Ref.TH.3499-CERN

FREE GRADED DIFFERENTIAL SUPERALGEBRAS *)

P. van Nieuwenhuizen +)
CERN -- Geneva

ABSTRACT

Two theorems of D. Sullivan on the structure of differential algebras are extended to the algebras mentioned in the title and a few applications of non-trivial cohomology classes to the gauging of extended groups are given. The applications are due to R. D'Auria, L. Castellani, P. Fré, F. Giani, K. Pilch and the author, and are discussed in more detail in the author's talk at the 1982 Chicago Meeting of the American Mathematical Society on group-theoretical methods in physics.

*) Invited talk at the August 1982 Istanbul meeting on group-theoretical methods in physics.

+) On leave from the Institute for Theoretical Physics, State University of New York at Stony Brook, N.Y.

Ref.TH.3499-CERN
20 December 1982

1. INTRODUCTION

Differential algebras have become an important tool in the construction of local Lagrangian field theories. They define the group or supergroup or even generalized supergroup ("groups" with arbitrary antisymmetric tensor gauge fields, see below), from which one hopes to obtain a local Lagrangian field theory. The task of obtaining a local Lagrangian field theory from a given (extended super) group is usually called "the gauging of groups". The first results were obtained by MacDowell and Mansouri [1] and by Chamseddine and West [2] for simple supergravity, by Townsend and the author for N = 2 extended supergravity [3] and by Kaku, Townsend and the author for simple conformal supergravity [4]. In all these cases the base manifold was ordinary Minkovski spacetime and one had to impose certain constraints on the curvatures by hand (just like in superspace supergravity), something which really should come out of the method by itself.

The meaning of these constraints became clear [4,5] and in particular in extended conformal supergravities new constraints were deduced in a systematic fashion [6].

There exists another method, the so-called "group manifold approach", according to which one takes an extended supergroup and following general rules, one turns the crank, without imposing constraints. Then either the program stops, or out comes *the action* of a local Lagrangian field theory. This program was originally proposed by Ne'eman and Regge [7], while it has been developed in particular by D'Auria and Fré [8]. At this point the program is completely geometrical as far as the construction of the action is concerned, but the construction of the transformation rules under which the action is invariant is incomplete in the following sense. When no auxiliary fields are known, the transformation rules follow directly only when all fields are on-shell (= satisfy their equations of motion) but to find their off-shell form one must add to them arbitrary terms proportional to equations of motion and fix those by requiring that the action be invariant. (This is not exactly the same as one does in the so-called component approach because here the action is already known.) When one knows the auxiliary fields, on

the other hand, the derivation of the transformation rules is again completely geometrical (it is then equivalent to the independence of the action from the particular hypersurface M chosen in the group manifold on which the action is defined [9]).

Below we will discuss the general form of the differential algebras used; in particular, we shall extend two theorems due to D. Sullivan [10] on ordinary differential algebras to differential superalgebras. Then we shall show examples of differential algebras which are used in applications. This discussion is based on work done with D'Auria, Fré, Castellani, Giani and Pilch [11]. For a review of how to obtain actions from a differential algebra we refer to two sets of lectures [9], while the issue of the transformation laws in the group manifold approach will be discussed in a forthcoming article by D'Auria, Fré, Townsend and the author [12]. What follows is self-contained, but it is only an element in the larger program of the group manifold approach.

2. DEFINITION OF FREE GRADED DIFFERENTIAL SUPERALGEBRAS

The differential algebras we consider are graded superalgebras. This means that they contain forms which have a grade k ($1 \leq k < \infty$) and are bosonic or fermionic (sometimes one also uses the words even and odd). Thus we have really a $Z_\infty \otimes Z_2$ grading. The algebras are generated by a finite number of generators. In an algebra one can add and multiply. We will need to add bosonic p-forms only to bosonic p-forms (idem for fermionic p-forms), but we will multiply any form with any other form. Multiplication is denoted, as usual, by the wedge symbol, and the product of a p-form with a q-form is a (p + q) form, which is bosonic when the p-form and q-form are both bosonic or fermionic, and which is fermionic if the p-form is bosonic (fermionic) while the q-form is fermionic (bosonic). The field over which the algebra is defined is the real number system. Obviously, multiplication of a given p-form by a real number does not change its grade nor its bose or fermi property.

The (anti)commutation relations of these forms are as for ordinary forms, except that one gets an extra minus sign when two fermionic forms are interchanged. Denoting a bosonic p-form by b_p and a fermionic p-form by f_p, we thus have

$$b_p \wedge b_q = (-)^{pq} b_q \wedge b_p \,, \; b_p \wedge f_q = (-)^{pq} f_q \wedge b_p \,, \; f_p \wedge f_q = (-)^{pq+1} f_q \wedge f_p \tag{1}$$

Except for these (anti)commutation properties, there are no other relations between the forms, so that we are dealing with free differential algebras. (In general relativity one uses these days also non-free differential algebras, see for example Harrison's talk at the Marcel Grossman meeting in Shanghai, 1982.)

The differential operator d satisfies Leibniz' rule $dd = 0$, and maps a bosonic p-form into a bosonic p + 1 form, and a fermionic p-form into a fermionic p + 1 form. When in Leibniz' rule d passes a form it acts as if it were a bosonic 1-form; for example

$$d\left(x_p \wedge x_q\right) = \left(dx_p\right) \wedge x_q + (-)^p x_p \wedge dx_q \tag{2}$$

independently of whether x_p is bosonic or fermionic.

Let us stress that we do not consider the fermionic forms as forms with a negative grade: all forms have positive grade p with $p \geq 1$. This will be useful when we prove certain theorems by induction. For algebras with 0-forms see [15].

3. AN EXAMPLE OF A DIFFERENTIAL ALGEBRA

Consider the following differential algebra

$$\begin{aligned}
d\omega^m{}_n &= -\omega^m{}_k \wedge \omega^k{}_n && (m,k = 0,3)\\
dV^m &= -\omega^m{}_n \wedge V^n + \bar{\psi}\gamma^m \wedge \psi && (\omega^{mn} = -\omega^{nm})\\
d\psi^a &= -\tfrac{1}{4}\,\omega^{mn}(\gamma_{mn})^a{}_b \wedge \psi^b && (a = 1,4)\\
dA &= \bar{\psi}\gamma_{mn} \wedge \psi \wedge V^m \wedge V^n &&
\end{aligned} \tag{3}$$

The $\omega^m{}_n$ and V^m are bosonic 1-forms, the ψ^a are fermionic 1-forms and A is a bosonic 3-form. The γ^m are Dirac matrices satisfying

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \;, \quad \eta^{mn} = (-,+,+,+) \tag{4}$$

and $\bar{\psi} = \psi^T C$ where $C\gamma^m C^{-1} = -(\gamma^m)^T$ and $C^T = -C$. (For a detailed discussion of the charge conjugation matrix C in arbitrary dimensions, Majorana spinors, etc, see Ref. [13]).

These forms can be interpreted as

$$\omega^{mn} = dx^\Lambda \omega_\Lambda{}^{mn} \;, V^m = dx^\Lambda V_\Lambda{}^m \;, \psi^a = dx^\Lambda \psi_\Lambda{}^a \tag{5}$$

where x^Λ are either Minkovski coordinates or group-manifold coordinates or superspace coordinates. The x^Λ and $\omega_\Lambda{}^{mn}$ are bosonic or fermionic such that ω^{mn} is always bosonic. We shall never need to use x^Λ and dx^Λ but only work with forms; thus our results apply equally well to the group manifold, to superspace or to Minkovski space.

The differential algebra in (3) should be *consistent*. By this we mean that since dd = 0, (if $d = dx^\Lambda \partial_\Lambda$, dd = 0 follows) also d acting on the right-hand side in (3) must vanish. For ordinary Lie algebras in terms of forms

$$db_1^a = f^a{}_{bc}\, b_1^b \wedge b_1^c \tag{6}$$

this is equivalent to the Jacobi identities for $f^a{}_{bc}$, but for extended Lie algebras (with p-forms where $p \geq 2$) this is a generalization of the Jacobi identities to the case of "extended groups".

The consistency of (3) can be shown as follows. For $d\omega^m{}_n$ one finds consistency at once

$$-d\omega^m{}_k \wedge \omega^k{}_n + \omega^m{}_k \wedge d\omega^k{}_n =$$

$$\omega^m{}_l \wedge \omega^l{}_k \wedge \omega^k{}_n - \omega^m{}_k \wedge \omega^k{}_l \wedge \omega^l{}_n = 0 \tag{7}$$

Also for $d\psi^a$ there is no complication. In matrix notation

$$-\frac{1}{4}\, d\omega^{mn}\gamma_{mn}\wedge\psi + \frac{1}{4}\omega^{mn}\gamma_{mn}\, d\psi = \tag{8}$$
$$\frac{1}{4}\omega^m{}_k\wedge\omega^{kn}\gamma_{mn}\wedge\psi - \frac{1}{16}\omega^{mn}\wedge\omega^{kl}\gamma_{mn}\gamma_{kl}\wedge\psi$$

Now $\omega^{mn}{}_\wedge\omega^{k\ell} = -\omega^{k\ell}{}_\wedge\omega^{mn}$ and

$$\frac{1}{2}\left[\gamma_{mn},\gamma_{k\ell}\right] = \eta_{nk}\gamma_{m\ell} + 3 \text{ terms due to } m\leftrightarrow n,\ k\leftrightarrow\ell \tag{9}$$

and again consistency follows. For dV^m we get

$$-d\omega^m{}_n\wedge V^n + \omega^m{}_n\wedge dV^n - 2\bar\psi\gamma^m\wedge d\psi =$$
$$= \omega^m{}_k\wedge\omega^k{}_n\wedge V^n + \omega^m{}_n\wedge\left(-\omega^n{}_\ell\wedge V^\ell + \bar\psi\gamma^n\wedge\psi\right) + \frac{1}{2}\bar\psi\wedge\gamma^m\gamma_{k\ell}\omega^{k\ell}\wedge\psi \tag{10}$$
$$= \omega^{k\ell}\wedge\left(\delta^m_k\,\bar\psi\gamma_\ell\wedge\psi - \frac{1}{2}\bar\psi\gamma^m\gamma_{k\ell}\wedge\psi\right)$$

Now $\bar\psi\gamma^m\gamma_{k\ell}\wedge\psi = \delta^m_k\bar\psi\gamma_\ell\wedge\psi - \delta^m_\ell\bar\psi\gamma_k\wedge\psi$ [13], so that also the consistency of $d\psi$ is shown.

The most interesting case is the consistency of dA. In

$$-2\bar\psi\gamma_{mn}\wedge d\psi\wedge V^m\wedge V^n + 2\,\bar\psi\gamma_{mn}\wedge\psi\wedge dV^m\wedge V^n \tag{11}$$

we can replace d by $\mathcal{D}$ where $\mathcal{D}$ is the SO(3,1) Lorentz covariant derivative because the extra terms cancel. From (3)

$$\mathcal{D}\psi = 0 \quad , \qquad \mathcal{D}V^m = \bar\psi\gamma^m\wedge\psi \tag{12}$$

Hence consistency requires that

$$\bar\psi\gamma_{mn}\wedge\psi\wedge\bar\psi\gamma^m\wedge\psi \tag{13}$$

This identity indeed holds; it is equivalent to the identity for gravitino fields (as opposed to forms) which supergravity practitioners know very well [13]

$$\bar\psi_\mu\gamma_{mn}\psi_\nu\bar\psi_\sigma\gamma^m\psi_\rho\,\epsilon^{\mu\nu\rho\sigma} = 0 \tag{14}$$

It is usually proved by laborious Fierz rearrangements, but a more group-theoretical technique exists which is simpler in the important applications of 10 or 11 dimensional models [14].

It is clear from this example why differential algebras form a starting point for the gauging of "extended groups". In d = 11 (d = dimension) supergravity a 3-index antisymmetric tensor appears, in addition to an elevenbein e_μ^m and a gravitino field ψ_μ^a ($a = 1, 32$, and $m,\mu = 0, 10$ in $d = 11$). Ordinary Lie algebras could not accommodate $A_{\mu\nu\rho}$; one would begin with, say $A_\mu{}^{\nu\rho}$ (or the 1-form $A^{\nu\rho}$), and a posteriori $A_\mu{}^{\nu\rho}$ should become totally antisymmetric by some mechanism. Forms allow one to start straight away with a totally antisymmetric $A_{\mu\nu\rho}$ (or the 3-form A).

4. BIANCHI IDENTITIES AND COVARIANT DERIVATIVES

We can define curvatures by bringing the right-hand sides of (3) to the left. For example

$$R(\omega)^{mn} \equiv d\omega^{mn} + \omega^m{}_k \wedge \omega^k{}_n \, , \; R(A) \equiv dA - \bar{\psi}\gamma_{mn} \wedge \psi \wedge V^m \wedge V^n \tag{15}$$

The generalized left-invariant forms are those forms for which all curvatures vanish (the classical vacuum). Physical fields are those forms for which the curvatures do not vanish.

Consider a differential algebra with a finite number of generators, and let the subset of p-form generators be labelled by an index a. Thus the generators consist of $(x_1^a, x_2^a, \ldots x_N^a)$. Let the curvatures be given by

$$R(x_k^a) \equiv dx_k^a + \sum \frac{1}{p} C\left(\begin{smallmatrix} a \\ k \end{smallmatrix} \middle| \begin{smallmatrix} \ell_1 \ldots \ell_p \\ a_1 \ldots a_p \end{smallmatrix}\right) x_{\ell_1}^{a_1} \wedge \ldots \wedge x_{\ell_p}^{a_p} \tag{16}$$

where the sum runs over all possible terms and where we assume that the generalized structure constants C have the same symmetry as induced by permuting the various x's in the wedge product. Of course $\ell_1 + \ldots \ell_p = k + 1$ but in principle all possible partitions can occur. This system is assumed to be consistent: if all $R(x_k^a) = 0$ then also $dR(x_k^a) = 0$.

If the $R(x^k_a)$ are non-vanishing, they still satisfy Bianchi identities. Consistency leads to the simple result

$$dR(x^a_k) = \sum C\left({}^a_k \middle|{}^{\ell_1 \dots \ell_p}_{a_1 \dots a_p}\right) R(x^{a_1}_{\ell_1}) \wedge x^{a_2}_{\ell_2} \wedge \dots x^{a_p}_{\ell_p} \tag{17}$$

This defines the covariant derivative ∇ in the adjoint representation of the extended group, and in particular

$$\nabla R(x^a_k) \equiv 0 \quad (\text{Bianchi identity}) \tag{18}$$

Notice that in $\nabla R(x^a_k)$ one finds in principle all $R(x^b_\ell)$ with $\ell < k$. To define the covariant derivative in the coadjoint representation we consider a set of d-k forms V^{d-k}_a (one V^{d-k}_a per generator x^a_k) and define for arbitrary d

$$d\left(T^a_k \wedge V^{d-k}_a\right) = \left(\nabla T^a_k\right) \wedge V^{d-k}_a + (-)^k T^a_k \wedge \nabla V^{d-k}_a \tag{19}$$

where ∇T^a_k is the covariant derivative in the adjoint representation. Later we shall show that ∇ contains a piece D which is the covariant derivative w.r.t. the algebra generated by the 1-form generators alone.

5. DIFFERENTIAL ALGEBRAS SPLIT

Any free graded differential superalgebra A can be decomposed into a contractible algebra C and a minimal algebra M according to D. Sullivan. The contractible algebra consists of *pairs* of k and k + 1 forms (both bosonic or both fermionic) satisfying

$$dx^a_k = x^a_{k+1} \quad , \quad dx^a_{k+1} = 0 \tag{20}$$

In the minimal algebra dx^a_k is equal to a sum of *products* of forms (or equal to zero) but never equal to a single k + 1 form generator. Denoting the algebras *generated* by all p-forms with $p \leq k$ by C^k and M^k we thus have

$$dC^k \subset C^{k+1} \quad , \quad dM^k \subset M^k \wedge M^k \tag{21}$$

Before proving this theorem, let us give an example. Let

$$db_1^1 = 0 \, , \, df_1^1 = f_2 + b_1^1 \wedge f_1^2 \, , \, df_1^2 = f_2 + b_1^1 \wedge f_1^1 \, , \, df_2 = b_1^1 \wedge f_2 \tag{22}$$

Clearly this algebra is consistent.

Redefining the generators

$$\hat{f}_2 = f_2 + b_1^1 \wedge f_1^2 \, , \, \hat{f}_1 = f_1^1 \, , \, \tilde{f}_1 = f_1^2 - f_1^1 \, , \, \tilde{b}_1 = b_1^1 \tag{23}$$

we find that the algebra splits

$$\begin{aligned} d\hat{f}_1 &= \hat{f}_2 \, , \quad d\hat{f}_2 = 0 \\ d\tilde{f}_1 &= \tilde{b}_1 \wedge (f_1^1 - f_1^2) = -\tilde{b}_1 \wedge \tilde{f}_1 \, , \quad d\tilde{b}_1 = 0 \end{aligned} \tag{24}$$

We will now prove the theorem straightforwardly at the level of 1, 2 and 3-forms. By then the general inductive proof will become clear, but we shall not present the latter here. Seeing what goes on at the 1, 2 and 3-form level will convince the reader that the theorem holds and is much more understandable.

<u>Proof for 1-forms</u>

The most general expression for dx_1^a where a runs over all independent generators of grade 1 (bosonic or fermionic) contains terms $M^a{}_b x_2^b$. Redefining a maximal set of independent $M^a{}_b x_2^b$ as x_2^a, and taking linear combinations of the x_1^a

$$dx_1^a = x_2^a + f^a{}_{bc} \, x_1^b \wedge x_1^c \tag{25}$$

By redefining $\hat{x}_2^a = x_2^a + f^a{}_{bc} x_1^b x_1^c$ we find

$$dx_1^a = \hat{x}_2^a \qquad \text{or} \qquad dx_1^a = f^a{}_{bc} \, x_1^b \wedge x_1^c \tag{26}$$

By taking linear combinations of the x_1^a we can achieve that the correspondence $dx_1^a = \hat{x}_2^a$ is 1-1. Let us denote the x_1^a, which satisfy $dx_1^a = \hat{x}_2^a$ by $\hat{x}_1^a$ and the rest by $\tilde{x}_1^a$. Thus

$$x_1^a = (\hat{x}_1^a, \tilde{x}_1^a) \, , \, d\hat{x}_1^a = \hat{x}_2^a \, , \, d\tilde{x}_1^a = f^a{}_{bc} \, x_1^b \wedge x_1^c \tag{27}$$

To show that at the level of 1-forms the algebra splits into a contractible part ($d\hat{x}_1^a = \hat{x}_2^a$) and a minimal part, we must show that the x_1^b and x_1^c in the expressions for $d\tilde{x}_1^a$ are of the kind $\tilde{x}_1$ and not $\hat{x}_1$. The proof is easy. Since $dd\tilde{x}_1^a$ should yield zero, due to dd = 0, we have

$$dd\tilde{x}_1^a = 0 = f^a{}_{bc}\left(dx_1^b \wedge x_1^c - x_1^b \wedge dx_1^c\right) \tag{28}$$

Noting that

$$d(\hat{x}_1^b \wedge \hat{x}_1^c) = \hat{x}_2^b \wedge \hat{x}_1^c - \hat{x}_1^b \wedge \hat{x}_2^c \tag{29}$$

$$d(\hat{x}_1^b \wedge \tilde{x}_1^c) = \hat{x}_2^b \wedge \tilde{x}_1^c - \hat{x}_1^b \wedge (f^c{}_{pq}\, x_1^p \wedge x_1^q) \tag{30}$$

$$d(\tilde{x}_1^b \wedge \tilde{x}_1^c) = (d\tilde{x}_1^b) \wedge \tilde{x}_1^c - \tilde{x}_1^b \wedge d\tilde{x}_1^c = \text{terms with three } x_1\text{'s} \tag{31}$$

it is clear that the three kinds of terms cannot help each other in cancelling. In fact,

$$f^a{}_{bc}\left(\hat{x}_2^b \hat{x}_1^c - \hat{x}_1^b \wedge \hat{x}_2^c\right) \tag{32}$$

never vanishes, because if $\hat{x}_2$ and $\hat{x}_1$ are both fermionic then $f^a{}_{bc}$ is symmetric in (b,c), while if one or both of $\hat{x}_2$ and $\hat{x}_1$ are bosonic then $\hat{x}_2$ and $\hat{x}_1$ commute and in that case $f^a{}_{bc}$ is antisymmetric in (bc). Hence, there cannot be terms with $\hat{x}_1 \wedge \hat{x}_1$ in $d\tilde{x}_1^a$.

The terms with $\hat{x}_2\tilde{x}_1$ in $d\tilde{x}_1^a$ are non-vanishing and must cancel by themselves. Since they only come from $d(\hat{x}_1\tilde{x}_1)$, also the $\hat{x}_1\tilde{x}_1$ terms in $d\tilde{x}_1$ must be absent. Hence, at the level of 1-forms, the algebra indeed splits: $\vec{x}_1 = (\vec{\tilde{x}},\vec{\hat{x}})$, $d\tilde{x}_1^a = f^a{}_{bc}\tilde{x}_1^b \wedge \tilde{x}_1^c$, $d\hat{x}_1^a = \hat{x}_2^a$.

Proof for 2-forms

We first look for 2-form generators (even or odd) for which dx_2^a contains a 3-form generator x_3 on the right-hand side. Thus $dx_2^a = x_3^a$ + more. By redefining x_3^a into $\hat{x}_3^a$, we obtain that $dx_2^a = \hat{x}_3^a$ and thus $d\hat{x}_3^a = 0$; moreover, we can make the

connection $(x_2^a, \hat{x}_3^a)$ again 1-1 by taking linear combinations of the x_2^a. Let us denote these pairs by $(\hat{\hat{x}}_2, \hat{x}_3)$ since the notation $\hat{x}_2$ was already reserved for $d\hat{x}_1 = \hat{x}_2$. The rest of the x_2 we denote by $\tilde{x}_2$. Thus

$$x_2^a = \left(\hat{x}_2^a, \hat{\hat{x}}_2^a, \tilde{x}_2^a\right), \; d\hat{x}_1^a = \hat{x}_2^a, \; d\hat{\hat{x}}_2^a = \hat{x}_3^a \tag{33}$$

We must again show that on the right-hand side of $d\tilde{x}_2^a$ one only finds $\tilde{x}_1 \wedge \tilde{x}_1 \wedge \tilde{x}_1$ or $\tilde{x}_1 \wedge \tilde{x}_2$ but never terms with $\hat{x}_1$ or with $\hat{x}_2$ or $\hat{\hat{x}}_2$. If we can show this then we have accomplished the decomposition into a contractible and minimal part also at the level of 2-forms.

We have in general

$$d\tilde{x}_2^a = f^a_{(2)bc}\, x_1^b \wedge x_2^c + x_1 \wedge x_1 \wedge x_1 \text{ terms} \tag{34}$$

Clearly x_2^c cannot be $\hat{\hat{x}}_2^c$ since otherwise the $\hat{x}_3^a$ terms in $dd\tilde{x}_2^a$ would not cancel.

Suppose the x_2^c were $\hat{x}_2^c$, and x_1^b were $\hat{x}_1^b$. In that case consistency would require that $f^a_{(2)bc}\, \hat{x}_2^b \hat{x}_2^c = 0$, hence $f^a_{(2)bc}$ is super-antisymmetric (antisymmetric except when both x_2^b and x_2^c are fermionic in which case $f^a_{(2)bc}$ is symmetric in bc). The super-antisymmetric part of $f^a_{(2)bc}\, \hat{x}_1^b \hat{x}_2^c$ can be written as a total derivative

$$f^a_{(2)bc}\, \hat{x}_1^b \wedge \hat{x}_2^c = -\frac{1}{2}\, d\left(f^a_{(2)bc}\, \hat{x}_1^b \wedge \hat{x}_1^c\right) \tag{35}$$

and by redefining $\tilde{x}_2^{a\prime} = \tilde{x}_2^a + \frac{1}{2}\, f^a_{(2)bc}\, \hat{x}_1^b \hat{x}_1^c$ we can eliminate all $\hat{x}_1 \hat{x}_2$ terms in $d\tilde{x}_2$. Thus, although the bosonic and fermionic forms behave exactly opposite ($x_1^a x_1^a$ is non-zero when x_1^a is fermionic but vanishes when x_1^a is bosonic), the notion of super-(anti)symmetry covers both cases.

Suppose next that there is a term $f^a_{(2)bc}\, \tilde{x}_1^b \hat{x}_2^c$ in $d\tilde{x}_2^a$. It can be written as

$$f^a_{(2)bc}\, \tilde{x}_1^b\, \hat{x}_2^c = -\, d\left(f^a_{(2)bc}\, \tilde{x}_1^b \wedge \hat{x}_1^c\right) + \tilde{x}_1 \tilde{x}_1 \tilde{x}_1 \text{ terms} \tag{36}$$

and by redefining $\tilde{x}_2^a$ once more, also such terms could be eliminated.

Hence at this point,

$$d\tilde{x}_2^a = f^a_{(2)bc}\, x_1^b \wedge \tilde{x}_2^c + g^a_{\ bcd}\, x_1^b \wedge x_1^c \wedge x_1^d \tag{37}$$

and we must show that all x_1 are $\tilde{x}_1$. This is easy; if there were one or more $\hat{x}_1$, the $\hat{x}_2 x_1 x_1$ or $\hat{x}_2\tilde{x}_2$ terms in $dd\tilde{x}_2^a$ would not cancel. Hence, also at the level of 2 forms the algebra splits into a contractible and minimal part.

$$\begin{gathered} x_2^a = \left(\hat{x}_2^a, \hat{\hat{x}}_2^a, \tilde{x}_2^a\right), \quad d\hat{x}_1^a = \hat{x}_2^a, \quad d\hat{\hat{x}}_2^a = \hat{x}_3^a \\ d\tilde{x}_2^a = f^a_{(2)bc}\, \tilde{x}_1^b \wedge \tilde{x}_2^c + g^a_{\ bcd}\, \tilde{x}_1^b \wedge \tilde{x}_1^c \wedge \tilde{x}_1^d \end{gathered} \tag{38}$$

<u>Proof for 3-forms</u>

As for x_2, we decompose the 3-forms x_3^a into three classes

$$\begin{gathered} x_3^a = \left(\hat{x}_3^a, \hat{\hat{x}}_3^a, \tilde{x}_3^a\right), \quad d\hat{\hat{x}}_2^a = \hat{x}_3^a, \quad d\hat{\hat{x}}_3^a = \hat{x}_4^a \\ d\tilde{x}_3^a = f^a_{(3)bc}\, x_1^b \wedge x_3^c + g^a_{\ bc}\, x_2^b \wedge x_2^c + h^a_{\ bcd}\, x_2^b \wedge x_1^c \wedge x_1^d + x_1x_1x_1x_1 \text{ terms} \end{gathered} \tag{39}$$

Consistency ($dd\tilde{x}_3^a = 0$) forbids $x_3 = \hat{\hat{x}}_3$. If $x_3 = \hat{x}_3$, then redefinition of $\tilde{x}_3^a$ in $d\tilde{x}_3^a$ can eliminate $\hat{x}_3 x_1$ terms. The x_1 in the $\tilde{x}_3 x_1$ term cannot be an $\hat{x}_1$, since $\tilde{x}_3 \wedge \hat{x}_2$ would not cancel ($d\hat{\hat{x}}_2 \wedge \hat{x}_2$ could only yield $\hat{x}_3 \wedge \hat{x}_2$, not $\tilde{x}_3 \wedge \hat{x}_2$). Hence the first term in $d\tilde{x}_3^a$ is a product of generators of the minimal part of the algebra.

Now the $x_2 \wedge x_2$ terms. None of these x_2 can be a $\hat{\hat{x}}_2$ since in $dd\tilde{x}_3^a = 0$ the $\hat{x}_3\hat{x}_2$ terms would not cancel. Again the notion of supersymmetry of $g^a_{\ bc}$ is helpful in treating all cases. Suppose both of these x_2 were an $\hat{x}_2$, then $\hat{x}_2 \wedge \hat{x}_2 = d\hat{x}_1 \wedge \hat{x}_2 = d(\hat{x}_1 \wedge \hat{x}_2)$ could be eliminated by redefining $\tilde{x}_3^a$. Similarly one could eliminate $\hat{x}_2 \wedge \tilde{x}_2$ terms because although we get now an extra term, namely $-\hat{x}_1 \wedge d\tilde{x}_2$, this extra term lies in the h-sector or in the 4 x_1 sector. Going on to the $x_2 \wedge x_1 \wedge x_1$ terms, $\hat{\hat{x}}_2$ is ruled out by consistency, as are the combinations $\hat{x}_2\hat{x}_1\hat{x}_1$ and $\hat{x}_2\hat{x}_1\tilde{x}_1$. The $\hat{x}_2\tilde{x}_1\tilde{x}_1$ can be removed by redefinition. Again the x_1's in the term with h cannot be $\hat{x}_1$ since $\tilde{x}_2\hat{x}_2x_1$ would not cancel. As to the 4 x_1 terms, also these must 4 $\tilde{x}_1$ terms.

This concludes the first three steps in the iterative proof that the algebra decomposes into a contractible and minimal part. We covered both cases of bosonic and fermionic forms. There are, of course, important differences (for example, $x_1^a \wedge x_1^a$ is non-vanishing for odd forms) but to cover both, the notion of super-(anti)symmetry turned out to be useful.

6. COHOMOLOGY CLASSES IN MINIMAL ALGEBRAS

Let A now be a minimal algebra and denote all generators of grade p by x_p^a. The x_p^a can be bosonic or fermionic. Dropping tildas from now on, we have

$$dx_p^a = (M_p)^a{}_b \wedge x_p^b + a_p \tag{40}$$

where the matrices M_p are 1-forms and a_p is a p + 1 form which is generated by the generators of grade p - 1 and less

$$(M_p)^a{}_b = f^a_{(p)\,cb}\; x_1^c \tag{41}$$

Consistency implies that (in matrix notation)

$$(dM_p)\wedge x_p - M_p \wedge dx_p + da_p = (dM_p - M_p \wedge M_p)\wedge x_p + (d - M_p)\, a_p = 0 \tag{42}$$

The terms with generators of grade p must vanish separately (we have a free algebra) and hence

$$dM_p - M_p \wedge M_p = 0\,, \qquad (d - M_p)\, a_p = 0 \tag{43}$$

It follows that $D(M_p) \equiv d - M_p$ is nilpotent, just like d

$$D(M_p)\, D(M_p) = 0\,, \qquad D(M_p)\, x_p = a_p \tag{44}$$

This implies two things. The matrices $f_{(p)c}$ for given p form a representation of the Lie superalgebra defined by the 1-forms x_1^c

$$[\, f_{(p)a}\,, f_{(p)b}\,] = f_{(p)c}\, f^c{}_{ab}\ ; \ dx_1^c = f^c{}_{ab}\, x_1^a \wedge x_1^b \tag{45}$$

Furthermore, a_p is closed under $D(M_p)$ but not necessarily exact in A^{p-1} [it is, of course, exact in A^p by definition: $a_p = D(M_p)x_p$]. Thus the a_p are elements of a cohomology class

$$a_p \in H^{p+1}(A^{p-1}, M_p) \tag{46}$$

In words: the a_p are p + 1 forms (bosonic or fermionic) constructed from the generators with grade p - 1, which are closed with relation to the derivative $d - M_p$ (where the M_p form a representation of A^1).

The physical relevance of non-trivial cohomology classes (forms which are closed but not exact) is that one can introduce new higher-order forms into the differential algebra and still maintain consistency. Indeed, let

$$a_p^a = \sum C\left(\begin{smallmatrix} a \\ p \end{smallmatrix}\middle|\begin{smallmatrix} p_1 \cdots p_\ell \\ a_1 \cdots a_\ell \end{smallmatrix}\right) x_{p_1}^{a_1} \wedge \ldots \wedge x_{p_\ell}^{a_\ell} \tag{47}$$

where $p_1 + \ldots + p_\ell = p + 1$ and all p_i have $p_i \leq p - 1$. If $D(M_p)a_p = 0$ we can add to the algebra a new p form y_p^a as follows

$$D(M_p)\, y_p^a = a_p^a \tag{48}$$

and consistency would hold due to $D(M_p)a_p = 0$.

In practice one begins with an ordinary Lie superalgebra (usually the super-Poincaré or super de Sitter algebra), finds non-trivial cohomology classes, and then one adds new higher-grade forms to the system. One example was given in (3): the 4-form [11]

$$a_3 = \bar{\psi}\gamma_{mn} \wedge \psi \wedge V^m \wedge V^n \tag{49}$$

is an element of $H^4(A^1, I)$. Numerous other examples can be found in Refs. [11,14]; a discussion is given by the author in the proceedings of the Chicago conference 1982.

7. TRIVIALIZING COHOMOLOGY CLASSES

An interesting development in field theory is the following possibility [14]: given a p-form generator y_p with p > 1 in a differential algebra, can one *add* new k-forms with k < p to the algebra such that if y_p is represented by a product of forms with grade < p, then dy_p is consistent identically. Consider again (3), and write [11]:

$$A = \alpha B^{mn} \wedge V_m \wedge V_n + \beta \bar{\Psi} \gamma^m \wedge \eta \wedge V_m + \gamma \bar{\Psi} \gamma_{mn} \wedge \psi \wedge B^{mn} + \delta B^{mk} \wedge B_{k\ell} \wedge B^{\ell}{}_m \tag{50}$$

The new forms are here a bosonic 1-form $B^{mn} = -B^{nm}$ and a fermionic 1-form η. Their differential relations (= Cartan-Maurer equations, since they are 1-forms) we take as

$$\mathcal{D} B^{mn} = \bar{\Psi} \gamma^{mn} \wedge \psi \tag{51}$$

$$\mathcal{D} \eta = \gamma^m \psi \wedge V_m + \epsilon \gamma^{mn} \psi \wedge B_{mn} \tag{52}$$

where $\mathcal{D}$ is the Lorentz covariant derivative. The consistency of (51, 52) follows from $\mathcal{D}\psi = 0$ and the identities in d = 4

$$\bar{\Psi} \gamma^m \wedge \psi \wedge \gamma_m \psi = 0 \quad , \quad \bar{\Psi} \gamma^{mn} \psi \wedge \gamma_{mn} \psi = 0 \tag{53}$$

Let us now turn to A in (50). We can compute dA in two ways: from (3) or by explicitly working out d on the right-hand side of (50), using (3), (51), and (52). We can arrange both results to be identical by fixing α, β, γ, δ, and ε appropriately. (There is actually a 2-parameter class of solutions.)

In d = 11 supergravity similar things happen. There one must introduce in addition to η and B^{ab} a five-index bosonic 1-form $B^{a_1 \ldots a_5}$, and one finds two discrete solutions. The 1-forms (the original ones plus the new ones) define a new superalgebra, which has now two "supersymmetry charges" Q^a and $\hat{Q}^a$, corresponding to ψ and η. The corresponding superalgebra has the following structure [14]

$$\{Q, Q\} \sim B^{\cdot\cdot} + B^{\cdot\cdot\cdot\cdot\cdot} + P \; ; \; \{Q, \hat{Q}\} = \{\hat{Q}, \hat{Q}\} = 0 \tag{54}$$

$$[Q, P \text{ or } B^{\cdot\cdot} \text{ or } B^{\cdot\cdot\cdot\cdot\cdot}] \sim \hat{Q} \; ; \; [\hat{Q}, P \text{ or } B^{\cdot\cdot} \text{ or } B^{\cdot\cdot\cdot\cdot\cdot}] = 0 \tag{55}$$

Thus $\hat{Q}^a$ is a kind of fermionic central charge: it commutes with as many generators as possible (being a spinor it does not commute with the Lorentz generators).

The idea to reformulate the theory with antisymmetric tensors into a theory with only ordinary gauge fields has not yet been worked out. In particular, whether both theories are equivalent is not known, but it would be interesting in either case.

It is not always possible to add new 1-forms such that a p-form can be represented by a product of 1-forms. We quote here a counter example given to us by D. Sullivan. The proof (if correct) is undoubtedly equal to his, but we were only able to construct the proof in the form below.

Theorem: in a simple ordinary Lie algebra there are non-trivial cohomology classes which remain non-trivial, no matter how many new 1-forms one adds to the differential algebra.

Proof: let the simple Lie algebra S have generators $x_1^1, \ldots x_1^n$ satisfying

$$dx_1^a = f^a{}_{bc}\, x_1^b \wedge x_1^c \tag{56}$$

Consistency implies that $f^a{}_{b[c} f^b{}_{k\ell]} = 0$, in other words, the Jacobi identities. The structure constants $f^a{}_{bc}$ are taken to be totally antisymmetric (S is simple).

The following form is closed

$$a_n = x_1^1 \wedge \ldots\ldots \wedge x_1^n \tag{57}$$

because a totally antisymmetric tensor in n dimensions with n + 1 indices vanishes. This form is not exact: if $a_n = db_{n-1}$ then

$$b_{n-1} = \sum \alpha_k\, x_1^1 \wedge \ldots \wedge x_1^{k-1} \wedge x_1^{k+1} \wedge \ldots \wedge x_1^n \tag{58}$$

However, db_{n-1} vanishes always because in dx_1^ℓ one never finds x_1^ℓ on the right-hand side (due to the antisymmetry of f_{abc}).

Let us now add an arbitrary number of new 1-forms λ^a, satisfying consistent Cartan-Maurer equations

$$d\lambda^a = g^a{}_{bc}\,\lambda^b \wedge \lambda^c + h^a{}_{bc}\,\lambda^b \wedge x^c + k^a{}_{bc}\,x^b \wedge x^c \tag{59}$$

We shall prove that one can never find a b_{n-1} constructed from the x^a and λ^a such that $db_{n-1} = a_n$.

We begin by noting that *any* ordinary Lie algebra (semisimple or not) can always be decomposed into a semidirect sum of a solvable part P and a semisimple part; the latter is, of course, the direct sum of simple parts

$$A = P \ni S_1 \oplus \ldots\ldots \oplus S_n \tag{60}$$

Let the generators of P be denoted by P_A and those of S_i by $S_{i,A}$. Then

$$[P, P] \sim P \;;\quad [P, S_i] \sim P \;;\quad [S_i, S_j] \sim \delta_{ij} S_j \tag{61}$$

The generators of the Lie algebra corresponding to (56) are linear combinations of the P_A's and $S_{i,A}$'s, and from (56) and (61) it follows that

$$\begin{aligned} X_a &= \hat{P}_a + \hat{S}_a \,, \quad [X_a, X_b] = X_c f^c{}_{ab} \\ [\hat{S}_a, \hat{S}_b] &= \hat{S}_c f^c{}_{ab} \\ [\hat{S}_a, \hat{P}_b] + [\hat{P}_a, \hat{S}_b] &+ [\hat{P}_a, \hat{P}_b] = \hat{P}_c f^c{}_{ab} \end{aligned} \tag{62}$$

Thus the $\hat{S}_a$ in X_a form a simple Lie algebra. Since a solvable Lie algebra has no simple subalgebra, the $\hat{S}_a$ can be identified with one of the factors S_i in (60).

Let us now go over to the dual Lie algebra (the algebra in terms of 1-forms). We have

$$x_1^a = \hat{p}_1^a + \hat{s}_1^a \,, \quad d\hat{s}_{1,i}^a \sim \hat{s}_{1,i} \wedge \hat{s}_{1,i} \,, \quad d\hat{p}_1 \sim \hat{p}_1 \wedge \hat{p}_1 + \hat{p}_1 \wedge \hat{s}_1 \tag{63}$$

Suppose we could find a b_{p-1} as a sum of products of p_1^a and $s_{1,i}^a$ 1-forms, such that $db_{n-1} = a_n$. In a_n one has a term $\hat{s}_1 \wedge \ldots \hat{s}_n$. In b_{n-1} one would have terms with and without p_1^a 1-forms, but dp_1^a produces at least one p_1^a, so the terms with p_1^a cannot produce the term $\hat{s}_1 \wedge \ldots\ \hat{s}_n$. However, nor can the terms without p_1^a

forms, because of the same argument as given below (58). Hence, even in the larger algebra generated by x_1^a and λ_1^a, the closed form a_n remains non-exact.

Acknowledgements

The author is grateful to D. Sullivan for showing and explaining him the two theorems in the text, and to R. Stora and B. Julia for discussions.

REFERENCES

[1] S. MacDowell and F. Mansouri, Phys. Rev. Lett. 38, 739 (1977).

[2] A.H. Chamseddine and P.C. West, Nucl. Phys. B129, 39 (1977).

[3] P.K. Townsend and P. van Nieuwenhuizen, Phys. Lett. 67B, 439 (1977).

[4] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Lett. 69B, 304 (1977) and Phys. Rev. Lett. 39, 1109 (1977) and especially Phys. Rev. D 17, 3179 (1978).

[5] P. van Nieuwenhuizen in the Proceedings of the 1980 Cambridge Nuffield Workshop.

[6] B. de Wit, in "Supergravity 82", Cambridge Univ. Press, S. Ferrara and J.G. Taylor editors and references therein.

[7] Y. Ne'eman and T. Regge, Phys. Lett. 74B, 54 (1978) and Riv. del Nuovo Cimento 1, 1 (1978).

[8] R. D'Auria and P. Fré, Nucl. Phys. B173, 456 (1980);
P. Fré, Nucl. Phys. B186, 44 (1981);
R. D'Auria, P. Fré and A.J. Da Silva, Nucl. Phys. B196, 205 (1982).

[9] P. van Nieuwenhuizen in Proceedings 1982 Erice School on (Super)gravity in higher dimensions and exact solutions, World Publishing Cy, V. de Sabata editor.
R. D'Auria, P. Fré and T. Regge in "Supergravity 81", Cambridge Univ. Press S. Ferrara and J.G. Taylor editors.

[10] D. Sullivan, "Infinitesimal Computations in Topology", reprints to be obtained from IHES, Bures sur Yvette, France.

[11] L. Castellani, P. Fré, F. Giani, K. Pilch and P. van Nieuwenhuizen, Phys. Rev. D26, 1481 (1982) and Ann. of Phys. (to be published).

[12] R. D'Auria, P. Fré, P.K. Townsend and P. van Nieuwenhuizen, CERN preprint TH 3495.

[13] See, for example, P. van Nieuwenhuizen, Physics Report 68, 189 (1981), pages 361-365.

[14] R. D'Auria and P. Fré, Nucl. Phys. B201, 101 (1982).

[15] P.K. Townsend and P. van Nieuwenhuizen, Ecole Normale Supérieure, preprint LPTENS 82/15.