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LARGE N FIELD THEORIES, QUENCHING AND
THE MASTER FIELD

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A B S T R A C T

A non-perturbative proof of the quenching prescription for large N field theories is given, showing its connection with the existence of a master field configuration which dominates the functional integral.

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The observation by Eguchi and Kawai ¹⁾ that in the large N limit lattice QCD can be reduced to only one site, has renewed the interest in the $N \rightarrow \infty$ limit.

The original proposal by Eguchi and Kawai has been modified and a set of rules, known under the name of "quenched" has been found to reproduce the perturbation expansion of the large N theory ²⁾⁻⁶⁾.

In this paper we show that these rules are true beyond perturbation theory and we provide a connection of the quenching prescription with another peculiar feature of the large N limit : the master field ⁷⁾.

We will discuss in detail the case of a traceless Hermitean matrix field $\Phi_{ik}(x)$ in d dimensions. For reasons which will be clear later, we will think of each of the two indices i, k of $\Phi_{ik}(x)$ as a collection of d indices, $i \equiv i_1, \dots, i_d$, $k \equiv k_1, \dots, k_d$. This means that, if each of these indices varies from 1 to n , the field $\Phi_{ik}(x)$ will have $n^{2d}-1$ independent components. In the following we will indicate n^d by N .

The theory which we want to discuss is described by the Euclidean functional integral :

$$Z = \lim_{N \rightarrow \infty} \int \prod_{i,k,x} \delta \Phi_{ik}(x) e^S \quad (1.a)$$

$$S = - \text{Tr} \int d^d x \left[\partial_\mu \Phi(x) \partial_\mu \Phi(x) + m^2 \Phi^2(x) + \frac{g_1}{\sqrt{N}} \Phi^3(x) + \frac{g_2}{N} \Phi^4(x) \right] \quad (1.b)$$

where we have rescaled the coupling constants in order to have the correct $N \rightarrow \infty$ limit.

We suppose the system enclosed in a space time cubic box of volume $V \equiv L^d$ with periodic boundary conditions. In order not to have problems with ultra-violet divergences, we will suppose that Pauli-Villars regulators have been added to the action.

As $N \rightarrow \infty$ the planar diagrams of (1) dominate and the expectation values of the product of invariant operators factorize into the product of the expectation values. This fact suggests ⁷⁾ that, as $N \rightarrow \infty$, only one path $\Phi_M(x)$, together with its orbit under the group $SU(N)$, dominates the integral in (1). The master field $\Phi_M(x)$ is then the saddle point of (1) as N becomes large. Of course, due to the fact that the measure itself, $\prod_{i,k,x} \delta \Phi_{ik}(x)$, depends on N , the master field will not be a point of stationarity of the classical action, but rather will represent the point where

the growth of the phase space volume and the decrease of the exponential of the action exactly balance each other⁸⁾. This situation is strongly reminiscent of the saddle points which occur in statistical mechanics when the equivalence between the microcanonical and the canonical ensemble is established.

If the master field were known, it would be possible to evaluate any expectation value of invariant objects, by simply computing them on the master field itself. What we want to show is that quenching is a procedure to replace the original theory by a simpler one with the same master field. More precisely, we will show that quenching consists of cutting a fraction of the degrees of freedom of the fluctuations around the master field, while at the same time compensating for the loss of entropy caused by this operation.

We recall that, due to the translation invariance of the theory (1), space time translations of $\Phi_M(x)$ are implemented by elements of the internal symmetry group⁹⁾. More precisely, we must have :

$$\Phi_M(x) = e^{i P \cdot x} \Phi_M(0) e^{-i P \cdot x} \quad (2)$$

where the matrices P_μ can always be chosen diagonal.

Due to the periodic boundary conditions, the possible values of the momenta in the theory (1) will be discrete ($p_k = \frac{2\pi}{L}k$) and the matrices P_μ can be written as follows^{*}):

$$(P_\mu)_{i_1 \dots i_d, \kappa_1 \dots \kappa_d} = \delta_{i_1 \kappa_1} \dots P_{i_\mu} \delta_{i_\mu \kappa_\mu} \dots \delta_{i_d \kappa_d} \quad (3)$$

That this choice of P_μ is the good one, follows from the fact that $\Phi_M(x)$ cannot be constant in virtue of the cluster property^{**)}:

$$\lim_{|x-y| \rightarrow \infty} \langle T_2(\Phi(x) \Phi(y)) \rangle = \lim_{|x-y| \rightarrow \infty} T_2(\Phi_M(x) \Phi_M(y)) = 0 \quad (4)$$

By Lorentz invariance, then, if $\Phi_M(x)$ is not constant, it must contain all the momenta with uniform distribution^{***)}.

*) We note that the matrix (3) at finite n will contain components of momenta going from $-(2\pi/L)n/2$ to $(2\pi/L)n/2$ and will span all the possible momenta in the limit $N(=n^d) \rightarrow \infty$.

**) We will suppose that the $SU(N)$ symmetry is not spontaneously broken.

***) This follows immediately from the expression of Green's functions,

$$\text{e.g. : } \langle T_2(\Phi(x) \Phi(0)) \rangle = \sum_{i \kappa} \Phi_{M i \kappa}(x) \Phi_{M \kappa i}(0) = \sum_{i \kappa} e^{i(P_i - P_\kappa) \cdot x} |\Phi_{M i \kappa}(0)|^2$$

The next step is to reduce the global symmetry of (1) from $SU(N)$ to the group $U^N(1)$ of all the unitary transformations which commute with the P_μ 's of Eq. (3). This can be done by introducing, à la Faddeev-Popov, a constraint which is automatically satisfied by a $\Phi_M(x)$ of the form (2). This constraint is

$$\int d^d x \Phi_{i\kappa}(x) = 0 \quad i \neq \kappa \quad (5)$$

The condition (5) is obviously satisfied by $\Phi_M(x)$ in virtue of its peculiar x dependence.

We now define $\Omega(\Phi)$:

$$\Omega(\Phi) \cdot \int_{g \in SU(N)/U^N(1)} \mathcal{D}^\mu(g) \prod_{i \neq \kappa} \delta \left[\int d^d x \Phi_{i\kappa}^g(x) \right] = 1 \quad (6)$$

$$\Phi_{i\kappa}^g(x) \equiv (U_g^+)_{i\ell} \Phi_{\ell m}(x) (U_g)_{m\kappa}$$

$\Omega(\Phi)$ is easily evaluated for a field configuration Φ satisfying (5) :

$$\Omega(\Phi) = \prod_{i \neq \kappa} \left| \int (\Phi_{i\kappa}(x) - \Phi_{\kappa\kappa}(x)) d^d x \right| \quad (7)$$

We then insert (6) into (1) and extract the volume of the manifold $SU(N)/U^N(1)$, obtaining

$$Z' = \lim_{N \rightarrow \infty} \int \prod_{i, \kappa; x} \pi \delta \Phi_{i\kappa}(x) \prod_{i \neq \kappa} \delta \left[\int \Phi_{i\kappa}(x) d^d x \right] e^{S'(\Phi)} \quad (8.a)$$

$$S'(\Phi) = S(\Phi) + \log \Omega(\Phi) \quad (8.b)$$

It is clear that, as far as the computation of Green's functions of invariant operators is concerned, the theory described by Eqs. (8.a) and (8.b) will be equivalent to the theory based on (1). In particular, (8.a)-(8.b) will have, in the large N limit, the same master field $\Phi_M(x)$. The only difference with (1) is that now the orbit of $\Phi_M(x)$ is on the group $U^N(1)$ of transformations commuting with the P_μ 's.

This means that, as $N \rightarrow \infty$, the integral in (8) will be concentrated on the manifold :

$$\Phi_M^g(x) \equiv U_g^+ \Phi_M(x) U_g \quad (9)$$

where, from now on, we will denote by $\tilde{g}$ the elements of $U^N(1)$.

Since, as $N \rightarrow \infty$, the manifold (9) will dominate the theory, it is convenient to parametrize the generic field configuration in (8) as :

$$\Phi(x) = \Phi_n^{\tilde{g}}(x) + \xi(x) \quad (10)$$

with $\tilde{g}$ given, as an implicit function of $\Phi(x)$, by the condition :

$$T_2 \int d^4x \left[(\Phi(x) - \Phi_n^{\tilde{g}}(x)) \cdot \frac{\partial \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_a} \right] = 0 \quad (11)$$

($a = 1, \dots, N$)

It is convenient to think of $\xi_{ik}(x)$ as a function which only contains Fourier components ranging from $-(2\pi/L)n/2$ to $(2\pi/L)n/2$. In the limit $N \rightarrow \infty$ we obviously get back the integration over all periodic $\xi_{ik}(x)$. In this way we naturally let N act as an ultra-violet cut-off.

To perform the change of variables $\Phi \rightarrow \{\tilde{g}, \xi\}$ defined by Eqs. (10) and (11), we again use a Faddeev-Popov technique by defining :

$$\Delta(\Phi) \cdot \int_{\tilde{g} \in U^N(1)} \mathcal{D}\mu(\tilde{g}) \prod_a \delta \left[T_2 \int d^4x \left[(\Phi(x) - \Phi_n^{\tilde{g}}(x)) \cdot \frac{\partial \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_a} \right] \right] = 1 \quad (12)$$

We would like to remark that in $\tilde{g}$ are also included the translations generated by P_μ 's. In fact, since the master field is not constant, we must also introduce space-time translation collective co-ordinates x_0 by $\Phi_M(x-x_0)$. It is easy to convince oneself that this amounts to writing down Eq. (12) with the product over the index a running over all $U^N(1)$.

The quantity $\Delta(\Phi)$ is given by :

$$\Delta(\Phi) = \left| \det_{a,b} \left(T_2 \int d^4x \left[\frac{\partial \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_a} \frac{\partial \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_b} - \xi(x) \frac{\partial^2 \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_a \partial \tilde{g}_b} \right] \right) \right| \quad (13)$$

Inserting (12) into (8.a) and changing the order of integration, we obtain :

$$Z' = \lim_{N \rightarrow \infty} \int \mathcal{D}\mu(\tilde{g}) \int \prod_{i,k} \delta \xi_{ik}(x) \prod_{i \neq k} \delta \left[\int d^4x \xi_{ik}(x) \right] e^{S'(\Phi_n^{\tilde{g}} + \xi)} \cdot \Delta(\Phi_n^{\tilde{g}} + \xi) \prod_a \delta \left[T_2 \int d^4x \left(\xi(x) \cdot \frac{\partial \Phi_n^{\tilde{g}}(x)}{\partial \tilde{g}_a} \right) \right] \quad (14)$$

The crucial step now is to reduce the degrees of freedom in Eq. (14). Instead of integrating over all the possible [from $-(2\pi/L)n/2$ to $(2\pi/L)n/2$] Fourier components of $\xi_{ik}(x)$, we choose to integrate only over one Fourier component for each i, k ; that is, we integrate over fluctuations of the form :

$$\tilde{\xi}_{ik}(x) = e^{i(P_i - P_k) \cdot x} \eta_{ik} \quad (15)$$

with η_{ik} independent of x .

Of course, through this operation we have reduced by a factor N the degrees of freedom of the original theory and therefore we reduced by the same factor the entropy of the system. This means that, if we change back the integration variable to $\tilde{\Phi} = \tilde{\Phi}_M(0) + \tilde{\xi}$, the balance between the volume of the phase space and the exponential decrease of the action will take place for values of $\tilde{\Phi}$ much smaller than the original master field. In order for the reduced system to have the same saddle point as before, we must make the exponential factor in Eq. (14) less steep by a factor N . This implies that, as $N \rightarrow \infty$, the system described by (14) is equivalent to the reduced system :

$$\tilde{Z} = \lim_{N \rightarrow \infty} \int \prod_{ik} d\eta_{ik} e^{\tilde{S}'(\tilde{\Phi}_M(0) + \tilde{\xi})/N} \tilde{\Delta}(\tilde{\Phi}_M(0) + \tilde{\xi}) \prod_a \delta \left[T_2 \left(\eta \cdot \frac{\partial \tilde{\Phi}_M(0)}{\partial \tilde{g}_a} \right) \right] \quad (16)$$

where we have defined :

$$\tilde{S}'(\tilde{\Phi}) \equiv -V T_2 \left\{ -[P_\mu, \tilde{\Phi}]^2 + m^2 \tilde{\Phi}^2 + \frac{g_1}{N} \tilde{\Phi}^3 + \frac{g_2}{N} \tilde{\Phi}^4 \right\} + \log \prod_{i \neq k} |\tilde{\Phi}_{ii} - \tilde{\Phi}_{kk}| \quad (17.a)$$

$$\tilde{\Delta}(\tilde{\Phi}_M(0) + \tilde{\xi}) \equiv \frac{1(\tilde{\Phi}_M(0) + \tilde{\xi})}{V^N} = \left| \det \left(T_2 \left\{ \frac{\partial \tilde{\Phi}_M(0)}{\partial \tilde{g}_a} \cdot \frac{\partial \tilde{\Phi}_M(0)}{\partial \tilde{g}_b} - \eta \cdot \frac{\partial^2 \tilde{\Phi}_M(0)}{\partial \tilde{g}_a \partial \tilde{g}_b} \right\} \right) \right| \quad (17.b)$$

Due to the particular choice (15) of the fluctuations $\tilde{\xi}$, the factor $\prod_{i \neq k} \delta [\tilde{\xi}_{ik}(x) d^d x]$ in Eq. (14) is evaluated at zero argument of the δ functions and can be dropped *).

*) To be precise one should first regularize the δ functions, but this of course does not make any difference.

An important remark is that since the group $U^N(1)$ commutes with the translations, it has been possible to perform in a trivial way the space-time integrals inside the δ -functions and the Faddeev-Popov determinants in (14) replacing them by the integrands evaluated at the origin. Had we not fixed the global gauge in Eq. (8.a), the group integration in Eq. (14) would have been extended over all $SU(N)$ and Eq. (16) would not be valid.

Our last step is to observe that, since (for constant matrices $\tilde{\Phi}$) :

$$\tilde{\Delta}(\tilde{\Phi}) \cdot \int \mathcal{D}\mu(\tilde{g}) \prod_a \pi \delta \left\{ \text{Tr} \left[(\tilde{\Phi} - \Phi_{N(0)}^{\tilde{g}}) \cdot \frac{\partial \Phi_{N(0)}^{\tilde{g}}}{\partial \tilde{g}_a} \right] \right\} = 1 \quad (18)$$

we have :

$$\mathcal{D}\mu(\tilde{g}) \prod_{i,k} \pi \delta \eta_{ik} \tilde{\Delta}(\Phi_{N(0)}^{\tilde{g}} + \eta) \prod_a \pi \delta \left[\text{Tr} \left(\eta \cdot \frac{\partial \Phi_{N(0)}^{\tilde{g}}}{\partial \tilde{g}_a} \right) \right] = \prod_{i,k} \pi \delta \tilde{\Phi}_{ik} \quad (19)$$

where :

$$\tilde{\Phi} = \Phi_{N(0)}^{\tilde{g}} + \eta \quad (20)$$

This means that all the factors (apart, of course, from the exponential one) in Eq. (16) conspire to give exactly the (Cartesian) measure over the constant matrix $\tilde{\Phi}$. In conclusion, we have thus shown the equivalence (as $N \rightarrow \infty$) between (1) and the reduced system :

$$\tilde{Z} = \lim_{N \rightarrow \infty} \int \prod_{i,k} \pi \delta \tilde{\Phi}_{ik} e^{\frac{\tilde{S}'(\tilde{\Phi})}{N}} \quad (21)$$

This equivalence implies in particular that the invariant Green's functions of (1) can be computed in the large N limit as :

$$\begin{aligned} \langle \text{Tr}(\Phi(x_1) \dots \Phi(x_p)) \rangle &= \\ &= \frac{\lim_{N \rightarrow \infty} \int \prod_{i,k} \pi \delta \tilde{\Phi}_{ik} e^{\frac{\tilde{S}'(\tilde{\Phi})}{N}} \text{Tr}(e^{iP \cdot x_1} \tilde{\Phi} e^{-iP \cdot x_1} \dots e^{iP \cdot x_p} \tilde{\Phi} e^{-iP \cdot x_p})}{\tilde{Z}} \quad (22) \end{aligned}$$

The quenched model described by (21) and (22) is equivalent to the non-integrated (over P_μ) version of quenching of Refs. 3) and 5) after the identification of n/L with the momentum cut-off $\Lambda/2\pi$. In fact, the only

difference occurs in the measure $\square(\tilde{\Phi})$ which, however, only involves a modification of the integration over the diagonal components Φ_{ii} . It is easy to see that in the infinite volume limit its contribution can safely be neglected.

An interesting point to notice is that, after a rescaling of the matrix $\tilde{\Phi}$ by $\sqrt{N}$ in Eq. (21), all the explicit N dependence disappears from the action and it is possible to write down directly the theory at $N=\infty$, which was not possible, of course, in the original theory (1).

As a last remark, we want to point out that it is possible to perform other kinds of reduction of degrees of freedom. For example, we could perform a partial quenching in d_q dimensions with $d_q < d$. This would amount to integrating over fluctuations of the form :

$$\sum_{i_1, \dots, i_d, \kappa_1, \dots, \kappa_d} (x) = \prod_{\mu=1}^{d_q} e^{i(P_{\mu}^i - P_{\mu}^{\kappa}) \cdot x_{\mu}} \eta_{i_1, \dots, i_d, \kappa_1, \dots, \kappa_d}(\underline{x}) \quad (23)$$

$$\underline{x} \equiv (x_{d_q+1}, \dots, x_d)$$

where η is a periodic function of the variables collectively indicated by $\underline{x}$. We would reduce the degrees of freedom by a factor n^{d_q} and the reduced theory is easy to write down.

In conclusion, the argument shown in this paper sheds light on the nature of the quenching prescription but seems to forbid a systematic theory of $1/N$ corrections within the scheme of quenching. In fact, in order to compute $1/N$ corrections, we would have to take into account the complete fluctuations around the master field and not only those of the restricted form (15).

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