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CONSTRUCTING EFFECTIVE ACTIONS FOR N = 1 SUPERSYMMETRY THEORIES

- I - SYMMETRY PRINCIPLES AND WARD IDENTITIES

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A B S T R A C T

The implications of the anomalous superconformal Ward identities for the construction of effective actions describing the low-energy dynamics of supersymmetry theories are investigated. The results apply to any N=1 supersymmetry theory with an anomaly structure,

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + 2 D_{\alpha} S = 0$$

where $V_{\alpha\dot{\alpha}}$ is the supercurrent and the chiral superfield S represents the anomaly.

1. - INTRODUCTION

Supersymmetry theories possess a rich and in many ways elegant mathematical structure. They exhibit remarkable renormalization properties, such as the all orders finiteness of the $N = 4$ super-Yang-Mills theory^{1),2)} and the non-renormalization theorems which prohibit perturbative spontaneous breaking of supersymmetry by radiative corrections in four dimensions³⁾. Yet they have still to find a secure place in our understanding of elementary particle physics.

One obstacle to such progress is our presently inadequate understanding of the physics of strongly interacting supersymmetric field theories. In particular we would like to understand the low-energy bound state spectrum of supersymmetric, asymptotically free gauge theories. Many questions arise here. Does supersymmetry remain unbroken at strong coupling? Are the chiral symmetries, including the R symmetry of the superconformal algebra, spontaneously broken? Does the low-energy spectrum comprise Goldstone bosons or massless composite fermions, or both, and what are their constituents?

Effective Lagrangians cannot answer these questions. However, they do provide an arena in which to examine the consequences and consistency of certain assumptions about the strong coupling dynamics. Further, by incorporating through the anomalous Ward identities all the information we have on the symmetry properties of the theory, they predict in principle testable relations amongst the masses, decay constants, and higher couplings of the bound states.

In this paper, I begin a programme of studying effective actions for $N = 1$ supersymmetric theories. I have divided my work into two parts, reflecting the distinction between the implications of the superconformal symmetries, which are valid in a very general class of theories, and the consequences of assumptions and exact theorems on the dynamical behaviour of specific theories. This paper is concerned with symmetry, specifically with the anomalous superconformal Ward identities, the restrictions they place on the effective action, and the general properties they imply. In a succeeding paper (hereafter referred to as II) I shall discuss the dynamics of supersymmetric gauge theories and the explicit construction of the effective action.

Effective Lagrangians for QCD have been investigated with considerable success by several authors⁴⁾ in recent years. In particular, they have clarified the rôle of the axial anomaly in resolving the $U(1)$ problem and determining the spectrum of pseudoscalar mesons. In a recent paper with Alvarez⁵⁾, I have shown that such effective Lagrangians may be extended to include pseudoscalar glueballs, and provide a description of meson-glueball mixing.

A first step in applying these ideas to supersymmetric theories has been taken by Veneziano and Yankielowicz (VY)⁶⁾, who have written down the simplest example of an effective Lagrangian for the pure $N = 1$ super-Yang-Mills theory. The present work owes much to this initiative.

The paper is organized as follows. The effective action is defined in Section 2 as the Legendre transform of the generating functional for connected Green's functions of the relevant composite operators, the supercurrent $V_{\alpha\dot{\alpha}}$ and the anomaly superfield S . The anomalous superconformal Ward identities in a functional formalism are constructed and extensively studied in Section 3. Section 4 describes an as yet incomplete attempt to find an effective action involving both S and $V_{\alpha\dot{\alpha}}$, which turns out to be a much more difficult task than that accomplished for QCD in Ref. 5).

The rest of the paper is concerned with constructing the effective action as a functional of the anomaly superfield S alone. In Section 5, it is explained why this requires only the information contained in the zero momentum Ward identities, and the complete restrictions of the anomalous superconformal symmetry on the form of the effective action are derived. Finally, and as an introduction to II, Section 6 contains two examples of effective actions, first the original VY model, then an action which gives rise to the most general effective potential. Interestingly, it may be proved purely on the basis of the anomalous Ward identities that this effective potential is always unbounded below.

2. - DEFINITION OF THE EFFECTIVE ACTION

The considerations of this paper apply to any $N = 1$ supersymmetric theory which has the anomaly structure

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + 2 D_{\alpha} S = 0 \quad (2.1)$$

where the real, axial vector superfield $V_{\alpha\dot{\alpha}}$ is the supercurrent containing all the symmetry currents of the superconformal algebra, and the chiral superfield S represents the anomaly⁷⁾⁻¹¹⁾. $V_{\alpha\dot{\alpha}}$ has length dimension $d = -3$ and $U_R(1)$ chiral weight $n = 0$, while S has $d = -3$ and $n = -2$. (For conventions, see Appendix A.)

Without further motivation I will assume that the low-energy degrees of freedom correspond to the component fields of the composite operators $V_{\alpha\dot{\alpha}}$ and S . The low-energy dynamics of the theory may therefore be approximated by an effective Lagrangian comprising only the composite fields $V_{\alpha\dot{\alpha}}$ and S and their supercovariant derivatives. The justifications for this assumption in specific

theories, together with the explicit form for $V_{\alpha\dot{\alpha}}$ and S in terms of the elementary fields, will be given in II. In fact, I will later restrict the Lagrangian further to depend only on the chiral field S , but for the moment the general case will be considered.

Given this assumption, the low-energy dynamics is determined by the Green's functions of the composite fields $V_{\alpha\dot{\alpha}}$ and S . It is the purpose of this paper to investigate the consequences of the superconformal symmetry for these Green's functions. I begin by defining the generating functional $W[J_V^{\alpha\dot{\alpha}}, J_S, J_{\bar{S}}]$ for connected Green's functions of the composite fields in the standard way by introducing sources into the path integral:

$$\begin{aligned} & \exp i W [J_V, J_S, J_{\bar{S}}] \\ &= \int \mathcal{D}A \exp i \left\{ I[A] + \int dx d\theta d\bar{\theta} J_V^{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}} \right. \\ & \quad \left. + \int dx d\theta J_S S + \int dx d\bar{\theta} J_{\bar{S}} \bar{S} \right\} \end{aligned} \quad (2.2)$$

Here, A denotes all the elementary fields, $I[A]$ is the action, and details of the regularization, renormalization and any gauge fixing are suppressed. $V_{\alpha\dot{\alpha}}$ and S are of course functions of A . The various functional derivatives of W evaluated at the source values $J_V = J_S = J_{\bar{S}} = 0$ yield the connected superfield Green's functions. The dimensions and chiral weights of the sources are as follows: J_V has $d = 1, n = 0$ while J_S and $J_{\bar{S}}$ have $d = 0, n = 0$.

Now in order to write the superconformal Ward identity satisfied by W , we must take into account the transformation properties of $V_{\alpha\dot{\alpha}}$ and S themselves. This results in additional, "contact", terms in the Ward identity, arising from the source field coupling terms in Eq. (2.2). These contact terms, denoted below by $\omega_{\alpha}(J)$, will be derived in the next section. The Ward identity following from Eq. (2.1) therefore takes the form,

$$\begin{aligned} & \left\{ \bar{D}^{\dot{\alpha}} \frac{d}{dJ_V^{\alpha\dot{\alpha}}} + 2 D^{\alpha} \frac{d}{dJ_S} + 2 \omega_{\alpha}(J_V) + 2 \omega_{\alpha}(J_S) \right\} \\ & \times W [J_V, J_S, J_{\bar{S}}] \end{aligned} \quad (2.3)$$

The effective action $\Gamma[V_{\alpha\dot{\alpha}}, S, \bar{S}]$ is defined as a Legendre transform of W , viz.

$$\Gamma[v, s, \bar{s}] = W[J_v, J_s, J_{\bar{s}}] - \int d^4x d\theta d\bar{\theta} J_v^{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}} - \int d^4x d\theta J_s s - \int d^4x d\bar{\theta} J_{\bar{s}} \bar{s} \quad (2.4)$$

The fields occurring in Γ are defined in the standard way as,

$$V_{\alpha\dot{\alpha}} = \frac{\delta W}{\delta J_v^{\alpha\dot{\alpha}}} \quad s = \frac{\delta W}{\delta J_s} \quad \bar{s} = \frac{\delta W}{\delta J_{\bar{s}}} \quad (2.5)$$

and as usual,

$$\frac{\delta \Gamma}{\delta V_{\alpha\dot{\alpha}}} = - J_v^{\alpha\dot{\alpha}} \quad \frac{\delta \Gamma}{\delta s} = - J_s \quad \frac{\delta \Gamma}{\delta \bar{s}} = - J_{\bar{s}} \quad (2.6)$$

We can then immediately write down the anomalous superconformal Ward identity for Γ ,

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + 2 D_{\alpha} s + \left(2 \omega_{\alpha}(v) + 2 \omega_{\alpha}(s) \right) \Gamma = 0 \quad (2.7)$$

The effective action defined in this way is the generating functional for 1PI (proper) vertices with respect to the propagators of the composite fields $V_{\alpha\dot{\alpha}}$ and s . The Green's functions are then constructed from tree diagrams using these proper vertices connected by the propagators. All the relevant information about the full quantum theory, in particular the analytic structure, is therefore contained in these proper vertices. The effective action (2.4) is not subsequently to be quantized, and loop diagrams are not to be included in making higher-point Green's functions.

I emphasize these points because they will be crucial to the interpretation of the supersymmetric effective action, and it is important to avoid confusion with other, quite different, objects existing in the literature which also go under the name of effective or phenomenological Lagrangians. Further discussion of this is given in II.

3. - SUPERCONFORMAL WARD IDENTITIES

In this section, I will discuss in some detail the superconformal Ward identities. In particular, the contact terms left unspecified in Eqs. (2.3) and (2.7) will be derived, so completing the Ward identities for $W[J_v, J_s, J_{\bar{s}}]$ and

and $\Gamma[V, S, \bar{S}]$. The general treatment of Ward identities given here follows that of Clark, Piguet and Sibold⁸⁾⁻¹¹⁾, who have discussed the construction and renormalization of the supercurrent for a variety of models. I will draw heavily on their results in what follows.

The first step is to introduce the symmetry operators Ω^G corresponding to particular generators G of the superconformal algebra. A brief review of the superconformal algebra and its action on the fields is given in Appendix A.

For a superfield $\Phi(x, \theta, \bar{\theta})$ in the real representation, the symmetry operator Ω^G (which acts on the effective action) is defined as

$$\Omega^G = -i \int dx d\theta d\bar{\theta} \delta^G \Phi \frac{\delta}{\delta \Phi} \quad (3.1)$$

where the field variation

$$\delta^G \Phi = i [G, \Phi] \quad (3.2)$$

For example,

$$\delta^R \Phi = i \left(n + \theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Phi \quad (3.3)$$

$$\delta^D \Phi = \left(-d + x^\lambda \partial_\lambda + \frac{1}{2} \theta \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Phi \quad (3.4)$$

$$\begin{aligned} \delta_\nu^K \Phi = & \left(2x_\nu \delta^D + 2x^\lambda \delta_{\nu\lambda}^M + x^2 \partial_\nu - 2x_\nu x^\lambda \partial_\lambda \right. \\ & \left. + \theta \sigma_\nu \bar{\theta} (2\delta^R + in) + \theta^2 \bar{\theta}^2 \partial_\nu \right) \Phi \end{aligned} \quad (3.5)$$

Similarly, for a chiral superfield $\phi(x, \theta)$ in the chiral representation, the symmetry operator is

$$\Omega^G = -i \int dx d\theta \delta^G \phi \frac{\delta}{\delta \phi} \quad (3.6)$$

Here,

$$\delta^R \phi = i \left(\frac{2}{3} d + \theta \frac{\partial}{\partial \theta} \right) \phi \quad (3.7)$$

$$\delta^D \phi = \left(-d + x^\lambda \partial_\lambda + \frac{1}{2} \theta \frac{\partial}{\partial \theta} \right) \phi \quad (3.8)$$

$$\begin{aligned} \delta^\kappa \phi = & \left(-2x_\nu d - x^2 \partial_\nu + 2x_\nu x^\lambda \partial_\lambda \right. \\ & \left. + x_\nu \theta \frac{\partial}{\partial \theta} - i x^\lambda \theta \sigma_{\nu\lambda} \frac{\partial}{\partial \theta} \right) \phi \end{aligned} \quad (3.9)$$

Notice that the superconformal relation $d = 3/2 n$ has been used here.

Similar expressions can be written down in terms of the sources to give the symmetry operators for the generating functional $W[J_V, J_S, J_{\bar{S}}]$.

The functional differential operators Ω^G themselves satisfy

$$[\Omega^G, \Xi] = -i \delta^G \Xi \quad (3.10)$$

and obey the same algebra as the generators G . They may therefore be grouped into x -independent superfields as follows:

$$\hat{Q}^R(\theta, \bar{\theta}) = Q^R - i\theta Q^Q + i\bar{\theta} \bar{Q}^Q - 2\theta\sigma^\lambda \bar{\theta} Q^P_\lambda \quad (3.11)$$

$$\hat{Q}^D(\theta, \bar{\theta}) = Q^D - \frac{1}{2}\theta Q^Q - \frac{1}{2}\bar{\theta} \bar{Q}^Q \quad (3.12)$$

$$\begin{aligned} \hat{Q}^\kappa_\nu(\theta, \bar{\theta}) = & Q^\kappa_\nu - \theta\sigma_{\nu\lambda} \bar{Q}^\lambda - Q^\lambda \sigma_{\nu\lambda} \bar{\theta} \\ & + \theta\sigma^\lambda \bar{\theta} (3g_{\lambda\nu} Q^R + \epsilon_{\lambda\rho\sigma} Q^{R\rho\sigma}) \\ & - i\bar{\theta}^2 \theta\sigma_\nu \bar{Q}^Q + iQ^Q \sigma_\nu \bar{\theta} \theta^2 - \theta^2 \bar{\theta}^2 Q^P_\nu \end{aligned} \quad (3.13)$$

We can of course write similar expressions $\hat{\Omega}^G$ for all the generators G . However, given the above superfield structure the action of all the superconformal generators is contained in the symmetry operators $\hat{\Omega}^R$, $\hat{\Omega}^D$ and $\hat{\Omega}_V^K$. Similarly, all the information in the complete supercurrent may be expressed in terms of the current superfields corresponding to R , D and K_V transformations.

The decomposition of the supercurrent $V_{\alpha\dot{\alpha}}$ into its component currents⁸⁾ is displayed in Appendix B. For our purposes it will be sufficient to note here that the current superfields for chiral, dilatation and conformal transformations are, (with $V_{\alpha\dot{\alpha}} = \frac{1}{2} \sigma_{\alpha\dot{\alpha}}^\mu V_\mu$),

$$\hat{R}_\mu = V_\mu \quad (3.14)$$

$$\hat{D}_\mu = x^\lambda \hat{T}_{\mu\lambda} \quad (3.15)$$

$$\hat{K}_{\mu\nu} = (2 x_\nu x_\lambda - g_{\nu\lambda} x^2) \hat{T}_{\mu}{}^\lambda \quad (3.16)$$

where,

$$\hat{T}_{\mu\nu} = -\frac{1}{16} (V_{\mu\nu} + V_{\nu\mu} - 2g_{\mu\nu} V_\lambda{}^\lambda) \quad (3.17)$$

with

$$V_{\mu\nu} = [D^\beta, \bar{D}^{\dot{\beta}}] \sigma_{\mu\beta\dot{\beta}} V_\nu \quad (3.18)$$

With this identification, the superfield Ward identity for chiral transformations is found from Eq. (2.7) by acting with D^α and subtracting the conjugate equation. This gives,

$$\partial^\mu \hat{R}_\mu - i (D^2 S - \bar{D}^2 \bar{S}) - i (\omega^R(V) + \omega^R(S, \bar{S})) \Gamma = 0 \quad (3.19)$$

where

$$\omega^R = D^\alpha \omega_\alpha - \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}} \quad (3.20)$$

is the local symmetry operator corresponding to $\hat{\Omega}^R$, i.e.,

$$\hat{Q}^R(\theta, \bar{\theta}) = \int dx \omega^R(x, \theta, \bar{\theta}) \quad (3.21)$$

Remember that Eq. (3.19) is a superfield equation. It contains as lowest ($\theta = \bar{\theta} = 0$) component the Ward identity for chiral transformations, its higher components giving the Ward identities for supersymmetry transformations and translations according to Eq. (3.11).

The superfield Ward identity for dilatations may be deduced similarly. This is (see Appendix B)

$$\partial^\mu \hat{J}_\mu + \frac{3}{2} (D^2 S + \bar{D}^2 \bar{S}) - i (\omega^D(V) + \omega^D(S, \bar{S})) \Gamma = 0 \quad (3.22)$$

where,

$$\omega^D = \omega_{Tr}^P + x^\nu \omega_\nu^P \quad (3.23)$$

with,

$$\omega_{Tr}^P = \frac{3}{2} i (D^\alpha \omega_\alpha + \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}) \quad (3.24)$$

$$\begin{aligned} \omega_\nu^P = -\frac{1}{16} \left\{ (D^2 \bar{D}_{\dot{\alpha}} \omega_\alpha + \bar{D}^{\dot{\alpha}} D_\alpha \bar{\omega}_{\dot{\alpha}}) \sigma_{\nu}^{\alpha\dot{\alpha}} \right. \\ \left. + \sigma_{\nu}^{\beta\dot{\beta}} [D_\beta, \bar{D}_{\dot{\beta}}] (D^\alpha \omega_\alpha - \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}) \right. \\ \left. + 8i \partial_\nu (D^\alpha \omega_\alpha + \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}) \right\} \end{aligned} \quad (3.25)$$

Finally, the Ward identity for conformal transformations is,

$$\partial^\mu \hat{K}_{\mu\nu} + 3x_\nu (D^2 S + \bar{D}^2 \bar{S}) - i \omega_\nu^K \Gamma = 0 \quad (3.26)$$

where

$$\omega_\nu^K = 2x^\nu \omega_{Tr}^P + 2x^\lambda \omega_{\nu\lambda}^M \quad (3.27)$$

The expression for the local symmetry operator of Lorentz transformations $\omega_{\nu\lambda}^M$ in terms of ω_α is not simple, and is not needed here.

As already mentioned, Eqs. (3.19), (3.22) and (3.26) together contain the same information as the more compact form of the Ward identity, Eq. (2.7). Which is easier to use will in practice depend on the circumstances, the $\bar{D}^{\dot{\alpha}}V_{\alpha\dot{\alpha}}$ expression proving more powerful in general, while the component current equations are more convenient for exploiting the zero-momentum Ward identities (see Section 5).

In writing the superconformal Ward identity in the form (2.7) it has of course been assumed that the local symmetry operator ω^R can be decomposed as shown in Eq. (3.20). This is verified by the following explicit construction.

I shall now derive expressions for the operators ω_α in terms of the fields and their functional derivatives for both chiral and real fields of arbitrary dimension. The required calculations are quite long, and details are relegated to Appendix C.

First, notice that for dimensional consistency in Eqs. (2.7) or (3.20) ω_α must have dimension $d = -7/2$. Further, since

$$\{Q^R, \omega_\alpha\} = -\omega_\alpha \quad (3.28)$$

ω_α has a fixed chiral weight $n = -1$.

Now consider the case of a chiral field ϕ . Since ω_α is required to be linear in ϕ and $\delta/\delta\phi$, the restriction that $d = -7/2$ ($n = -1$ is then automatically satisfied) restricts ω_α to be of the form

$$\omega_\alpha = a D_\alpha \phi \frac{\delta}{\delta\phi} + b \phi D_\alpha \frac{\delta}{\delta\phi} \quad (3.29)$$

Substituting into the identity,

$$Q^R = -i \int d^4x d\theta \delta^R \phi \frac{\delta}{\delta\phi} - i \int d^4x d\bar{\theta} \delta^R \bar{\phi} \frac{\delta}{\delta\bar{\phi}} \quad (3.30)$$

$$= \int d^4x (D^\alpha \omega_\alpha - \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}) \Big|_{\theta=\bar{\theta}=0} \quad (3.31)$$

determines the coefficients uniquely to be $a = n + 2$, $b = n$. Thus,

$$\omega_\alpha(\phi) = n D_\alpha \left(\phi \frac{\delta}{\delta\phi} \right) + 2 D_\alpha \phi \frac{\delta}{\delta\phi} \quad (3.32)$$

Because of the superconformal relation $d = 3/2 n$ between the dimension and chiral weight of a chiral superfield no new information would be found by also checking the identity for Ω^D in this case. The Ω^R identity determines the unique $\omega_\alpha(\phi)$.

This is not true for a real superfield ϕ , and here both identities must be used to determine $\omega_\alpha(\phi)$. The conditions that $\omega_\alpha(\phi)$ has $d = -7/2$, $n = -1$, and is linear in ϕ and $\delta/\delta\phi$ restrict ω_α to be a linear combination of ten possible terms involving different orderings of $\bar{D}_\alpha$, $\bar{D}^2$, D_α , ϕ and $\delta/\delta\phi$.

Satisfying the identity,

$$Q^R = -i \int dx d\theta d\bar{\theta} \int^R \mathbb{I} \frac{\delta}{\delta \mathbb{I}} \quad (3.33)$$

$$= \int dx \left(D^\alpha \omega_\alpha - \bar{D}_\alpha \bar{\omega}^{\dot{\alpha}} \right) \Big|_{\theta = \bar{\theta} = 0} \quad (3.34)$$

then imposes five conditions on the coefficients, while the further restrictions from the dilatation identity

$$Q^D = -i \int dx d\theta d\bar{\theta} \int^D \mathbb{I} \frac{\delta}{\delta \mathbb{I}} \quad (3.35)$$

$$= \int dx \omega^D \Big|_{\theta = \bar{\theta} = 0} \quad (3.36)$$

with ω^D given in terms of ω_α by Eqs. (3.23) to (3.25), fix the coefficients uniquely (see Appendix C). The result^{*} for a real field ϕ of dimension d ($n = 0$ since ϕ is real) is,

$$\begin{aligned} \omega_\alpha(\mathbb{I}) = & -d \left(\mathbb{I} \bar{D}^2 D_\alpha \frac{\delta}{\delta \mathbb{I}} - \bar{D}^2 \mathbb{I} D_\alpha \frac{\delta}{\delta \mathbb{I}} \right) \\ & + (d+2) \left(D_\alpha \mathbb{I} \bar{D}^2 \frac{\delta}{\delta \mathbb{I}} - \bar{D}^2 D_\alpha \mathbb{I} \frac{\delta}{\delta \mathbb{I}} \right) \end{aligned} \quad (3.37)$$

^{*} Contact terms for particular examples of real fields are given in Refs. 9) and 10). However, in those cases BRS invariance could be applied to simplify the results. This restriction does not apply in general, and in particular is not helpful for the supercurrent, and so we are forced to a more general approach. The results here agree with those of Refs. 9) and 10) in the special cases considered there.

These expressions have been derived in terms of fields ϕ and Φ . The results may equally well be used to write the contact terms for sources, required for the Ward identity for the generating functional W . To derive the corresponding form to (3.32) for a chiral source J_ϕ coupled to ϕ , simply make the replacements $\delta/\delta\phi \rightarrow -J_\phi$, $\phi \rightarrow \delta/\delta J_\phi$. This gives

$$\omega_\alpha(\mathcal{J}_\phi) = -(n_\phi + 2) \mathcal{D}_\alpha \left(\mathcal{J}_\phi \frac{\delta}{\delta \mathcal{J}_\phi} \right) + 2 \mathcal{D}_\alpha \mathcal{J}_\phi \frac{\delta}{\delta \mathcal{J}_\phi} \quad (3.38)$$

This is consistent, since it yields the operator ω_α for a chiral field with the dimension and weight appropriate for J_ϕ , viz. $d_{J_\phi} = -(d_\phi + 3)$, $n_{J_\phi} = -(n_\phi + 2)$.

Similarly, substituting $\delta/\delta\Phi \rightarrow -J_\Phi$, $\Phi \rightarrow \delta/\delta J_\Phi$ into Eq. (3.37) gives

$$\begin{aligned} \omega_\alpha(\mathcal{J}_\Phi) = & (d_\Phi + 2) \left(\mathcal{J}_\Phi \bar{\mathcal{D}}^2 \mathcal{D}_\alpha \frac{\delta}{\delta \mathcal{J}_\Phi} - \bar{\mathcal{D}}^2 \mathcal{J}_\Phi \mathcal{D}_\alpha \frac{\delta}{\delta \mathcal{J}_\Phi} \right) \\ & - d_\Phi \left(\mathcal{D}_\alpha \mathcal{J}_\Phi \bar{\mathcal{D}}^2 \frac{\delta}{\delta \mathcal{J}_\Phi} - \bar{\mathcal{D}}^2 \mathcal{D}_\alpha \mathcal{J}_\Phi \frac{\delta}{\delta \mathcal{J}_\Phi} \right) \end{aligned} \quad (3.39)$$

the appropriate form for a real field of dimension $d_{J_\Phi} = -(d_\Phi + 2)$.

This completes the discussion of the purely symmetry aspects of the superconformal Ward identities. I close this section by collecting the results to write the Ward identities for the generating functional $W[J_V, J_S, J_\Phi]$ and the effective action $\Gamma[V, S, \bar{S}]$.

Since $J_V^{\alpha\dot{\alpha}}$ has $d = 1$, $n = 0$ and J_S has $d = 0$, $n = 0$, the Ward identity for W is,

$$\left\{ \bar{\mathcal{D}}^{\dot{\alpha}} \frac{\delta}{\delta \mathcal{J}_V^{\alpha\dot{\alpha}}} + 2 \mathcal{D}_\alpha \frac{\delta}{\delta \mathcal{J}_S} + 2 \omega_\alpha(\mathcal{J}_V) + 2 \omega_\alpha(\mathcal{J}_S) \right\} W = 0 \quad (3.40)$$

with,

$$\omega_\alpha(\mathcal{J}_S) = 2 \mathcal{D}_\alpha \mathcal{J}_S \frac{\delta}{\delta \mathcal{J}_S} \quad (3.41)$$

and

$$\begin{aligned} \omega_\alpha(J_V) = & - \left(J_V^{\beta\dot{\beta}} \bar{D}^2 D_\alpha \frac{\delta}{\delta J_V^{\beta\dot{\beta}}} - \bar{D}^2 J_V^{\beta\dot{\beta}} D_\alpha \frac{\delta}{\delta J_V^{\beta\dot{\beta}}} \right) \\ & + 3 \left(D_\alpha J_V^{\beta\dot{\beta}} \bar{D}^2 \frac{\delta}{\delta J_V^{\beta\dot{\beta}}} - \bar{D}^2 D_\alpha J_V^{\beta\dot{\beta}} \frac{\delta}{\delta J_V^{\beta\dot{\beta}}} \right) \end{aligned} \quad (3.42)$$

Finally, recalling that $V_{\alpha\dot{\alpha}}$ has $d = -3$, $n = 0$ and S has $d = -3$, $n = -2$, the Ward identity for the effective action Γ is found to be,

$$\bar{D}^2 V_{\alpha\dot{\alpha}} + 2 D_\alpha S + \left(2 \omega_\alpha(V) + 2 \omega_\alpha(S) \right) \Gamma = 0 \quad (3.43)$$

with

$$\omega_\alpha(S) = - 2 S D_\alpha \frac{\delta}{\delta S} \quad (3.44)$$

and

$$\begin{aligned} \omega_\alpha(V) = & 3 \left(V^{\beta\dot{\beta}} \bar{D}^2 D_\alpha \frac{\delta}{\delta V^{\beta\dot{\beta}}} - \bar{D}^2 V^{\beta\dot{\beta}} D_\alpha \frac{\delta}{\delta V^{\beta\dot{\beta}}} \right) \\ & - \left(D_\alpha V^{\beta\dot{\beta}} \bar{D}^2 \frac{\delta}{\delta V^{\beta\dot{\beta}}} - \bar{D}^2 D_\alpha V^{\beta\dot{\beta}} \frac{\delta}{\delta V^{\beta\dot{\beta}}} \right) \end{aligned} \quad (3.45)$$

4. - TOWARDS $\Gamma[V_{\alpha\dot{\alpha}}, S, \bar{S}]$

The Ward identities for $W[J_V, J_S, J_{\bar{S}}]$ or $\Gamma[V, S, \bar{S}]$ contain all the superconformal symmetry information of the theory. The various relations amongst Green's functions implied by the symmetry may be deduced directly by taking further functional derivatives of these equations.

However, in order to implement a low-energy approximation to the dynamics it is convenient to construct the effective action explicitly. The Ward identities place considerable restrictions on the allowed structure of Γ , and these restrictions are the subject of the rest of this paper. The succeeding paper, II, will deal with the dynamical assumptions and approximations which must be made on physical grounds for specific theories in order to determine a simple and predictive effective action.

In this section I will discuss the issues involved in attempting to write an effective action involving both the chiral field S and the supercurrent $V_{\alpha\dot{\alpha}}$. This attempt has not yet proved successful, although the methods shown here go a long way towards solving the problem. Then, in the following section, it is shown how to construct an effective action for S only in a consistent way by using only the restricted information in the zero-momentum Ward identities.

The starting point is the Ward identity (3.43) for Γ . An immediate difficulty is the inhomogeneity of this equation - it comprises a functional differential operator acting on Γ and a term independent of Γ . This suggests splitting Γ into two parts, the sum of a particular integral and a complementary functional, viz.

$$\Gamma = \Gamma_0 + \Gamma_1 \quad (4.1)$$

where Γ_0 is any particular solution of the first order functional differential equation,

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} + 2 D_{\alpha} S + \left(2 \omega_{\alpha}(V) + 2 \omega_{\alpha}(S) \right) \Gamma_0 = 0 \quad (4.2)$$

and its conjugate, while Γ_1 is the general solution of the homogeneous equation

$$\left(\omega_{\alpha}(V) + \omega_{\alpha}(S) \right) \Gamma_1 = 0 \quad (4.3)$$

and its conjugate.

Notice here that the contact term for $\omega_{\alpha}(V)$ may be written with an explicit factor of $\bar{D}^{\dot{\alpha}}$ taken outside, viz.

$$\begin{aligned} \omega_{\alpha}(V) = \bar{D}^{\dot{\alpha}} \left\{ 3 \left(\bar{D}_{\dot{\alpha}} V D_{\alpha} \frac{\delta}{\delta V} - V \bar{D}_{\dot{\alpha}} D_{\alpha} \frac{\delta}{\delta V} \right) \right. \\ \left. - \left(\bar{D}_{\dot{\alpha}} D_{\alpha} V \frac{\delta}{\delta V} + D_{\alpha} V \bar{D}_{\dot{\alpha}} \frac{\delta}{\delta V} \right) \right\} \end{aligned} \quad (4.4)$$

Here, and subsequently, the contracted indices on V , $\delta/\delta V$ are omitted.

Consider first just the S dependent part of Eq. (4.2) and attempt to solve,

$$D_{\alpha} S + \omega_{\alpha}(S) \Gamma_0 = 0 \quad (4.5)$$

that is,

$$D_\alpha S - 2 S D_\alpha \frac{\delta \Gamma_0}{\delta S} = 0 \quad (4.6)$$

The solution is straightforward. Equation (4.6) implies

$$- 2 S D_\alpha \left(\frac{\delta \Gamma_0}{\delta S} - \frac{1}{2} \log S/\mu^2 \right) = 0 \quad (4.7)$$

where μ is an arbitrary constant with dimensions of mass, and a particular solution is found by equating the expression in brackets to zero. This gives (including the conjugate equation),

$$\Gamma_0 = \int d^4x d\theta \frac{1}{2} S (\log S/\mu^2 - 1) + \int d^4x d\bar{\theta} \frac{1}{2} \bar{S} (\log \bar{S}/\mu^2 - 1) \quad (4.8)$$

Unfortunately, I have not yet found a particular integral for the full Ward identity including the field $V_{\alpha\dot{\alpha}}$. The main difficulty in solving this equation appears to be the different way in which the indices appear in the inhomogeneous and homogeneous terms, despite the encouraging result (4.4).

To continue, let us nevertheless suppose that such a complete particular integral $\Gamma_0[V, S, \bar{S}]$ can be found. Then we require only the most general solution Γ_1 of the homogeneous equation (4.3). Again, I do not have the solution. However, the following approach reduces the problem.

An equation such as (4.3) can be converted into an invariance property of the complementary functional Γ_1 under an infinitesimal transformation. To uncover this, multiply Eq. (4.3) by a spinor superfield $X^\alpha(x, \theta, \bar{\theta})$ and integrate over the superspace co-ordinates. An integration by parts then gives

$$\begin{aligned} 0 = \int d^4x d\theta d\bar{\theta} \left\{ \right. & 2 \bar{D}^2 D_\alpha (S X_\alpha) \frac{\delta \Gamma_1}{\delta S} \\ & + 3 \left(D^\alpha (\bar{D}^{\dot{\alpha}} V \bar{D}_{\dot{\alpha}} X_\alpha) + D^\alpha \bar{D}^{\dot{\alpha}} (V \bar{D}_{\dot{\alpha}} X_\alpha) \right) \frac{\delta \Gamma_1}{\delta V} \\ & \left. - \left(\bar{D}^{\dot{\alpha}} D^\alpha V \bar{D}_{\dot{\alpha}} X_\alpha + \bar{D}^{\dot{\alpha}} (D^\alpha V \bar{D}_{\dot{\alpha}} X_\alpha) \right) \frac{\delta \Gamma_1}{\delta V} \right\} \quad (4.9) \end{aligned}$$

Recalling that Γ_1 also satisfies the conjugate equation, we see that it must be invariant under the transformations

$$\begin{aligned}
 S &\rightarrow S + 2 \bar{D}^2 D^\alpha (S \chi_\alpha) \\
 \bar{S} &\rightarrow \bar{S} + 2 D^2 \bar{D}_{\dot{\alpha}} (S \bar{\chi}^{\dot{\alpha}}) \\
 V_{\beta\dot{\beta}} &\rightarrow V_{\beta\dot{\beta}} + 3 \left(D^\alpha (\bar{D}_{\dot{\alpha}} V_{\beta\dot{\beta}} \bar{D}_{\dot{\alpha}} \chi_\alpha) + D^\alpha \bar{D}_{\dot{\alpha}} (V_{\beta\dot{\beta}} \bar{D}_{\dot{\alpha}} \chi_\alpha) \right) \\
 &\quad - \left((\bar{D}_{\dot{\alpha}} D^\alpha V_{\beta\dot{\beta}}) \bar{D}_{\dot{\alpha}} \chi_\alpha + \bar{D}_{\dot{\alpha}} (D^\alpha V_{\beta\dot{\beta}} \bar{D}_{\dot{\alpha}} \chi_\alpha) \right) \\
 &\quad + \text{conjugate}
 \end{aligned} \tag{4.10}$$

This form of the transformation for $V_{\beta\dot{\beta}}$ shows explicitly that it depends only on the covariant derivative of the parameter, viz. $\bar{D}_{\dot{\alpha}} \chi_\alpha$ and $D_\alpha \bar{\chi}_{\dot{\alpha}}$. The complementary functional $\Gamma_1[V, S, \bar{S}]$ is then the most general action which can be constructed from invariants under the transformations (4.10).

It is not clear at present how to make further progress. In the analogous, but much simpler, problem of deriving an effective action for QCD including the axial vector current j_μ^5 , it was possible⁵⁾ to define a novel Legendre transform giving rise to a "symmetric" effective action $\tilde{\Gamma}$, closely related to the generating functional of proper vertices, which possessed a simple local $U(1)$ symmetry. Here the trick was to define a non-standard transform, first introduced by Zumino, such that the current derivative term in the Ward identity for $\tilde{\Gamma}$ appeared as $\partial_\mu (\delta \tilde{\Gamma} / \delta j_\mu^5)$, and the identity was homogeneous. It was therefore not necessary to find a particular integral. However, here the current $V_{\alpha\dot{\alpha}}$ transforms non-trivially under the superconformal group and the occurrence of contact terms for $V_{\alpha\dot{\alpha}}$ prevents the Zumino transform from producing a homogeneous Ward identity. Unless another non-standard transform can be found, it therefore seems that we must use the method described here, and solve Eq. (4.2) for the particular integral.

The symmetry transformations (4.10) are unfortunately also rather complicated, although in view of the wealth of symmetry information they contain, this is perhaps not surprising. There is every reason to believe that the problem of constructing invariants is tractable. It is worth remarking that the supercurrent contains amongst its components the spin 2 field $T_{\mu\nu}$. This suggests that the required action for $V_{\alpha\dot{\alpha}}$ may well be related to that of (conformal) supergravity. There are indeed some hints in the structures of Eqs. (4.2) and (4.10) to support this view.

To conclude this section, the problem of constructing an effective action $\Gamma[V, S, \bar{S}]$ involving both the chiral and supercurrent fields remains unsolved, despite the progress made here. I have included this section in the hope that others more expert in the formalism of superspace and supergravity may recognize familiar structures, and that the programme outlined here may be completed.

5. - ZERO MOMENTUM WARD IDENTITIES AND $\Gamma[S, \bar{S}]$

I now turn to the simpler question of finding an effective action as a functional of the chiral field S alone. In any case, this is presumably more immediately realistic as the low-energy degrees of freedom for the theories of interest are expected to be those contained in the anomaly superfield S .

The problem here is how to leave out $V_{\alpha\dot{\alpha}}$ in a consistent way. This question may be studied in the analogous example of the QCD effective action of Ref. 5). There we find that the reduced action with $\hat{j}_{\mu}^5$ eliminated no longer exhibits the local $U(1)$ symmetry, but remains invariant under global transformations. Whereas the local symmetry follows from the full Ward identity, the global symmetry is implied by the Ward identities at zero momentum.

This observation motivates the following approach. The complete Ward identity (3.43) is completely equivalent to the set of identities (3.19), (3.22) and (3.26) for $\hat{R}_{\mu}$, $\hat{D}_{\mu}$ and $\hat{K}_{\mu\nu}$. Consider the zero-momentum limit of these identities, i.e., integrate over space-time. The current divergence term is eliminated, leaving

$$-i \int d^4x (\partial^2 S - \bar{\partial}^2 \bar{S}) - i (\hat{Q}^R(V) + \hat{Q}^R(S, \bar{S})) \Gamma = 0 \quad (5.1)$$

etc. We may now consistently look for solutions for Γ independent of V and so drop the $\hat{Q}(V)$ term.

In other words, all the symmetry information needed to construct the effective action as a functional of S alone is contained in the zero-momentum Ward identities. The extra information provided by the full identities is only relevant for determining the dependence of Γ on V .

We therefore have to solve the following equations for $\Gamma[S, \bar{S}]$,

$$\hat{Q}^R(S, \bar{S}) \Gamma + \int d^4x (\partial^2 S - \bar{\partial}^2 \bar{S}) = 0 \quad (5.2)$$

$$\hat{Q}^D(S, \bar{S}) \Gamma + \frac{3}{2} i \int d^4x (\partial^2 S + \bar{\partial}^2 \bar{S}) = 0 \quad (5.3)$$

$$\hat{\Omega}_0^K (S, \bar{S}) \Gamma + 3i \int dx x_\alpha (D^2 S + \bar{D}^2 \bar{S}) = 0 \quad (5.4)$$

Using Eqs. (3.20) and (3.21) to express $\hat{\Omega}^R$ in terms of ω_α and the explicit form (3.44) for $\omega_\alpha(S)$, the first of these reduces to

$$\int dx \left(D^2 S - 2S D^2 \frac{\delta \Gamma}{\delta S} - 2D^\alpha S D_\alpha \frac{\delta \Gamma}{\delta S} \right) + \text{conj} = 0 \quad (5.5)$$

which is clearly satisfied by the particular integral $\Gamma_0[S, \bar{S}]$ of Eq. (4.8). The complementary functional is then required to satisfy

$$\int dx \left(S D^2 \frac{\delta \Gamma_1}{\delta S} + D^\alpha S D_\alpha \frac{\delta \Gamma_1}{\delta S} \right) + \text{conj} = 0 \quad (5.6)$$

Similar results hold for D and K_V transformations. In practice, however, since neither $\hat{\Omega}^D$ nor $\hat{\Omega}_V^K$ have simple expressions in terms of ω_α , it is more useful to use the definitions of $\hat{\Omega}$ in terms of field variations. Consider the three cases separately.

i) Chiral Transformations

The lowest component of $\hat{\Omega}^R \Gamma$ is simply $\Omega^R \Gamma$, while the anomaly term is independent of $\theta, \bar{\theta}$. A particular integral is therefore found by solving

$$0 = \hat{\Omega}^R (S, \bar{S}) \Gamma_0 + \int dx (D^2 S - \bar{D}^2 \bar{S}) \quad (5.7)$$

$$= \int dx d\theta \left(-i \delta^R S \frac{\delta \Gamma_0}{\delta S} + S \right) + \text{conj} \quad (5.8)$$

This follows from Eq. (3.6) and the relation $\int dx D^2 S = \int dx d\theta S$. Look for a solution $\Gamma_0 = \int dx d\theta f(S) + \text{conj}$. Substituting for $\delta^R S$ (recall S has $n = -2$) then gives,

$$0 = \int dx d\theta \left(-2S \frac{\partial f}{\partial S} + \theta \frac{\partial S}{\partial \theta} \frac{\partial f}{\partial S} + S \right) \quad (5.9)$$

$$= \int dx d\theta \left(-2s \frac{\partial f}{\partial s} + \theta \frac{\partial f}{\partial \theta} + s \right) \quad (5.10)$$

$$= \int dx d\theta \left(-2s \frac{\partial f}{\partial s} + 2f + s \right) \quad (5.11)$$

after an integration by parts. The same equation can be derived by considering the finite transformation (Appendix A) but the infinitesimal approach is better suited to conformal transformations. The equation

$$s \frac{\partial f}{\partial s} - f - \frac{1}{2} s = 0 \quad (5.12)$$

is easily solved:

$$f = \frac{1}{2} s (\log s/\mu^2 - 1) \quad (5.13)$$

yielding the by now familiar particular integral.

The complementary functional satisfies,

$$\hat{Q}^R \Gamma_1 [s, \bar{s}] = 0 \quad (5.14)$$

The superfield structure means that we need only check the lowest component, i.e., $\Omega^R \Gamma_1 = 0$. This criterion is easily understood - it simply states that the chiral weight of Γ_1 must be zero. To see this explicitly, consider a Γ_1 of the form $\int dx d\theta d\bar{\theta} \phi(S, \bar{S})$. Then,

$$\hat{Q}^R \Gamma_1 = \int dx d\theta d\bar{\theta} \left(n_{\frac{1}{2}} + \theta \frac{\partial}{\partial \theta} - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \phi \quad (5.15)$$

$$= n_{\frac{1}{2}} \Gamma_1 \quad (5.16)$$

So any function $\phi(S, \bar{S})$ with $n_{\frac{1}{2}} = 0$ (as independently required by reality) satisfies the constraint.

ii) Dilatations

The discussion here is almost identical. The particular integral satisfies

$$0 = \int dx d\theta \left(-i s^\lambda s \frac{\partial \Gamma_0}{\partial s} + \frac{3}{2} i s \right) + \text{conj} \quad (5.17)$$

and choosing it of the same form as before, we find

$$0 = -i \int dx d\theta \left(\left(3s + x^\lambda \frac{\partial s}{\partial x^\lambda} + \frac{1}{2} \theta \frac{\partial s}{\partial \theta} \right) \frac{\partial f}{\partial s} - \frac{3}{2} s \right) \quad (5.18)$$

$$= -i \int dx d\theta \left(3s \frac{\partial f}{\partial s} - 3f - \frac{3}{2} s \right) \quad (5.19)$$

giving the same equation, (5.12), as before for f . This is not surprising since for a chiral field the R weight and dimension are not independent.

The complementary functional is required to satisfy

$$\hat{Q}^\lambda \Gamma_1 = 0 \quad (5.20)$$

Choosing Γ_1 of the previous form, we find

$$\hat{Q}^\lambda \Gamma_1 = -i \int dx d\theta d\bar{\theta} \left(-d + x^\lambda \frac{\partial}{\partial x^\lambda} + \frac{1}{2} \theta \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Phi \quad (5.21)$$

$$= i \left(d_{\frac{1}{2}} + 2 \right) \Gamma_1 \quad (5.22)$$

Any function $\Phi(S, \bar{S})$ with $d_\Phi = -2$ is therefore consistent with dilatation invariance.

iii) Conformal Transformations

The Ward identity (5.4) for conformal transformations is a little different in that the anomaly term has higher components in $\theta, \bar{\theta}$. As always, however, the superfield structure ensures that only the lowest component need be calculated explicitly to verify the formula.

The particular integral therefore satisfies,

$$0 = Q_V^K(S, \bar{S}) \Gamma_0 + 3i \int dx x_\nu (D^2 S + \bar{D}^2 \bar{S}) \Big|_{\theta=\bar{\theta}=0} \quad (5.23)$$

$$= \int dx d\theta \left(-i \delta_V^K S \frac{\delta \Gamma_0}{\delta S} + 3i x_\nu S \right) + \text{conj} \quad (5.24)$$

Substituting for $\delta_V^K S$ and performing various integrations by parts, gives

$$0 = \int dx d\theta x_\nu \left(-6 S \frac{\partial f}{\partial S} + 6 f + 3 S \right) \quad (5.25)$$

So the same function f gives a particular integral for all three Ward identities.

To determine the restrictions on the complementary functional, evaluate

$$Q_V^K \Gamma_1 = -i \int dx d\theta d\bar{\theta} \left(2 x_\nu S^D + 2 x^\lambda S_{\lambda}^M + x^2 \partial_\nu - 2 x_\nu x^\lambda \partial_\lambda \right. \quad (5.26)$$

$$\left. + \theta \sigma_\nu \bar{\theta} (2 S^R + i n) + \theta^2 \bar{\theta}^2 \partial_\nu \right) \Xi$$

$$= -i \int dx d\theta d\bar{\theta} \left(-2(d+2) x_\nu + 3 i n \theta \sigma_\nu \bar{\theta} \right) \Xi \quad (5.27)$$

So conformal invariance requires that in the complementary functional, Φ should have $d = -2$ and $n = 0$. No new restrictions therefore emerge from the conformal Ward identity.

As a double-check on the formalism, I have also calculated the $\bar{\theta}$ component of Eq. (5.4) explicitly, this being the Ward identity for the conformal supersymmetry current $S_{\mu\alpha}$. The expected result holds - the particular integral function f is required to satisfy Eq. (5.12), while the requirement on Φ is weaker, namely that $2(d_\Phi + 2) + 3n_\Phi = 0$.

Summarizing, the zero-momentum superconformal Ward identities have been shown to be satisfied by an effective action $\Gamma[S, \bar{S}]$ of the form

$$\begin{aligned} \Gamma [s, \bar{s}] &= \int d^4x d\theta \frac{1}{2} S \left(\log \frac{S}{\mu^3} - 1 \right) + \int d^4x d\bar{\theta} \frac{1}{2} \bar{S} \left(\log \frac{\bar{S}}{\mu^3} - 1 \right) \\ &+ \int d^4x d\theta d\bar{\theta} \Xi (s, \bar{s}) \end{aligned} \quad (5.28)$$

where ϕ is a real superfield with $d = -2$, $n = 0$.

To close this section, there are two further points which must be made. First, the complementary functional in Eq. (5.28) has been assumed to be the integral of a real scalar superfield. To be more general we should really consider the possibility of Γ_1 being the integral of a chiral superfield, viz.

$$\Gamma_1 [s, \bar{s}] = \int d^4x d\theta \phi(s, \bar{s}) + \text{conj} \quad (5.29)$$

with ϕ chiral and Γ_1 real. This is a more general form than above, although we will not exploit the extra freedom in the examples which follow. For a complementary functional of this form, it is easy to check that the Ward identities require $d_\phi = -3$, $n_\phi = -2$.

Finally, there is an additional and rather subtle restriction imposed by the superconformal algebra. The expressions used throughout for field variations assume that the fields transform covariantly under the superconformal algebra. In particular, it is required that any chiral (antichiral) field $\phi(\bar{\phi})$ must satisfy the conformal supersymmetry relation $d = 3/2 n$ ($d = -3/2 n$).

We are therefore led to impose the extra condition of superconformal covariance. This is a non-trivial restriction in the examples where supersymmetric covariant derivatives occur. For example, consider the antichiral field $D^2 S$. This has $d = -4$, $n = 0$ and is not a covariant quantity. In fact, it is easily checked that to maintain covariance under the action of D^2 , $\bar{D}^2$, any chiral field must have dimension $d = -1$ and chiral weight $n = -2/3$, while real fields must have $d = -2$, $n = 0$. This suggests that it will be simpler to use $S^{1/3}$ rather than S as the basic field in constructing Γ .

The superconformal covariance condition is not straightforward to impose in practice since it is of course always possible to write a covariant expression as a sum of individually non-covariant terms. However, if we make the natural assumption that the most general covariant action can be constructed from individually covariant terms, we may satisfy the condition simply by only ever writing down covariant terms.

This concludes the discussion of the consequences of the zero-momentum superconformal Ward identities for the structure of the effective action $\Gamma[S, \bar{S}]$. In the next section, special examples of actions satisfying these requirements will be displayed.

6. - THE SIMPLEST EFFECTIVE ACTION AND THE GENERAL EFFECTIVE POTENTIAL

While the anomalous Ward identities place powerful constraints on the allowed form of the effective action, they by no means determine it completely. Further, dynamical, information specific to a particular theory must be incorporated before a simple and predictive effective action can be found. This additional dynamical input will take the form of general theorems [e.g., Witten's index criterion for supersymmetry breaking¹²⁾], specified assumptions (e.g., the presence or absence of chiral symmetry breaking and the associated selection of relevant low-energy degrees of freedom), and a low-energy approximation.

These dynamical issues, for theories of physical interest such as supersymmetric QCD, will be the subject of II. In this final section, I will provide an introduction to this work by considering two simple examples of effective actions consistent with the Ward identities which display interesting features.

i) The simplest effective action (VY model)

The form of the particular integral $\Gamma_0[S, \bar{S}]$ has been fixed once and for all by the Ward identities, and so the remaining freedom rests entirely in the choice of the complementary functional $\Gamma_1[S, \bar{S}]$. The simplest choice is to take Γ_1 to be a functional of the chiral fields S and $\bar{S}$ only, with no covariant derivatives.

It is easy to see that the requirements of reality, dimensionality and chiral weight discussed in the last section then determine Γ_1 uniquely to be

$$\Gamma_1[S, \bar{S}] = \alpha \int d^4x d\theta d\bar{\theta} (S \bar{S})^{1/3} \quad (6.1)$$

α is an undetermined dimensionless constant. Interestingly, and in accordance with the remarks on covariance at the end of Section 5, we see that the chiral field enters as the $d = -1$ function $S^{1/3}$.

The complete effective action is

$$\begin{aligned} \Gamma[S, \bar{S}] = & \int d^4x d\theta \frac{1}{2} S \left(\log \frac{S}{\mu^2} - 1 \right) + \int d^4x d\bar{\theta} \frac{1}{2} \bar{S} \left(\log \frac{\bar{S}}{\mu^2} - 1 \right) \\ & + \alpha \int d^4x d\theta d\bar{\theta} (S \bar{S})^{1/3} \end{aligned} \quad (6.2)$$

This is the action first written down by Veneziano and Yankielowicz⁶⁾, who have discussed some of its immediate properties. I will return to this analysis in II, but for completeness I give here the main results, taken from Ref. 6).

Expanding S in components,

$$S(x, \theta) = \phi(x) + \theta^\alpha \chi_\alpha(x) + \theta^2 F(x) \quad (6.3)$$

the $(S\bar{S})^{1/3}$ term in the action provides standard kinetic terms for $\phi(x)$ and $\chi_\alpha(x)$, but not for the auxiliary field F which can therefore be eliminated using its equation of motion.

The bosonic potential $V(\phi)$, which receives contributions from both Γ_0 and Γ_1 is then bounded below, and has a supersymmetric minimum [$V(\phi) = 0$] at $\phi = \mu^3$. In supersymmetric Yang-Mills theory, where $\phi(x)$ is proportional to $\bar{\lambda}\lambda$ with λ the gluino field, this indicates that the chiral $U_R(1)$ symmetry is spontaneously broken, whereas the global supersymmetry remains unbroken.

The low-energy spectrum is a Wess-Zumino scalar multiplet comprising a massive scalar and a massive pseudoscalar boson together with a Majorana fermion of equal mass. The anomaly allows the would-be pseudoscalar Goldstone boson corresponding to the spontaneous breaking of the $U_R(1)$ symmetry to acquire a mass, and the unbroken supersymmetry then requires the existence of the degenerate mass scalar and fermion.

ii) The most general effective potential

I will now write down the most general action terms allowed by the anomalous Ward identities which can contribute to the effective potential.

Notice first that any term involving more than two covariant derivatives acting on S gives rise to a space-time derivative and so cannot contribute to the potential. This is an immediate consequence of the (anti-) commutation relations for covariant derivatives and the chirality of S .

It may also be seen that terms involving only one covariant derivative acting on S , e.g., $D^\alpha S D_\alpha S$, may be excluded by the requirement of superconformal covariance. This follows from the identity, $D^\alpha S D_\alpha S = \frac{1}{2} D^2(S^2) - S D^2 S$. This covariance requirement further suggests that we use $S^{1/3}$ rather than S itself as the basic field. It is then not difficult to see that the complementary functional giving rise to the most general effective potential is an algebraic function of the variables $S^{1/3}$, $\bar{S}^{1/3}$, $D^2 S^{1/3}$, $\bar{D}^2 \bar{S}^{1/3}$.

Then, imposing the condition that with

$$\Gamma_1 = \int dx d\theta d\bar{\theta} \quad \Xi \left(s^{1/3}, \bar{s}^{1/3}, D^2 s^{1/3}, \bar{D}^2 \bar{s}^{1/3} \right) \quad (6.4)$$

Φ should be real with $d_\Phi = -2$, $n_\Phi = 0$, the effective action is determined to be,

$$\begin{aligned} \Gamma[s, \bar{s}] &= \int dx d\theta \frac{1}{2} s \left(\log \frac{s}{\mu^2} - 1 \right) + \int dx d\bar{\theta} \frac{1}{2} \bar{s} \left(\log \frac{\bar{s}}{\mu^2} - 1 \right) \\ &+ \int dx d\theta d\bar{\theta} \quad (s \bar{s})^{1/3} \quad f(x, \bar{x}) \end{aligned} \quad (6.5)$$

where X is the dimensionless chiral combination,

$$X = s^{1/3} \left(\bar{D}^2 \bar{s}^{1/3} \right)^{-1/2} \quad (6.6)$$

and $f(x, y)$ is an arbitrary function of two variables, satisfying the reality condition,

$$f^*(x, y) = f(y, x) \quad (6.7)$$

A full analysis of this action will be given in II. Here, I simply wish to point out an interesting and general property of the potential derived from Eq. (6.5).

Defining the effective potential V by

$$\Gamma[s, \bar{s}] = \int dx \left(-V(\phi, X, F) + \text{spacetime derivative terms} \right) \quad (6.8)$$

and expanding S in components, we find the bosonic (i.e., $X = 0$) part of the potential to be,

$$\begin{aligned} V(\phi, F) &= 72 \int dx \quad \hat{\phi}^4 \left((z + z^*) \log \hat{\phi} \right. \\ &\quad \left. + \frac{1}{3} i \theta (z - z^*) - g(z, z^*) \right) \end{aligned} \quad (6.9)$$

where $\hat{\phi}^2 = (\phi\phi^*)^{1/3}$, the angle θ is the phase of $\phi(x)$, and the dimensionless field $z = \frac{1}{12} (\phi\phi^*)^{-2/3} F$. $g(z, z^*)$ is an arbitrary real function related, in a rather non-trivial way, to the function f appearing in the action. The derivation of Eq. (6.9) is given in II.

Clearly, without further specifying f , or g , it is difficult to extract much information from this potential. In particular it is quite feasible that for a broad class of functions the potential may only exhibit local minima with $V > 0$, indicative of supersymmetry breaking.

The potential does however have one completely general property, which may be seen from Eq. (6.9) - it is unbounded below. Remarkably, this is true for any choice of the arbitrary function $g(z, z^*)$. To see this consider the large $\hat{\phi}$ behaviour of V . This is dominated by the $(z + z^*) \hat{\phi}^4 \log \hat{\phi}$ term (which of course is why the function g is irrelevant here), which is negative and unbounded as $\hat{\phi} \rightarrow \infty$ for negative values of $\text{Re } z$.

Notice that this behaviour did not appear in the Veneziano-Yankielowicz model, because there the absence of kinetic terms for the auxiliary field F allowed it to be eliminated in terms of ϕ by its equation of motion. In the present, more general, situation, this corresponds to restricting to a line in $(z, \hat{\phi})$ space - a line along which V remains positive. The elimination of F does not however appear possible in the general case.

Whether this curious behaviour of the potential is or is not a problem raises many interesting questions concerning the interpretation and physical implications of the effective action. In this paper we have come as far as possible utilizing only the symmetry information contained in the anomalous superconformal Ward identities. To go further, we must face up to the much more difficult problems of the dynamical behaviour of supersymmetric gauge theories. These issues will be addressed in the succeeding paper of this series.

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APPENDIX A - THE SUPERCONFORMAL ALGEBRA

The superconformal algebra is the graded Lie algebra comprising the usual conformal group generators, translations P_μ , Lorentz transformations $M_{\mu\nu}$, dilatations D , conformal transformations K_μ , together with the generator R of chiral $U_R(1)$ transformations, and the grading generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ of translational supersymmetry and $S_\alpha, \bar{S}_{\dot{\alpha}}$ of conformal supersymmetry.

They satisfy the (anti-) commutation relations

$$\begin{aligned} [M_{\mu\nu}, P_\lambda] &= i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) & [D, P_\mu] &= -i P_\mu \\ [M_{\mu\nu}, K_\lambda] &= i(K_\mu g_{\nu\lambda} - K_\nu g_{\mu\lambda}) & [D, K_\mu] &= i K_\mu \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(M_{\nu\sigma} g_{\mu\rho} - M_{\nu\rho} g_{\mu\sigma} + M_{\mu\sigma} g_{\nu\rho} - M_{\mu\rho} g_{\nu\sigma}) \\ [P_\mu, K_\nu] &= 2i(D g_{\mu\nu} - M_{\mu\nu}) \\ [P_\mu, P_\nu] &= [K_\mu, K_\nu] = [D, M_{\mu\nu}] = 0 \end{aligned} \quad (A.1)$$

$$[R, P_\mu] = [R, K_\mu] = [R, D] = [R, M_{\mu\nu}] = 0 \quad (A.2)$$

$$\begin{aligned} [R, Q_\alpha] &= Q_\alpha & [R, S_\alpha] &= -S_\alpha \\ [R, \bar{Q}_{\dot{\alpha}}] &= -\bar{Q}_{\dot{\alpha}} & [R, \bar{S}_{\dot{\alpha}}] &= \bar{S}_{\dot{\alpha}} \\ [P^\mu, Q_\alpha] &= 0 & [P^\mu, S_\alpha] &= -i\sigma^\mu_{\alpha\dot{\alpha}} \bar{Q}^{\dot{\alpha}} \\ [P^\mu, \bar{Q}_{\dot{\alpha}}] &= 0 & [P^\mu, \bar{S}_{\dot{\alpha}}] &= -i\bar{Q}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \\ [M^{\mu\nu}, Q_\alpha] &= -\frac{1}{2}\sigma^{\mu\nu}_\alpha{}^\beta Q_\beta & [M^{\mu\nu}, S_\alpha] &= -\frac{1}{2}\sigma^{\mu\nu}_\alpha{}^\beta S_\beta \\ [M^{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= -\frac{1}{2}\bar{\sigma}^{\mu\nu}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{Q}_{\dot{\beta}} & [M^{\mu\nu}, \bar{S}_{\dot{\alpha}}] &= -\frac{1}{2}\bar{\sigma}^{\mu\nu}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{S}_{\dot{\beta}} \end{aligned} \quad (A.3)$$

$$\begin{aligned}
 [D, Q_\alpha] &= -\frac{1}{2} i Q_\alpha & [D, S_\alpha] &= \frac{1}{2} i S_\alpha \\
 [D, \bar{Q}_{\dot{\alpha}}] &= -\frac{1}{2} i \bar{Q}_{\dot{\alpha}} & [D, \bar{S}_{\dot{\alpha}}] &= \frac{1}{2} i \bar{S}_{\dot{\alpha}} & (A.3) \\
 [K^\mu, Q_\alpha] &= -i \sigma^\mu_{\alpha\dot{\alpha}} \bar{S}^{\dot{\alpha}} & [K^\mu, S_\alpha] &= 0 & \text{cont.} \\
 [K^\mu, \bar{Q}_{\dot{\alpha}}] &= -i S^\alpha \sigma^\mu_{\alpha\dot{\alpha}} & [K^\mu, \bar{S}_{\dot{\alpha}}] &= 0
 \end{aligned}$$

and,

$$\begin{aligned}
 \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} &= 2 \sigma^\mu_{\alpha\dot{\alpha}} P_\mu & \{S_\alpha, \bar{S}_{\dot{\alpha}}\} &= 2 \sigma^\mu_{\alpha\dot{\alpha}} K_\mu \\
 \{Q_\alpha, S_\beta\} &= i \left(\sigma^\mu_{\alpha\beta} M_{\mu\nu} + 2i \epsilon_{\alpha\beta} D - 3 \epsilon_{\alpha\beta} R \right) & (A.4) \\
 \{\bar{Q}_{\dot{\alpha}}, \bar{S}_{\dot{\beta}}\} &= -i \left(\bar{\sigma}^\mu_{\dot{\alpha}\dot{\beta}} M_{\mu\nu} - 2i \epsilon_{\alpha\beta} D - 3 \epsilon_{\alpha\beta} R \right) \\
 \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = \{S_\alpha, S_\beta\} = \{\bar{S}_{\dot{\alpha}}, \bar{S}_{\dot{\beta}}\} = 0 \\
 \{Q_\alpha, \bar{S}_{\dot{\alpha}}\} &= \{\bar{Q}_{\dot{\alpha}}, S_\alpha\} = 0
 \end{aligned}$$

Notice that from the $\{Q_\alpha, S_\beta\}$ relation it follows that for a chiral superfield ϕ , i.e., satisfying $\bar{D}_{\dot{\alpha}}\phi = 0$, the length dimension d and $U_R(1)$ chiral weight n are not independent, but are related by $d = 3/2 n$. Similarly for an antichiral field $\bar{\phi}$, for which $D_\alpha\bar{\phi} = 0$, $d = -3/2 n$.

[Apart from using d to denote length dimension, I follow the conventions and notations of Clark, Piguet and Sibold listed in Appendix A of Ref. 8). The integration symbol $\int dx d\theta d\bar{\theta}$ here denotes $\int d^4x d^2\theta d^2\bar{\theta}$.]

These generators can be represented by linear differential operators in superspace acting on superfields. The infinitesimal variation $\delta^G \Phi$ of a superfield Φ under the action of the generator G is defined by,

$$-i \delta^G \Phi = [G, \Phi] \quad (A.5)$$

For an arbitrary superfield $\Phi(x, \theta, \bar{\theta})$ of dimension d and chiral weight n in the real representation, the infinitesimal transformations under the superconformal algebra are,

$$\begin{aligned}
\delta_\mu^P \Xi &= \partial_\mu \Xi \\
\delta_{\mu\nu}^M \Xi &= \left(x_\mu \partial_\nu - x_\nu \partial_\mu - \frac{1}{2} i \theta \sigma_{\mu\nu} \frac{\partial}{\partial \theta} + \frac{1}{2} i \bar{\theta} \bar{\sigma}_{\mu\nu} \frac{\partial}{\partial \bar{\theta}} \right) \Xi \\
\delta^D \Xi &= \left(-d + x^\lambda \partial_\lambda + \frac{1}{2} \theta \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Xi \\
\delta_\mu^K \Xi &= \left(2 x_\mu \delta^D + 2 x^\nu \delta_{\mu\nu}^M + x^2 \partial_\mu - 2 x_\mu x^\lambda \partial_\lambda \right. \\
&\quad \left. + \theta \sigma_\mu \bar{\theta} (2 \delta^R + i n) + \theta^2 \bar{\theta}^2 \partial_\mu \right) \Xi \\
\delta^R \Xi &= i \left(n + \theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Xi \\
\delta_\alpha^Q \Xi &= \left(\frac{\partial}{\partial \theta^\alpha} + i \sigma_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) \Xi \\
\delta_{\dot{\alpha}}^{\bar{Q}} \Xi &= \left(-\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma_{\alpha\dot{\alpha}} \partial_\mu \right) \Xi \\
\delta_\alpha^S \Xi &= \left(-x_\mu \sigma_{\alpha\dot{\alpha}} \delta^{\bar{Q}\dot{\alpha}} + 2 \theta_\alpha \delta^R - i \theta^2 \bar{D}_\alpha + 2i \left(d + \frac{1}{2} n \right) \theta_\alpha \right) \Xi \\
\delta_{\dot{\alpha}}^{\bar{S}} \Xi &= \left(-x_\mu \delta^{Q\alpha} \sigma_{\alpha\dot{\alpha}} + 2 \bar{\theta}_{\dot{\alpha}} \delta^R + i \bar{\theta}^2 D_{\dot{\alpha}} + 2i \left(-d + \frac{1}{2} n \right) \bar{\theta}_{\dot{\alpha}} \right) \Xi
\end{aligned} \tag{A.6}$$

For a real field Φ , $n = 0$.

For a chiral superfield $\phi(x, \theta)$ in the chiral representation the corresponding transformations are,

$$\begin{aligned}
\delta_\mu^P \phi &= \partial_\mu \phi \\
\delta_{\mu\nu}^M \phi &= \left(x_\mu \partial_\nu - x_\nu \partial_\mu - \frac{1}{2} i \theta \sigma_{\mu\nu} \frac{\partial}{\partial \theta} \right) \phi \\
\delta^D \phi &= \left(-d + x^\lambda \partial_\lambda + \frac{1}{2} \theta \frac{\partial}{\partial \theta} \right) \phi \\
\delta_\mu^K \phi &= \left(2 x_\mu (-d + x^\lambda \partial_\lambda) - x^2 \partial_\mu + x_\mu \theta \frac{\partial}{\partial \theta} - i x^\lambda \theta \sigma_{\lambda\mu} \frac{\partial}{\partial \theta} \right) \phi
\end{aligned} \tag{A.7}$$

$$\delta^R \phi = i \left(\frac{2}{3} d + \theta \frac{\partial}{\partial \theta} \right) \phi$$

$$\delta_\alpha^Q \phi = \frac{\partial}{\partial \theta^\alpha} \phi$$

$$\delta_{\dot{\alpha}}^{\bar{Q}} \phi = -2i \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \phi \quad (A.7) \text{ cont.}$$

$$\delta_\alpha^S \phi = \left(-x_\mu \sigma_{\alpha\dot{\alpha}}^\mu \delta_{\dot{\alpha}}^{\bar{Q}} + 4i d \theta_\alpha - 2i \theta^2 \frac{\partial}{\partial \theta^\alpha} \right) \phi$$

$$\delta_{\dot{\alpha}}^{\bar{S}} \phi = -x_\mu \delta^Q \sigma_{\alpha\dot{\alpha}}^\mu \phi$$

where the superconformal relation $d = 3/2 n$ is used. Similar results hold for the antichiral representation.

My conventions are such that under finite R and D transformations, a superfield of dimension d and chiral weight n transforms as,

$$\mathbb{I}(x, \theta, \bar{\theta}) \xrightarrow{R} e^{i n \alpha} \mathbb{I}(x, \theta e^{i\alpha}, \bar{\theta} e^{-i\alpha}) \quad (A.8)$$

$$\mathbb{I}(x, \theta, \bar{\theta}) \xrightarrow{D} e^{-d\beta} \mathbb{I}(x e^\beta, \theta e^{\beta/2}, \bar{\theta} e^{\beta/2}) \quad (A.9)$$

For convenience, I collect here the dimensions and chiral weights of some of the quantities considered in the text.

	V	J _V	S	J _S	$\bar{S}$	J $\bar{S}$	θ	$\bar{\theta}$	D	$\bar{D}$
d	-3	1	-3	0	-3	0	1/2	1/2	-1/2	-1/2
n	0	0	-2	0	2	0	-1	1	1	-1

The symmetry operators $\hat{\Omega}^R$, $\hat{\Omega}^D$ and $\hat{\Omega}_V^K$ were listed in Section 3. For completeness, the remaining ones are given here, the derivation following directly from the algebra, Eqs. (A.1) to (A.4)

$$\hat{\Omega}_\mu^P = R_\mu^P \quad (A.10)$$

$$\hat{Q}_{\mu\nu}^M = Q_{\mu\nu}^M + \frac{1}{2} i \theta \sigma_{\mu\nu} Q^Q + \frac{1}{2} i \bar{\theta} \bar{\sigma}_{\mu\nu} \bar{Q}^Q \quad (A.11)$$

$$- \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} \theta \sigma^{\rho\lambda} \bar{Q}^P$$

$$\hat{Q}_\alpha^Q = Q_\alpha^Q - 2i \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\theta}^{\dot{\alpha}} Q_\mu^P \quad (A.12)$$

$$\begin{aligned} \hat{Q}_\alpha^S &= Q_\alpha^S - \theta^\beta \left(\sigma_{\alpha\beta}^{\mu\nu} Q_{\mu\nu}^M + 3 \epsilon_{\alpha\beta} Q^R - 2i \epsilon_{\alpha\beta} Q^D \right) \\ &\quad - 2i \theta^2 Q_\alpha^Q - i \theta \sigma^{\mu\nu} \bar{\theta} \left(\sigma_{\mu\alpha\dot{\alpha}} \bar{Q}^{Q\dot{\alpha}} \right) - 2 \theta^2 \sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\theta}^{\dot{\alpha}} Q_\mu^P \end{aligned} \quad (A.13)$$

APPENDIX B - SUPERCONFORMAL CURRENTS

The Ward identity for the supercurrent $V_{\alpha\dot{\alpha}}$ contains all the superconformal symmetry information of the theory. To display this explicitly, it is necessary to show how the individual superconformal currents are obtained from $V_{\alpha\dot{\alpha}}$, and to write the corresponding Ward identities. This analysis has been carried out by Clark, Piguet and Sibold⁸⁾, and I quote their results here.

The superfield currents are defined by

$$\hat{R}_\mu = V_\mu \quad (B.1)$$

$$\hat{Q}_{\mu\alpha} = i \left(D_\alpha V_\mu - (\sigma_\mu \bar{\sigma}^\nu D)_\alpha V_\nu \right) \quad (B.2)$$

$$\hat{T}_{\mu\nu} = -\frac{1}{16} \left(V_{\mu\nu} + V_{\nu\mu} - 2g_{\mu\nu} V_\lambda{}^\lambda \right) \quad (B.3)$$

with

$$V_{\mu\nu} = [D^\rho, \bar{D}^{\dot{\rho}}] \sigma_{\mu\rho\dot{\rho}} V_\nu \quad (B.4)$$

$$\hat{M}_{\mu\nu\lambda} = x_\lambda \hat{T}_{\mu\nu} - x_\nu \hat{T}_{\mu\lambda} \quad (B.5)$$

$$\hat{D}_\mu = x^\lambda \hat{T}_{\mu\lambda} \quad (B.6)$$

$$\hat{K}_{\mu\nu} = (2x_\nu x^\lambda - g_\nu{}^\lambda x^2) \hat{T}_{\mu\lambda} \quad (B.7)$$

$$\hat{S}_{\mu\alpha} = i x_\lambda \sigma_{\alpha\dot{\alpha}}{}^\lambda \hat{Q}_{\mu}{}^{\dot{\alpha}} \quad (B.8)$$

Here, $\hat{R}_\mu$ is the superfield whose lowest $(\theta = \bar{\theta} = 0)$ component is the current corresponding to R symmetry, etc. It is proved in Ref. 8) that these x-moment currents are the only possible ones obtainable from $V_{\alpha\dot{\alpha}}$ which can be conserved.

These currents satisfy the following Ward identities, derived from Eqs. (2.7) and (B.1) to (B.8):

$$\partial^\mu \hat{R}_\mu - i (D^2 S - \bar{D}^2 \bar{S}) - i \omega^R \Gamma = 0 \quad (B.9)$$

where the contact term ω^R is the local symmetry operator corresponding to $\hat{\Omega}^R$

$$\partial^\mu \hat{Q}_{\mu\alpha} - i \omega_\alpha^Q \Gamma = 0 \quad (B.10)$$

$$\partial^\mu \hat{T}_{\mu\nu} - i \omega_\nu^P \Gamma = 0 \quad (B.11)$$

$$\partial^\mu M_{\mu\nu\lambda} - i \omega_{\nu\lambda}^M \Gamma = 0 \quad (B.12)$$

$$\partial^\mu \hat{D}_\mu + \frac{3}{2} (D^2 S + \bar{D}^2 \bar{S}) - i \omega^D \Gamma = 0 \quad (B.13)$$

$$\partial^\mu \hat{K}_{\mu\nu} + 3 \pi_\nu (D^2 S + \bar{D}^2 \bar{S}) - i \omega_\nu^K \Gamma = 0 \quad (B.14)$$

$$\partial^\mu \hat{S}_{\mu\alpha} + 12 i D_\alpha S - i \omega_\alpha^S \Gamma = 0 \quad (B.15)$$

From these results the following two trace identities can be deduced,

$$\hat{T}_\mu + \frac{3}{2} (D^2 S + \bar{D}^2 \bar{S}) - i \omega_\mu^P \Gamma = 0 \quad (B.16)$$

$$\hat{Q}_\mu^\alpha \sigma_{\alpha\dot{\alpha}}^\mu - 12 i \bar{D}_{\dot{\alpha}} \bar{S} - i \omega_{\mu\tau}^Q \Gamma = 0 \quad (B.17)$$

In many supersymmetry theories, e.g., the massive Wess-Zumino model and supersymmetric QED, the above definitions yield improved conserved currents, i.e., currents which at the tree level are conserved up to soft breaking terms. This requires that S should be soft at the tree level. However, in supersymmetric Yang-Mills theory, this is not true, if the supercurrent is required to be BRS invariant. In this model, the non-conformal nature of the gauge-fixing terms

means that S does not vanish at the tree level, and so the supercurrent is not improved¹⁰⁾. Nevertheless, the definitions (B.1) to (B.8) and identities (B.9) to (B.17) remain valid, and the supercurrent $V_{\alpha\dot{\alpha}}$ still includes all the superconformal symmetry content of the model.

APPENDIX C - DERIVATION OF THE CONTACT TERMS

In this appendix some details of the calculation of the contact terms ω_α described in Section 3 will be given. First consider the chiral superfield case. As discussed in the text, dimensionality and linearity arguments restrict the form of ω_α to,

$$\omega_\alpha(\phi) = a D_\alpha \phi \frac{\delta}{\delta \phi} + b \phi D_\alpha \frac{\delta}{\delta \phi} \quad (C.1)$$

and we require to check Eqs. (3.30) and (3.31). Using the simple identity,

$$\int dx D^\alpha A = \frac{1}{2} \int dx d\theta \theta^\alpha A + \frac{1}{2} \theta^\alpha \int dx d\theta A \quad (C.2)$$

we may evaluate,

$$\left. \int dx D^\alpha \omega_\alpha \right|_{\theta=\bar{\theta}=0} = \frac{1}{2} \int dx d\theta \theta^\alpha \omega_\alpha \Big|_{\bar{\theta}=0} \quad (C.3)$$

$$= \frac{1}{2} \int dx d\theta \left((a-b) \theta^\alpha \frac{\partial \phi}{\partial \theta^\alpha} + 2b \phi \right) \frac{\delta}{\delta \phi} \quad (C.4)$$

after an integration by parts. Subtracting the conjugate expression and comparing with (3.30), we find that the identity is satisfied with the identification $a - b = 2$, $b = n$.

Next consider the case of a real superfield $\Phi(x, \theta, \bar{\theta})$. As noted in the text, the dimensionality, chiral weight and linearity arguments restrict ω_α to be a linear combination of the ten independent terms which can be made from $\bar{D}^{\dot{\alpha}}$, $\bar{D}_{\dot{\alpha}}$, D_α , ϕ and $\delta/\delta\phi$, viz.

$$\begin{aligned} \frac{1}{2} \omega_\alpha(\Phi) = & a_1 \bar{D}^2 D_\alpha \Phi \frac{\delta}{\delta \Phi} + a_2 D_\alpha \bar{D}^2 \Phi \frac{\delta}{\delta \Phi} \\ & + b_1 \bar{D}^2 \Phi D_\alpha \frac{\delta}{\delta \Phi} + b_2 \bar{D}_\alpha D_\alpha \Phi \bar{D}^{\dot{\alpha}} \frac{\delta}{\delta \Phi} + b_3 D_\alpha \bar{D}_\alpha \Phi \bar{D}^{\dot{\alpha}} \frac{\delta}{\delta \Phi} \\ & + c_1 D_\alpha \Phi \bar{D}^2 \frac{\delta}{\delta \Phi} + c_2 \bar{D}^{\dot{\alpha}} \Phi \bar{D}_\alpha D_\alpha \frac{\delta}{\delta \Phi} + c_3 \bar{D}^{\dot{\alpha}} \Phi D_\alpha \bar{D}_\alpha \frac{\delta}{\delta \Phi} \\ & + d_1 \Phi \bar{D}^2 D_\alpha \frac{\delta}{\delta \Phi} + d_2 \Phi D_\alpha \bar{D}^2 \frac{\delta}{\delta \Phi} \end{aligned} \quad (C.5)$$

First we must check the R identity (3.33), (3.34). Inserting the above form for ω_α , using the identity (C.2) twice, and repeatedly integrating by parts gives,

$$\begin{aligned}
 & \int dx \left(D^\alpha \omega_\alpha - \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}} \right) \Big|_{\theta=\bar{\theta}=0} \\
 &= \int dx d\theta d\bar{\theta} \left\{ (c_1 + d_1 + d_3) \int^R \frac{d}{d\bar{x}} \right. \\
 & \quad + (-4d_3 - c_2 + b_2 + c_3 + b_3) 2i \theta^\alpha \bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \int^R \frac{d}{d\bar{x}} \left. \right\} \quad (C.6) \\
 & \quad + (a_1 + c_1 - d_3 + b_2 - c_3) \left[[\bar{D}^2 D_\alpha \int^R \frac{d}{d\bar{x}}] \right] \\
 & \quad + (d_3 + a_3 + c_3 - b_3) \left[[D_\alpha \bar{D}^2 \int^R \frac{d}{d\bar{x}}] \right] \\
 & \quad + (b_1 + d_1 + d_3 + c_2 - c_3) \left[[\bar{D}^2 \int^R \frac{d}{d\bar{x}} D_\alpha] \right]
 \end{aligned}$$

where here the symbol $[[X_\alpha]]$ means the contribution to $\int dx (D^\alpha \omega_\alpha - \bar{D}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}}) \Big|_{\theta=\bar{\theta}=0}$ from a term $\frac{1}{2} \omega_\alpha = X_\alpha$. These do not need to be evaluated further. To satisfy the identity (3.33), (3.34) requires $(c_1 + d_1 + d_3) = 1$, and all other combinations of the coefficients in brackets in Eq. (C.6) must vanish.

Now consider dilatations. Here we must evaluate

$$\Omega^D = \int dx \omega^D \Big|_{\theta=\bar{\theta}=0} \quad (C.7)$$

with ω^D given by Eqs. (3.23) to (3.25). This expression for Ω^D can be simplified. Using the identity (C.2) twice, and repeated use of the commutation relations for covariant derivatives with $x_{\alpha\dot{\alpha}}$, θ_α , and $\bar{\theta}_{\dot{\alpha}}$ gives,

$$\Omega^D = -\frac{i}{8} \int dx d\theta d\bar{\theta} \left(x^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \omega_\alpha + \frac{i}{2} \bar{\theta}^2 \theta^\alpha \omega_\alpha \right) + \text{conj.} \quad (C.8)$$

Then, substituting for ω_α , a long calculation eventually yields,

$$\begin{aligned}
 \Omega^D &= -i \int dx d\theta d\bar{\theta} \left\{ (2c_1 + 2d_1 - 2d_3 + c_2 + b_2 + c_3 + b_3) x^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \int^R \frac{d}{d\bar{x}} \right. \\
 & \quad + (c_1 + d_1 + 5d_3 + c_2 + 3c_3) \frac{1}{2} \left(\theta^\alpha \frac{\partial}{\partial \theta^\alpha} - \bar{\theta}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} \right) \int^R \frac{d}{d\bar{x}} \left. \right\} \quad (C.9)
 \end{aligned}$$

$$\begin{aligned}
 & + (2d_1 - 6d_3) \quad \bar{\mathbb{E}} \frac{d}{d\bar{\mathbb{E}}} \quad \Big\} \\
 & + (c_1 + a_1 - d_3 + b_2 + \frac{1}{2}c_3 + \frac{1}{2}b_3) \quad [[\bar{D}^2 D_\alpha \bar{\mathbb{E}} \quad \frac{d}{d\bar{\mathbb{E}}}]] \\
 & + (d_3 + a_3 - \frac{1}{2}c_3 + \frac{1}{2}b_3) \quad [[D_\alpha \bar{D}^2 \bar{\mathbb{E}} \quad \frac{d}{d\bar{\mathbb{E}}}]] \quad (C.9) \\
 & + (b_1 + d_1 + d_3 + c_2 + c_3) \quad [[\bar{D}^2 \bar{\mathbb{E}} \quad D_\alpha \frac{d}{d\bar{\mathbb{E}}}]] \quad \text{cont.}
 \end{aligned}$$

where again the symbol $[[X_\alpha]]$ denotes the contribution to Ω^D of a term $\frac{1}{2}\omega_\alpha = X_\alpha$. To satisfy the identity (3.35), (3.36) we require,

$$(2d_1 - 6d_3) = -d \quad (d = \text{length dimension of } \Phi) \quad (C.10)$$

$$c_1 + d_1 - d_3 + \frac{1}{2}(c_2 + b_2 + c_3 + b_3) = 1 \quad (C.11)$$

$$c_1 + d_1 + 5d_3 + c_2 + 3c_3 = 1 \quad (C.12)$$

and the other combinations in brackets all to be zero.

From these calculations we therefore find a total of 11 simultaneous equations to be satisfied by the parameters. They have a unique solution,

$$\begin{aligned}
 a_3 &= b_2 = b_3 = c_2 = c_3 = d_3 = 0 \\
 a_1 &= -c_1 = -\frac{1}{2}(d+2) \\
 b_1 &= -d_1 = \frac{1}{2}d
 \end{aligned} \quad (C.13)$$

This completes the proof of Eq. (3.37) for $\omega_\alpha(\Phi)$, and demonstrates its uniqueness.

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