = \bar{b}_p^{(h)}(m) + (d-m) \bar{b}_p^{(h)}(m) + \frac{1}{2}(d-m)(d-m+1) \bar{b}_p^{(h)}(m) \quad (4.26)$$

Thus all we are to know ~~the~~ gravitational, vector and scalar contributions, e.g., for $m = 4$. However, one can notice from (4.5)-(4.8) and (4.17)-(4.20) that (4.26) is valid only on the gravitational mass shell $R_{\mu\nu} = 0$. The reason is that d -dimensional background gravitational gauge contains quantum vectors and scalars, being written in terms of m -dimensional fields and therefore the expressions in different sides of (4.26) are computed in different gauges. This is in contrast with (4.25) which holds for any external metric.

(4c) quantization of gravitino in d-dimensions

The gravitino lagrangian on d -dimensional metric background is defined by

$$\mathcal{L} = \bar{\psi}_M \gamma^{\mu\nu\kappa} \partial_\nu \psi_\kappa \quad (4.27)$$

where $\mathcal{D}_M \Psi_N = \partial_M \Psi_N + \Gamma_{KM}^N \Psi_K + \frac{1}{4} \delta_{AB} \omega_M^{AB} \Psi_N$
 and γ -matrix algebra is given in Appendix B. The action is
 invariant under $\delta \Psi_M = \mathcal{D}_M \epsilon$ up to the terms vanishing on $R_{MN}=0$.
 That is why if we fix a background gauge and calculate the effective
 action it will be gauge independent only on shell $R_{MN}=0$
 (for instance, $\alpha_1, \alpha_3, \alpha_4$ in (4.2) will be gauge dependent, cf. [63]).
 The main idea in choosing the gauge is to obtain a simple propa-
 gator, i.e. to bring a gravitino operator to the general form
 (2.2). This turns to be somewhat non-trivial for $d \neq 4$. Indeed,
 let us follow the well-known $d=4$ recipe [64], taking the gauge
 $\delta_M \Psi_M = \zeta^{(v)}$ and averaging over ζ with the help of the $\hat{\mathcal{D}} =$
 $= \delta_M \mathcal{D}_M$ - operator

$$\mathcal{L} + \mathcal{L}_{g.e.} = \bar{\Psi}_M D_{MN}^{(f)} \Psi_N, \quad D_{MN}^{(f)} = \sum_{MKN}^{(f)} \mathcal{D}_K, \quad (4.28)$$

$$\sum_{MKN}^{(f)} = \delta_{MKN} + \xi \delta_M \delta_K \delta_N = -\frac{1}{2} (\delta_N \delta_K \delta_M + \eta \delta_M \delta_K \delta_N), \quad \eta = -2(\xi + \frac{1}{2}) \quad (4.29)$$

where ξ (or η) is a gauge parameter. The corresponding parti-
 tion function is (assuming Ψ_M to be Majorana)

$$Z = \sqrt{\det D^{(f)}} / \sqrt{(\det \hat{\mathcal{D}})^3} \quad (4.30)$$

The usual $d=4$ gauge choice is $\eta = 0$ ($\xi = -\frac{1}{2}$) [64, 63],
 because it provides diagonality of the " $\mathcal{D}^2$ "-term in the
 "squared" $D^{(f)}$ -operator. This is based essentially on the identity
 (B.6) showing the absence of "non-diagonal" terms for $d=4$. One
 can easily convince himself that non-diagonal terms however are
 unavoidable for any ξ and $d \neq 4$.

The solution of this problem comes from the observation that
 if we substitute $D^{(f)}$ by the operator $\tilde{D}_{MN} = \Lambda_{MK} D_{KL}^{(f)} \Lambda_{LN}$,
 where $\Lambda = \bar{\Lambda}^T$ is an algebraic operator, then we do not change

the non-trivial dependence of $\bar{Z}$ on the metric ($\log \det \Lambda \sim \delta^{(d)/2}$).
Namely, let us take

$$\Lambda_{MN} = g_{MN} + a \delta_M \delta_N, \quad \bar{\Lambda}_{MN} = \Lambda_{NM} (\bar{\delta}_M = -\delta_M) \quad (4.31)$$

The idea is to choose constants a and ξ in order to simplify $\bar{D}$. Direct computation shows that (cf. Appendix B)

$$\bar{D}_{MK} = \left[g_{MK} \delta_N - \frac{1}{2} (\delta \delta_M \delta_N \delta_K + \gamma_C \delta_M g_{KN}) \right] \partial_N,$$

where $\delta = \gamma (1 + a d)^2 - (2a + a^2 d)(d-4) + 2a^2(d-2) - 1$,
 $C = 1 + a(d-2)$. The condition $\delta = C = 0$ has only one solution

$$a = -\frac{1}{d-2}, \quad \gamma = \frac{1}{2}(d-4), \quad \text{i.e. } \xi = -\frac{1}{4}(d-2). \quad (4.32)$$

Hence we conclude that for all $d > 2$ there exists the "standart gauge" (4.32), where the gravitino operator has its simplest form^{*}

$$\begin{aligned} D_{MN} &= g_{MN} \hat{D}, \\ \Delta_{3/2, MN} &= -(\hat{D}^2)_{MN} = -g_{MN} \hat{D}^2 + \frac{R}{4} g_{MN} - \frac{1}{2} \delta_{KL} R_{MN}^{KL}. \end{aligned} \quad (4.33)$$

As a result, we can use this operator, for example, in off shell calculations of divergences. It is important to note that this observation simplifies even the $d = 4$ calculations, e.g. of the gravitino contribution in anomalies (compare with the approaches of refs. [64, 65] ^{and} in off shell divergences (cf. [63]).

All left is to use (4.33) with (2.2), (2.7)-(2.9) in order to calculate the gravitino contribution in $\bar{\mathcal{L}}_P$'s

$$\bar{\mathcal{L}}_P^{(3/2)} = -\frac{1}{\delta} \left[\bar{\mathcal{L}}_P(\Delta_{1/2}) - 3 \bar{\mathcal{L}}_P(\Delta_{3/2}) \right], \quad (4.34)$$

^{*} thus the usual $d=4$ gauge $\xi = 0$ appears to be distinguished by its connection with the "standart" one.

where $\Delta_{1/2} = -\hat{D}^2 = -\partial^2 + \frac{R}{4}$ and $\delta = 1, 2, 4$ for Dirac, Majorana and Majorana-Weyl cases as in (3.7). The final expressions for $\bar{\phi}_p^{(1/2)}$ are (of. (4.1)-(4.3)) ($\nu = 2^{(d/2)}$)

$$N = -\frac{\nu}{\delta} (d-3), \quad \rho = -\frac{1}{12} N, \quad (4.35)$$

$$\alpha_1 = \frac{1}{180} N + \frac{\nu}{\delta} \frac{(d-19)}{96}, \quad \alpha_2 = -\frac{1}{180} N, \quad \alpha_3 = \frac{1}{288} N, \quad \alpha_4 = -\frac{1}{120} N, \quad (4.36)$$

$$\phi_1 = \frac{1}{15120} N, \quad \phi_2 = \frac{1}{3240} N + \frac{\nu}{\delta} \frac{(d+5)}{1440}, \quad (4.37)$$

while for $\bar{\phi}_p^{(1/2)} = -\frac{1}{\delta} \bar{\phi}_p(\Delta_{1/2})$ we get

$$N = -\frac{\nu}{\delta}, \quad \alpha_1 = \frac{1}{180} N + \frac{\nu}{\delta} \frac{1}{96}, \quad \phi_2 = \frac{1}{3240} N + \frac{\nu}{\delta} \frac{1}{1440}, \quad (4.38)$$

with all other coefficients having the same structure as above.

If $d=4$ these values are the same as in [64-65] (α_1) and in [61]

($\alpha_2, \alpha_3, \alpha_4, \rho$). Finally, let us note that the following "reduction" relation holds; $\bar{\phi}_p^{(1/2)}(d) = \bar{\phi}_p^{(1/2)}(m) + (d-m) \bar{\phi}_p^{(1/2)}(m)$

(of. (4.25), (4.26)) if we omit the ν/δ -factors.

(4d) Applications

How it is possible to compute the leading L^{d-p} , $p \leq 6$ one-loop gravitational infinities (i.e. ϕ_p , $p \leq 6$) for different systems of fields. Remarkable cancellations are known to occur in $d=4$ supergravities and thus it is natural to start with the maximal possible one in $d > 4$, i.e. $SG_{11}^4(66,2)$ (maximal SG 's in $d < 11$ can be obtained by a reduction, while all others - by a reduction and truncation, of this theory). It contains one graviton $g_{\mu\nu}$, one Majorana gravitino ψ_μ and one antisymmetric gauge tensor A_{MN} . Making use of (4.5)-(4.8), (4.17)-(4.20) and (4.35)-(4.37) we find for $\phi_p = \phi_p^{(1)} + \bar{\phi}_p^{(1)} + \bar{\phi}_p^{(1/2)}$ at $d=11$:

$$N = 44 + 84 - 128 = 0, \quad \rho = -\frac{85}{2}, \quad \alpha_1 = \frac{1}{180} (149 + 219 - 368) = 0,$$

$$\alpha_2 = 7, \alpha_3 = \frac{421}{24}, \alpha_4 = -\frac{85}{12}, \epsilon_1 = 0, \epsilon_2 = \frac{1}{180}(3+93-96) = 0,$$

and so, on shell ($R_{\mu\nu} = 0$)

$$\bar{b}_0 = \bar{b}_2 = \bar{b}_4 = \bar{b}_6 = 0 \quad (4.39)$$

Thus there is no $L^{11}, \dots, L^5$ one-loop divergences in eleven dimensional supergravity. ^{*}/Using the Lemma (2.14) we conclude that (4.39) as the property of d=11 theory implies the same one for all lower dimensional theories which can be obtained by the reduction, i.e. for all maximal supergravities. For example, (4.39) is valid for SG_4^8 . Then the result $b_4 = 0$ is recognized as the vanishing of anomalies in the version of SG_4^8 directly following from dimensional reduction (i.e. containing $63 A, 7 A_{\mu\nu}, 1 A_{\mu\nu\lambda}$) [31;33]. Thus we understand the absence of anomalies (or topological counter-terms) in SG_4^8 as a consequence of the absence of L^2 -infinities in SG_{11}^1 ^{xxx}/. The new non-trivial result is $b_6(SG_4^8) = 0$ (which holds again only for the "reduction" version of SG_4^8 because $\bar{b}_6^{(1)}(4) \neq 0$, $\bar{b}_6^{(2)}(4) - \bar{b}_6^{(1)}(4) \neq 0$ cf. sect. 4b). ^{xxx}/

Next we pass to maximal supergravities in d=5,6,7. According to (2.6) and (4.39) we conclude that SG_5^8 [68], SG_6^4 and SG_7^4 [69]

^z/ We remind that $\beta, \alpha_2, \alpha_3, \alpha_4$ are gauge dependent; there exists a supergauge where they are zeroes and so (4.39) is valid off shell.

^{xx}/ Let us note that $\alpha_2(SG_4^8) = 0$ off shell result of ref. [63] does not contradict $\alpha_2(SG_{11}^1) \neq 0$ because the d=4 and d=11 gauges are different.

^{xxx}/ to illustrate the meaning of b_6 in d=4 case suppose we give a large mass M to all fields in the theory. Then expanding the effective action in powers of $1/M$ we get $\frac{1}{M^2} b_6$ as a first non-trivial term (for applications of this fact see, e.g., [19,67]). Also, the vanishing of b_6 is connected with the vanishing of 3-particle

(see next page)

are one-loop finite in corresponding number of dimensions and hence provide the first examples of $d > 4$ -dimensional one-loop finite gravitational theories (recall that pure gravity is not finite if $d > 4$ and so the finiteness of these theories is due to cancellations and not merely to non-existence of non-zero on shell invariants as in the case of $d=4$ SG_5). Therefore, it is this class of theories which may be considered as a natural candidate for Kaluza-Klein program. As for on shell finiteness of SG_{11}^4 conjectured in [5,6], it seems rather unprobable in view of $\beta(SG_{11}^4) \neq 0$ property (which is supported by the analogy with SYM_{10}^4 case and also by non-zero result for 4-particle amplitude in SG_7^6 [17, 18]).

One important clarification is needed concerning above finiteness conclusion. It is irreducible supergravity which is finite if it is finite in the gravitational sector. Given the vanishing of gravitational infinities we are to check that the theory, obtained by reduction from irreducible supergravity, is again an irreducible one. It appears that this property is valid only for maximal supergravities (i.e. one can use reduction but not truncation). In other cases irreducibility is lost because reduction breaks the $O(d)$ (Lorentz) symmetry, essential for irreducibility in d dimensions. If we take irreducible but non-maximal SG in $d > 4$ and reduce it to $d=4$, ^{we} obtain a $d=4$ SG plus some matter multiplets (for instance, $SG_5^4 \rightarrow SG_4^2 + YM_4^2$ -multiplet [70]), a theory, known to be one loop infinite [71]. However, if we are interested not only in $d=4$ reduced theory we may question about quantum properties of non-maximal supergravities as they are in d

amplitude, i.e. with the absence of R^3 -terms in expansion of effective action.

dimensions. Then a distinguished candidate is $N=1$, $d=10$ supergravity [72 - 74], connected with the closed sector of superstring theory [1, 16]. It contains one graviton, one Majorana-Weyl gravitino, one antisymmetric tensor A_{MN} , one scalar and one Majorana-Weyl spinor. The contributions of all these fields in $\mathcal{L}_p$ can be computed using the formulas of section 4 (taking $d=10$ and $\delta=4$) with the final conclusion that (4.39) is valid also for SG_{10}^1 . This result cannot be considered simply as consequence of (4.39) for SG_{11}^1 , because to relate SG_{11}^1 and SG_{10}^1 we are not only to reduce but also to truncate the former. ^{xx}/Eq. (4.39) is also true for all theories following from SG_{10}^1 by the reduction, for example, for $d=4$ $SG_4^1 + 6 SYM^1$ - one [72] ($\mathcal{L}_4=0$ corresponds to the vanishing of anomalies in this theory, first noted in [33]). However, keeping in mind the previous discussion, one should refrain from considering (4.39) as an indication of finiteness of corresponding theories in $d \leq 7$, because being obtained from a non-maximal $d=10$ supergravity they have reducible supersymmetry and thus may be infinite in spite of finiteness in gravitational sector (compare with the opposite belief for $d=4$ case in [22]). ^{xxx}/ Thus (4.39) is to be considered mainly as a property of SG_{10}^1 itself, implying the absence of $\mathcal{L}^{10}, \dots, \mathcal{L}^7$ divergences in this theory. ^{xxx}/ One can readily check that (4.39) is

^{xx}/ curiously, $\alpha_2(SG_{10}^1) = 10-4=6$, while $\alpha_1(SG_{11}^1) = 11-4=7$.

^{xx}/ one more counter-example to the claim of possibility to obtain a finite theory by reducing non-maximal supergravity is provided by $SG_5^1 \rightarrow SG_4^1 + 4 SYM^1$ [75], known to be infinite [71].

^{xxx}/ in view of the connection with string theory one may ask about some analogous properties for $d=26$ gravity; however they are apparently absent, predicting problems in the closed boson string loop calculations.

valid also for SYM_{10}^4 on the gravitational background and hence (4.39) is also true for the coupled $SG_{10}^4 + SYM_{10}^4$ theory, coinciding with the $\alpha' \rightarrow 0$ limit of type I superstring theory [16]. One more remark utilizes independence of $\mathcal{G}_p$, $p \leq 6$ on the choice of representation of antisymmetric tensors in $d > 6$. Namely, there exists the version of SG_{10}^4 with $A_{M_1 \dots M_6}$ instead of A_{MN} [73]. Using the $p < d$ equivalence relation in (4.21) we conclude that all $\mathcal{G}_p$, $p < 10$ are the same in both theories (i.e., we again have (4.39) and no anomalies in $d=4$). The analogous remark is true also for (probably non-existing [76]) version of SG_{11}^4 with $A_{M_1 \dots M_6}$. The moral is that duality transformations in higher dimensions do not influence the infinities (but not finite parts) of reduced theories.

Finally, we are going to illustrate the $p > d$ case in the quantum (in) equivalence relation (4.21) on the example of $\mathcal{G}_6$ for SG_4^N . As is known [74, 32], it is possible to establish $\mathcal{G}_N = 0$ for all $N = 3, \dots, 8$ by suitable duality transformations of scalars. Then a natural question is whether the spectrum, which gives $\mathcal{G}_N = 0$, provides also $\mathcal{G}_6 = 0$. The answer is "yes" for $N = 4, 8$ but "no" for $N = 5, 6$ (suggesting that $SG_4^{7,6}$ with the anomaly free spectrum actually cannot be constructed). Numerically we get: (1) $\alpha_1(N) = -\frac{1}{2}(N-3)$, $\sigma_2(N) = -\frac{1}{6}$ for $N \geq 3$ if all spin zero particles are represented by scalars; (11) $\alpha_1(A_{\mu\nu}) - \alpha_1(A) = \frac{1}{2}$, $\alpha_1(A_{\mu\nu\lambda}) - 0 = -1$, $\sigma_2(A_{\mu\nu}) - \sigma_2(A) = \frac{1}{6}$, $\sigma_2(A_{\mu\nu\lambda}) - 0 = -1$, and thus it is not possible to arrange the spectrum so that $\alpha_1 = \sigma_2 = 0$ except for $N = 4$ and 8 . The result $\mathcal{G}_N(N-3) = 0$ but $\mathcal{G}_6(N=3) \neq 0$ can probably be understood from the helicity sum rules: $\mathcal{G}_{2k} \sim \sum_{\lambda} (-1)^{2\lambda} d(\lambda) \lambda^k = 0$, $k < N$ [41, 5].

5. Concluding remarks

In this paper we considered quantum properties of theories obtained by the simplest dimensional reduction (with internal dimensions being $S^1 \times \dots \times S^1$). The question left open is about properties of differently reduced theories, given those of higher dimensional theory. The general strategy to provide an answer is to study the relation of d-dim and reduced quantum theories separately in each particular case of reduction. This recipe is, of course, rather in-constructive. That is why we give examples of possible more explicit answers starting with idea that different reductions correspond to different choices of vacuum in d-dimensions (and assuming the knowledge of d-dim results for arbitrary backgrounds). ^{*}/ Let us confine our discussion to d=11 supergravity. Several d=11 internal vacuum spaces were already considered in the literature: (a) $N^7 = S^1 \times \dots \times S^1$ [3]; (b) $N^7 =$ "flat group" space [12]; (c) $N^7 = (SU_1 \times SU_2 \times U_1)/K$, e.g. $CP^2 \times S^1 \times S^1$ [4]; (d) $N^7 = S^7$ [77-80]. Different reductions (neglecting all massive modes) correspond to different versions of "N=8", d=4 SG. Only in the first case it is known that reduction preserves supersymmetry. If this is also true for the fourth case (d) the resulting theory may be connected with the $SO(8)$ gauged version of SG_4 [81]. To give an idea how one can use d=11 results in analysis of this theory let us present the following heuristic argument for the zero value of its β -function, starting with the $b_Y = 0$ property of SG_{11}^1 . To realize the vacuum $M^4 \times S^7$ as a

^{*}/ It should be understood that we are speaking about quantum versions of differently reduced theories and not about d-dim theory quantized near different vacua.

classical solution for SG_{11}^1 we are to assume a non-zero A_{MNK} -background [77-78]. Suppose we calculated $\mathcal{L}(SG_{11}^1)$ with $g_{MN} \neq \delta_{MN}$, $A_{MNK} \neq 0$. Then in view of supersymmetry we again must have $\mathcal{L}_Y = 0$ on shell. But this implies the absence of Λ -term (on shell)-renormalization in corresponding d=4 theory and thus [82] zero β -function. Our conjecture is that the $\mathcal{L}_Y = 0$ property of SG_{11}^1 is in fact the origin of not only the vanishing of anomalies in "reduced" version of SG_Y^1 but also of vanishing of β -function in the gauged version. π

It may be instructive to present another (even less rigorous) variant of above argument. Naively, one could hope for the following understanding of F_{MN}^2 -renormalization in non-abelian Kaluza-Klein theories. Using the standart ansatz (2.11) (with $\mathcal{P}_Y$ being a non-trivial, corresponding to the coset internal space, of. [10]) on shell we have: $\bar{\mathcal{L}}_Y = \alpha_1 R_{MNPK}^2$, $R_{\dots}^2 \rightarrow (R_{\dots} + FF + \Lambda)^2 \rightarrow R_{\dots}^2 + RFF + F^2\Lambda + \Lambda F^2$. It is the last $\sim F_{MN}^2$ term that contributes in the β -function and hence $\alpha_1 = 0$ implies $\beta = 0$. However, a word of caution is needed here: $\alpha_1 \sim \beta$ does not actually hold in Kaluza-Klein theory. Really, as we already observed in sect.2 for the simple reduction, if there are non-diagonal metric components in (2.11), divergences of reduced theory are not exhausted by d-dim curvature invariants (i.e. there may be additional F_{MN}^2 -contributions). Though this question should be studied separately for the coset reduction, we may speculate that

π / another hint for this relation is provided by the analogy in sum rules which are connected with these two properties of (of. [5]). Note, however, that it is still unclear, if one can construct a version of SG_Y^1 having simultaneously zero anomalies and β -function (the version of ref. [8], employs only scalars).

in a supersymmetrical theory (with gravitational and gauge sectors being interrelated) the above argument is after all a correct one.

Our final comment is about supersymmetry breaking reductions. In general, coset reduction (a) changes the number of degrees of freedom (in contrast with soft supersymmetry breaking by the mass terms) and so one cannot hope for some good quantum properties (for instance, the resulting d=4 theories will have $\mathcal{L}'$, $\mathcal{L}^2$ and at least topological $\log \mathcal{L}^2$ divergences). At the same time it was proved [83] that "generalized dimensional reduction" (b) produces one-loop on shell renormalizable d=4 theory. To understand if it is possible to provide some simple d=11 explanation of this fact one should specially study the impact of this reduction on the quantum theory.

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APPENDIX A

Notations and useful formulas

Throughout this paper we employed the euclidean metric g_{MN} and the following curvature conventions

$$R^M{}_{NPQ} = \partial_P \Gamma^M{}_{NQ} - \dots, R_{NQ} = R^M{}_{NMQ}, R^{MN} = g^{MK} R_{KQ}^{MN} \quad (A.1)$$

where $M, N, \dots = 1, \dots, d$. Two other groups of indices are $\mu, \nu = 1, \dots, n$, $i, j = 1, 2, \dots, d-n$, $n < d$. Using the matrix notations

$$A_\mu^{bc} = A_\mu^a f^{bac}, \quad f_{acd} f_{bcd} = C_2 \delta_{ab} \quad (A.2)$$

and defining the gauge strength invariants

$$J_1 = \text{tr} (\partial_\mu F_{\mu\nu})^2, \quad J_2 = \text{tr} (\partial_\mu F_{\mu\nu} \partial_\nu F_{\mu\nu}),$$

$$J_3 = \text{tr} (\partial_\mu F_{MN})^2, \quad J_4 = \text{tr} (F_{MN} F_{\mu\nu} F_{\mu\nu}), \quad (A.3)$$

one can prove that

$$J_2 = \frac{1}{2} J_1, \quad J_3 = \frac{1}{2} J_1 + 2 J_4 - \frac{1}{2} R_{MNKL} \text{tr} (F_{MN} F_{KL}) + \text{div} \quad (A.4)$$

Assuming the "mass shell" condition $R_{MN} = 0$ one can establish analogous relations for the curvature invariants (cf. [57, 59])

$$I_1 = R^{MN}_{PQ} R^{PQ}_{KS} R^{KS}_{MN}, \quad I_2 = R^{MN}_{PQ} R^{QS}_{NK} R^{KM}_{SP}, \quad (A.5)$$

$$I_3 = R^{PM}_{NQ} R^{QS}_{NK} R^{KM}_{SP}, \quad I_4 = R_{MNPQ} R^{MP}_{KS} R^{NQ}_{KS},$$

$$I_5 = R_{MNPQ} D^2 R_{MNPQ}, \quad (A.6)$$

$$I_4 = \frac{1}{2} I_1, \quad I_3 = I_2 - \frac{1}{4} I_1, \quad I_5 = -I_1 - 4 I_2.$$

Let us also introduce

$$E = I_1 - 2 I_2, \quad (A.7)$$

connected with the Euler number [84] ($d = 2m$)

$$\begin{aligned} \chi_d &= \frac{1}{2 \cdot 4 \cdots d} \cdot \frac{1}{(4\pi)^{d/2}} \int R^{M_1 M_2}_{N_1 N_2} \cdots R^{M_{d-1} M_d}_{N_{d-1} N_d} \epsilon^{M_1 \cdots M_d} \\ &= \epsilon_{M_1 \cdots M_d} \sqrt{g} d^d x = \frac{1}{(4\pi)^{d/2}} \int E_d \sqrt{g} d^d x \end{aligned} \quad (A.8)$$

Namely, if $d=4$ and $d=6$

$$E_4 = \frac{1}{2} R^* R^*, \quad E_6 = \frac{2}{3} E \quad (A.9)$$

APPENDIX B

Identities for γ -matrices in d-dimensions

Let γ_M be $\gamma \times \gamma$ ($\gamma = 2^{d/2}$) Dirac matrices. If

$$\gamma_{M_1 \cdots M_n} = \gamma_{[M_1 \cdots M_n]} = \frac{1}{n!} (\gamma_{M_1} \cdots \gamma_{M_n} + \dots) \quad (B.1)$$

(the same "weighted" convention is used throughout also for symmetrization), then

$$\gamma_M \gamma_N = g_{MN} + \gamma_{MN}, \quad \gamma_M \gamma_N \gamma_K = g_{MN} \gamma_K - g_{MK} \gamma_N + g_{NK} \gamma_M + \gamma_{MNK}, \quad (B.2)$$

$$\begin{aligned} \gamma_M \gamma_N \gamma_K \gamma_S &= g_{MN} g_{KS} - g_{MK} g_{NS} + g_{NK} g_{MS} + \gamma_{MN} g_{KS} + g_{MN} \gamma_{KS} \\ &\quad - \gamma_{MK} g_{NS} - g_{MK} \gamma_{NS} + \gamma_{MS} g_{NK} + g_{MS} \gamma_{NK} + \gamma_{MNKS}, \end{aligned} \quad (B.3)$$

$$\gamma_M \gamma_N \gamma_M = (2-d) \gamma_N, \quad \gamma_M \gamma_N \gamma_K \gamma_M = d g_{NK} + (d-4) \gamma_{NK}, \quad (B.4)$$

$$\delta_m \delta_n \delta_k \delta_s \delta_m = (2-d)(g_{nk} \delta_s - g_{ns} \delta_k + g_{ks} \delta_n) + (6-d) \delta_{nks}, \quad (B.5)$$

$$\delta_m \delta_n \delta_k \delta_s \delta_p \delta_m = d(g_{nk} g_{sp} - g_{ns} g_{kp} + g_{np} g_{ks}) + \quad (B.6)$$

$$+ (d-4)(g_{nk} \delta_{sp} + g_{sp} \delta_{nk} - \delta_{ns} g_{kp} - \delta_{kp} g_{ns} + g_{ks} \delta_{np} + g_{np} \delta_{ks}) + (d-8) \delta_{nks p} \quad (B.7)$$

$$t_2 1 = \nu = 2^{[d/2]}, \quad t_2 \delta_m = 0, \quad t_2 (\delta_m \delta_n) = \nu g_{mn}, \quad (B.8)$$

In particular, if $\delta \cdot F = \delta_{mn} F_{mn}$, then

$$t_2 (\delta \cdot F)^2 = -2\nu F_{mn} F_{mn}, \quad t_2 (\delta \cdot F)^3 = 8\nu F_{mn} F_{nk} F_{km} \quad (B.9)$$

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