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DIMENSIONAL REDUCTION OF WEYL, MAJORANA
AND MAJORANA-WEYL SPINORS

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A B S T R A C T

We discuss the dimensional reduction for Weyl, Majorana, or Majorana-Weyl spinors coupled to pure d dimensional ($d > 4$) gravity. The only case where a realistic four-dimensional low energy spectrum for the fermions may be obtained, is for a Majorana-Weyl spinor in $d = 2 \bmod 8$ dimensions. Chiral massless fermions are not excluded in this case. The minimal number of dimensions for the construction of a realistic theory out of pure gravity is $d = 18$.

1. INTRODUCTION

The unification of four-dimensional gravity and gauge interactions within pure higher dimensional gravity¹⁾ seems to work well in the bosonic sector. Attractive unifying gauge groups for weak, electromagnetic and strong interactions can easily be obtained²⁾ by spontaneous compactification. There is enough freedom in the model to account for a variety of interesting ground states³⁾ and spontaneous symmetry breaking may be described in this context. For the fermionic sector, however, no satisfactory solution has been found so far. Coupling a spinor to higher dimensional gravity gives, in many cases, masses of the order of the Planck mass to all four-dimensional fermions. Nevertheless, some models admitting massless fermions have been discovered. The simplest case, five-dimensional gravity with one dimension forming a compact space S^1 with radius of the order of the Planck length, has a massless fermion which is a singlet with respect to the gauge group. More complicated compact spaces for the internal dimensions, with negative curvature (using conventions where the curvature of the hypersphere is positive) and admitting various numbers of massless fermions have been discussed by Hitchin⁴⁾. In all these cases the fermions are vector-like with respect to any gauge group which is a subgroup of higher dimensional coordinate transformations and Lorentz rotations. The only case with massless chiral fermions in the context of pure higher-dimensional gravity is more a technical example²⁾. The corresponding ground state is unsatisfactory since it has a boundary. Zero mass fermions can be obtained from dimensional reduction in the context of supersymmetry, where generalized spin $3/2$ fields and supplementary bosonic degrees of freedom have to be added to higher dimensional gravity. An interesting possibility⁵⁾ is $N = 8$ supersymmetry with gauged $SO(8)$ symmetry obtained from local $N = 1$ supergravity in eleven dimensions, where the internal dimensions are compactified to S^7 . The massless fermions transform vector-like with respect to $SO(8)$, but they are in chiral representations with respect to the hidden local $SU(8)$ symmetry of this theory. Unfortunately, the elementary massless fermions of this theory are in representations which do not allow the construction of a realistic low-energy theory. Another interesting approach⁶⁾ starts with higher dimensional gravity plus gauge interactions. Chiral four-dimensional fermions can be obtained for special cases. We feel, however, that the introduction of supplementary gauge fields in higher dimensions is not very satisfactory from the conceptional point of view. It destroys the main purpose for introducing supplementary dimensions, which is the description of four-dimensional gauge interactions out of pure higher dimensional gravity. In this paper we will concentrate on the question of whether chiral massless fermions can be obtained from pure higher-dimensional gravity coupled to a spinorial matter field.

What do we need to construct a realistic four-dimensional low energy theory? Quarks and charged leptons are in complex representations with respect to the unbroken gauge group $SU(3)_C \times U(1)_{em}$. They are massive Dirac spinors. The left-handed neutrino is in a real representation with respect to $SU(3)_C \times U(1)_{em}$. It is either a massless Weyl spinor or it may have a small Majorana mass and transform as a Majorana spinor with respect to the Lorentz group. In the limit of an unbroken $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group, all observed fermions belong to complex representations with respect to this group. They are Weyl spinors. Complex (with respect to the gauge group) Weyl spinors are called chiral fermions. The combination of gauge symmetry and Lorentz symmetry forbids any mass term for chiral fermions. Any attempt to construct a four-dimensional theory with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ out of higher dimensional gravity must therefore start with a space admitting chiral four-dimensional fermions after dimensional reduction.

It has been shown⁷⁾ that dimensional reduction of a d dimensional spinor cannot lead to chiral four-dimensional fermions if a d dimensional mass term is allowed by d dimensional general coordinate and Lorentz transformations. The only remaining cases are a Majorana spinor for $d = 9 \bmod 8$, a Weyl spinor for $d = 6, 8 \bmod 8$ and a Majorana-Weyl spinor for $d = 2 \bmod 8$. We are thus led to study the dimensional reduction of Majorana, Weyl and Majorana-Weyl spinors, and this is the purpose of this paper.

We do not hereby restrict our discussion to the dimensional reduction to four dimensions. We consider the more general case of pure d dimensional gravity coupled to a spinorial matter field where the ground state is a direct product of a compact D dimensional space admitting a symmetry group G and an arbitrary E dimensional space. Indeed, we have the interesting possibility that dimensional reduction can be done by steps: the first step leads to a theory with E dimensional ($E > 4$) gravity, gauge interactions and spinors. In a second step, one considers the dimensional reduction following from spontaneous compactification of $E-4$ dimensions. If the first step admits massless E dimensional fermions with certain transformation properties with respect to the gauge group G , chiral four-dimensional fermions may be obtained by applying the ideas of Ref. 6.

In Section 2, we discuss the dimensional reduction for Dirac spinors for arbitrary D and E . We find a degeneracy in the E dimensional mass spectrum which is unacceptable from the point of view of phenomenology, but necessary for a consistent treatment of Weyl and/or Majorana spinors. In Section 3 the dimensional reduction of Majorana, Weyl and Majorana-Weyl spinors is described. The only cases where four-dimensional chiral fermions could finally be obtained is for a Weyl spinor for $d = 6 \bmod 8$ or a Majorana-Weyl spinor for $d = 2 \bmod 8$. The

unwanted degeneracy of the mass spectrum disappears only for the second case. Any attempt to construct a realistic four-dimensional theory out of pure higher-dimensional gravity coupled to a spinorial matter field must start with a Majorana-Weyl spinor in $d = 2 \bmod 8$ dimensions! The minimal dimension for such a theory is $d = 18$. Throughout this paper we use the same conventions as in Ref. 7, where the reader is referred to for supplementary details on d dimensional Majorana/Weyl constraints.

2. DIMENSIONAL REDUCTION OF DIRAC SPINORS

Transformation properties

We start with the dimensional reduction of a Dirac spinor in d dimensions. One dimension is timelike and the others are spacelike. The ground state is assumed to be a direct product of a D dimensional compact space K^D with spacelike coordinates y^α and an E dimensional space with coordinates x^μ . The action is invariant with respect to the symmetry group of general coordinate transformations and Lorentz-transformations in d dimensions, gen_d . We further assume that K^D admits a continuous symmetry group G . This is the subgroup of gen_D transformations leaving the vielbein of the ground state invariant. The group G acts as a gauge symmetry in the reduced E dimensional theory. The vielbein of the D dimensional space, e_α^a , transforms under an infinitesimal transformation $y^\alpha \rightarrow y^\alpha + \xi^\alpha$

$$\delta_\xi e_\alpha^a = -\partial_\alpha \xi^\beta e_\beta^a - \xi^\beta \partial_\beta e_\alpha^a \quad (1)$$

and with respect to an infinitesimal rotation of the D dimensional Lorentz-frame

$$\delta_\epsilon e_\alpha^a = -e_\alpha^b \epsilon_b^a \quad (2)$$

Here ξ^α and $\epsilon_{ba} \equiv -\epsilon_{ab}$ are arbitrary functions of y and x and the Lorentz indices a, b are raised and lowered with $\eta_{ab} = \text{diag}(-1, -1, -1, \dots -1)$. The metric of K^D is left invariant by the transformations $\xi^\alpha = K_z^\alpha(y) \theta^z(x)$ where the K_z^α are the Killing vectors corresponding to the generators of the symmetry G . Then the vielbein of K^D , $\hat{e}_\alpha^a(y)$, is invariant under combined coordinate and Lorentz transformations provided*)

$$\mathcal{E}_{ab} = -\theta^z (\hat{e}_a^\alpha \hat{e}_{\beta b} \partial_\alpha K_z^\beta + \hat{e}_a^\alpha K_z^\beta \partial_\beta \hat{e}_{\alpha b}) \equiv \hat{\mathcal{E}}_{ab} \quad (3)$$

The d dimensional Dirac spinor has $2^{\lfloor d/2 \rfloor}$ components and transforms under G transformations according to

*) The antisymmetry $\hat{\mathcal{E}}_{ab} = -\hat{\mathcal{E}}_{ba}$ may be checked using the invariance of the ground state metric.

$$S_G \psi = -\theta^2 S_Z \psi \quad (4)$$

$$S_Z = K_Z^\alpha \partial_\alpha - \frac{1}{2} (\dot{e}_a^\alpha \dot{e}_{\beta b} \partial_\alpha K_Z^\beta + \dot{e}_a^\alpha K_Z^\beta \partial_\beta \dot{e}_{\alpha b}) \Sigma^{ab} \quad (5)$$

Here

$$\Sigma^{ab} = -\frac{1}{4} [\gamma^a, \gamma^b] \quad (6)$$

are the D dimensional Lorentz generators and γ^a the corresponding Dirac matrices. For simplicity, we neglect in the further discussion all bosonic excitations above the ground state except the graviton. This corresponds to an ansatz for the vielbein $e_\alpha^a = \dot{e}_\alpha^a(y)$, $e_\mu^m = e_\mu^m(x)$, 0 otherwise. The kinetic term for the Dirac spinor in the action is then given by

$$S_\psi = \int d^D y d^E x e_D e_E L_\psi \quad (7)$$

$$L_\psi = \frac{i}{2} \bar{\psi} (\gamma^\mu D_\mu + \gamma^\alpha D_\alpha) \psi + h.c. \quad (8)$$

Here e_E and e_D are the determinants of $\dot{e}_\alpha^a$ and e_μ^m , respectively, $\gamma^\alpha = \gamma^a \dot{e}_a^\alpha$ and D_α are the covariant derivatives

$$D_\alpha \psi = \left(\partial_\alpha - \frac{1}{2} \dot{\omega}_{\alpha ab} \Sigma^{ab} \right) \psi \quad (9)$$

with $\dot{\omega}_{\alpha ab}$ the spin connection constructed from $\dot{e}_\alpha^a$.

Dimensional reduction now consists in reducing the spinor ψ with respect to G representations and carrying out the y integration in the action in order to obtain an equivalent E dimensional action. An arbitrary spinor $\psi_{AM}(y, x)$ (we denote the $2^{[d/2]}$ spinor components by a double index $A = 1 \dots \bar{A}$, $M = 1 \dots \bar{M}$, $\bar{A}\bar{M} = 2^{[d/2]}$) can always be expanded

$$\psi_{AM}(y, x) = \psi_{iA}(y) \varphi_{iM}(x) \quad (10)$$

(A summation over doubled indices is always implied.) The $\psi_i(y)$ form a complete set of $\bar{A}$ component functions so that every $\bar{A}$ component function $\psi(y)$ can be written as $a_i \psi_i(y)$ with a_i complex and $\psi_i(y)$ linearly independent. The $\varphi_i(x)$ are arbitrary complex Grassmann variables. Of course, the summation involves infinitely many ψ_i . The Lorentz generators $\sum^{mn}$ relating to the E dimensional space commute

with the corresponding generators Σ^{ab} for the D dimensional space. One can always find a representation with

$$\begin{aligned}\Sigma^{mn} &= \left(-\frac{1}{4} [\Gamma^m, \Gamma^n] \otimes \mathbb{1}_{\bar{A}} \right) \\ \Sigma^{ab} &= \left(\mathbb{1}_{\bar{M}} \otimes -\frac{1}{4} [\Gamma^a, \Gamma^b] \right)\end{aligned}\quad (11)$$

where the $\Gamma^m(\Gamma^a)$ are Dirac matrices acting on $\bar{M}(\bar{A})$ component fields. An appropriate choice for the d dimensional Dirac matrices $\gamma^{\hat{m}} = (\gamma^m, \gamma^a)$ is given by

$$\begin{aligned}\gamma^m &= (\Gamma^m \otimes S_{\bar{A}}) \\ \gamma^a &= (S_{\bar{M}} \otimes \Gamma^a)\end{aligned}\quad (12)$$

with $S_{\bar{A}} S_{\bar{A}} = \mathbb{1}_{\bar{A}}$, $S_{\bar{M}} S_{\bar{M}} = \mathbb{1}_{\bar{M}}$, $S_{\bar{A}}^+ = S_{\bar{A}}$, $S_{\bar{M}}^+ = S_{\bar{M}}$, and $\{S_{\bar{A}}, \Gamma^a\} = 0$, $[S_{\bar{M}}, \Gamma^m] = 0$ or $[S_{\bar{A}}, \Gamma^a] = 0$, $\{S_{\bar{M}}, \Gamma^m\} = 0$. For E even, we choose $\bar{A} = 2^{E/2}$, $\bar{M} = 2^{[D/2]}$, $S_{\bar{A}} = \mathbb{1}_{\bar{A}}$, $S_{\bar{M}} = \bar{\Gamma}_E = \eta \Gamma^0 \Gamma^1 \dots \Gamma^{E-1}$. For E odd, no $2^{(E-1)/2}$ dimensional representation for the Γ^m exists admitting a matrix $S_{\bar{M}}$ anticommuting with all of them. If D is also odd, we can still have $S_{\bar{A}} = \mathbb{1}_{\bar{A}}$ with $\bar{A} = 2^{(D-1)/2}$. One has $\bar{M} = 2^{(E+1)/2}$, admitting a maximal set of E+2 anticommuting Γ matrices. We may choose the first E of these for the matrices Γ^m and the last for $S_{\bar{M}}$. For E odd and D even, a possible choice is given by $\bar{A} = 2^{(E-1)/2}$, $\bar{M} = 2^{D/2}$, $S_{\bar{M}} = \mathbb{1}_{\bar{M}}$, $S_{\bar{A}} = \bar{\Gamma}_D = \eta \Gamma^1 \dots \Gamma^D$.

Under gen_E transformations $\psi(y, x)$ transforms as

$$\delta_E \psi(y, x) = \psi_i(y) \left(-\xi^\gamma \partial_\gamma - \frac{1}{2} \epsilon_{mn} \Sigma^{mn} \right) \varphi_i(x) \quad (13)$$

For E or D even every field $\phi_i(x)$ transforms as an E dimensional Dirac spinor. For E even, it decomposes into two inequivalent Weyl spinors. For both E and D odd, $\phi_i(x)$ contains two independent E dimensional Dirac spinors since it has $2^{(E+1)/2}$ complex components. This is a consequence of the decomposition of d dimensional Dirac spinors with respect to the Lorentz-groups $\ell_E \times \ell_D$:

$$\begin{aligned}2^{[\frac{d}{2}]} &\rightarrow (2^{[\frac{E}{2}]}, 2^{[\frac{D}{2}]}) \quad \text{for E or D even} \\ 2^{\frac{d}{2}} &\rightarrow 2 \times (2^{\frac{E-1}{2}}, 2^{\frac{D-1}{2}}) \quad \text{for E and D odd}\end{aligned}\quad (14)$$

Next we consider the action of the gauge group G on the E dimensional spinors ϕ_i :

$$\begin{aligned}\delta_G \psi(y, x) &= \left(-\theta^2 K_2^a \partial_a - \frac{1}{2} \tilde{E}_{ab} \Sigma^{ab} \right) \psi_i(y) \phi_i(x) \\ &= -\theta^2 S_Z \psi_i(y) \phi_i(x) \\ &= \psi_j(y) \left(-\theta^2 (T_Z)_{ji} \phi_i(x) \right) \\ &= \psi_i(y) \delta_G \phi_i(x)\end{aligned}\tag{15}$$

Here we use the fact that applying the operator S_Z on $\psi_i(y)$ yields a new function $\psi'_i(y)$ which can be expanded in the old functions $\psi_j(y)$: $\psi'_i(y) = \psi_j(y) (T_Z)_{ji}$ with complex constant $(T_Z)_{ji}$. Thus G acts as a linear transformation on the ϕ_i :

$$\delta_G \phi_i(x) = -\theta^2 (T_Z)_{ij} \phi_j\tag{16}$$

The generators T_Z fulfil of course the commutation relations characterizing the group G .

So far we have not specified our choice of the $\psi_i(y)$. We expand in the non-linear representations of G contained in $\psi(y)^{*}$)

$$\psi_i(y) \equiv \psi_{nHk}(y)\tag{17}$$

with

$$S_Z \psi_{nHk}(y) = \psi_{n'Hk}(y) (T_Z)_{n'n}\tag{18}$$

$[(T_Z)_{ij}]$ is given by $\delta_{HH'} \delta_{kk'} (T_Z)_{nn'}$.] The sum over i involves a sum over all irreducible G representations H appearing in the expansion of $\psi(y)$, n denotes the n_H components of H and $k = 1 \dots k_H$ counts how many times H is contained in $\psi(y)$. The corresponding fields $\phi_{nHk}(x)$ are linear realizations of the representations H .

One can always normalize the $\psi_{nHk}(y)$ according to

$$\int d^D y e_D(y) \psi_{n'H'k'}(y)^+ \psi_{nHk}(y) = \delta_{n'n} \delta_{H'H} \delta_{k'k}\tag{19}$$

Indeed, the integral

$$\int d^D y e_D \psi^+ \psi = \int d^D y e_D \psi_{n'H'k'}^+ \psi_{nHk} a_{n'H'k'}^* a_{nHk}\tag{20}$$

*) For details of such an expansion on homogeneous spaces, see Ref. 8.

is invariant under gen_D transformations, and therefore also invariant with respect to G . Then Schur's Lemma implies

$$\int d^D y e_D \psi_{n'H'z'}^\dagger \psi_{nHz} = \delta_{n'n} \delta_{H'H} F_{z'z}^{(H)} \quad (21)$$

Furthermore, $F^{(H)}$ is hermitean

$$(F_{z'z}^{(H)})^* = F_{zz'}^{(H)} \quad (22)$$

and we may re-define the ψ_{nHk} by multiplication with a unitary $k_H \times k_H$ matrix, thus diagonalizing $F^{(H)}$:

$$F_{zz'}^{(H)} = f^{(H)} \delta_{zz'} \quad (23)$$

Since $f^{(H)}$ is positive, a rescaling of ψ_{nHk} finally gives Eq. (19). The normalization (19) implies that the generators $(T_z)_{nn'}$ are antihermitean as required for the infinitesimal generators of unitary transformations.

The sum of all representations ϕ_{Hk} forms a *self-conjugate* reducible representation of G . This follows from the existence of a D dimensional charge conjugation operator. If $\psi_{nHk}(y)$ is contained in the expansion of $\psi(y)$, its complex conjugate $\psi_{nHk}(y)^*$ is also contained in this expansion^{*)}. Defining

$$e_D \psi_{nHz} = B_D^{-1} \psi_{nHz}^* = \psi_{nHz}^c \quad (24)$$

with B_D fulfilling

$$(\Sigma^{ab})^* = B_D \Sigma^{ab} B_D^{-1} \quad (25)$$

one has

$$S_z \psi_{nHz}^c = \psi_{n'H'z}^c (T_z^*)_{n'n} \quad (26)$$

We distinguish complex, real and pseudoreal representations of G , which we denote by ψ_{nCk} , ψ_{nRk} and ψ_{nPk} , respectively. The corresponding E dimensional fields ϕ_{nHk} are often called complex, real or pseudoreal spinors.

We call ψ_{nCk} a complex representation if T_z and T_z^* belong to inequivalent representations. Then ψ_{nCk}^c is an independent representation in the expansion of $\psi(y)$ and we denote it by $\psi_{\bar{C}k}$. The field $\phi_{\bar{C}k}$ belongs to the complex conjugate representation of ϕ_{Ck} . It has the same transformation properties as ϕ_{Ck}^* , but it is an independent E dimensional Dirac spinor. For all complex representations C

) This holds for all unconstrained $\psi(y)$ and for all $\psi(y)$ obeying constraints invariant under complex conjugation. For example, if $\psi(y)$ obeys a constraint $\psi(y_0) = \psi(y_1)$, $\psi^(y)$ is also consistent with this constraint.

there is an equal number of complex conjugate representations $\bar{C}$ and the sum of both is a self-conjugate (reducible) representation.

If T_z and T_z^* belong to equivalent representations, we have

$$T_z^* = N T_z N^{-1}, \quad N^\dagger N = 1 \quad (27)$$

where N is a unitary $n_H \times n_H$ matrix. The spinor

$$\psi_{nH\mathbb{R}}^{CN} = \mathcal{C}_{DN} \psi_{nH\mathbb{R}} = \psi_{n'H\mathbb{R}}^C N_{n'm} \quad (28)$$

has the same transformation properties as ψ_{nHk} :

$$S_z \psi_{nH\mathbb{R}}^{CN} = \psi_{n'H\mathbb{R}}^{CN} (T_z)_{n'm} \quad (29)$$

Using involutive properties and an appropriate normalization gives

$$N^* N = \alpha = \pm 1 \quad (30)$$

For $N^* N = 1$, we call H a real representation with respect to G , whereas for $N^* N = -1$ it is called pseudoreal. For real representations ψ_{Rk} we can always choose a basis where $N = 1$. (This is done by taking a new basis $\psi'_{nRk} = \psi_{n'Rk}^{(N^{-\frac{1}{2}})_{n'n}}$.) In this basis, the generators T_z are real. Applying the operation $\mathcal{C}_{DN}$ twice yields

$$\mathcal{C}_{DN} (\mathcal{C}_{DN} \psi) = \alpha \varepsilon_D \psi \quad (31)$$

with

$$B_D^* B_D = \varepsilon_D = \pm 1. \quad (32)$$

Since the sum of all representations ϕ_{Hk} is self-conjugate both with respect to G and gen_E , no chiral four-dimensional fermions can be obtained from dimensional reduction of a Dirac spinor.

The mass matrix

The mass matrix for the E dimensional fermions is obtained from the term involving $i\gamma^{\alpha D_\alpha}$ in the action (7). Applying the operator $\Gamma^{\alpha D_\alpha}$ on a function $\psi_{n_0 H_0 k_0}(y)$ yields a new function $\psi'_{n_0 H_0 k_0}(y)$ which can be expanded in terms of the $\psi_{nHk}(y)$. Since $\Gamma^{\alpha D_\alpha}$ is a G singlet operator, $[\Gamma^{\alpha D_\alpha}, S_z] = 0$, one has

$$\begin{aligned} \Gamma^\alpha D_\alpha \psi_{mH\bar{z}}(y) &= \psi_{mH\bar{z}'}(y) M_{\bar{z}'\bar{z}}^{(H)} \quad \text{for } S_{\bar{A}} = 1 \\ i\bar{\Gamma}_D^\alpha \Gamma^\alpha D_\alpha \psi_{mH\bar{z}}(y) &= \psi_{mH\bar{z}'}(y) M_{\bar{z}'\bar{z}}^{(H)} \quad \text{for } S_{\bar{A}} = \bar{\Gamma}_D \end{aligned} \quad (33)$$

with a complex constant matrix $M_{\bar{k}'\bar{k}}^{(H)}$ for every representation H contained in $\psi(y)$.
We now expand L_ψ in the action (8) in terms of the $\psi_{nH\bar{k}}$:

$$\begin{aligned} L_\psi &= \frac{\gamma}{2} \psi_{mH\bar{z}}^\dagger \psi_{m'H'\bar{z}'} \bar{\varphi}_{mH\bar{z}} S_{\bar{H}} M_{\bar{z}''\bar{z}'}^{(H')} \varphi_{m'H'\bar{z}'} \\ &\quad + \frac{i}{2} \psi_{mH\bar{z}}^\dagger \psi_{m'H'\bar{z}'} \bar{\varphi}_{mH\bar{z}} \Gamma^\mu D_\mu \varphi_{m'H'\bar{z}'} + \text{h.c.} \end{aligned} \quad (34)$$

where $\gamma = 1$ for $S_{\bar{A}} = \bar{\Gamma}_D$ and $\gamma = i$ for $S_{\bar{A}} = 1$. For this, we used the following definitions

$$\begin{aligned} \bar{\psi}(y, x) &= \psi^\dagger(y, x) D_d \\ \bar{\varphi}(x) &= \varphi^\dagger(x) D_E \\ (\gamma^{\hat{m}})^\dagger &= D_d \gamma^{\hat{m}} D_d^{-1} \\ (\Gamma^m)^\dagger &= D_E \Gamma^m D_E^{-1} \\ D_d &= (D_E \otimes S_{\bar{A}}) \end{aligned} \quad (35)$$

(We may choose $D_E \equiv \Gamma^0$, $D_d \equiv \gamma^0$.) We carry out the integration over the y coordinates and obtain for the E dimensional action:

$$S = \int d^E x \, e_E (L_{E, \text{kin}} + L_{E, H}) \quad (36)$$

$$L_{E, \text{kin}} = \frac{i}{2} \bar{\varphi}_{mH\bar{z}} \Gamma^\mu D_\mu \varphi_{mH\bar{z}} + \text{h.c.} \quad (37)$$

$$L_{E, H} = \frac{\gamma}{2} \bar{\varphi}_{mH\bar{z}} S_{\bar{H}} M_{\bar{z}\bar{z}'}^{(H)} \varphi_{mH\bar{z}'} + \text{h.c.} \quad (38)$$

It contains the usual kinetic term for all E dimensional Dirac spinors and a mass term.

The matrix $M^{(H)}$ is hermitean for all representations H. This can be seen*) using

$$\begin{aligned} \int d^D y e_D \psi_{nH\mathbf{z}}^\dagger \Gamma^\alpha D_\alpha \psi_{m'H'\mathbf{z}'} &= \delta_{mm'} \delta_{HH'} M_{\mathbf{z}\mathbf{z}'}^{(H)} \quad \text{for } S_{\bar{A}} = 1 \\ \int d^D y e_D \psi_{nH\mathbf{z}}^\dagger i \bar{\Gamma}_D \Gamma^\alpha D_\alpha \psi_{m'H'\mathbf{z}'} &= \delta_{mm'} \delta_{HH'} M_{\mathbf{z}\mathbf{z}'}^{(H)} \quad \text{for } S_{\bar{A}} = \bar{\Gamma}_D \end{aligned} \quad (39)$$

We diagonalize $M^{(H)}$ by using unitary $k_H \times k_H$ matrices for a redefinition of the fields $\psi_{nH\mathbf{k}}$ consistent with the normalization (19):

$$M_{\mathbf{z}\mathbf{z}'}^{(H)} = M^{(H,\mathbf{z})} \delta_{\mathbf{z}\mathbf{z}'} \quad , \quad M^{(H,\mathbf{z})} = M^{(H,\mathbf{z}) *} \quad (40)$$

In general, we expect the eigenvalues $M^{(H,\mathbf{k})}$ to be discrete. We use them for a relabelling of the $\psi_{nH\mathbf{k}}$,

$$\psi_{nH\mathbf{z}} \equiv \psi_{nH M_j} \quad , \quad \Gamma^\alpha D_\alpha \psi_{nH M_j} = M \psi_{nH M_j} \quad (41)$$

(and a corresponding modification for the case $S_{\bar{A}} = \bar{\Gamma}_D$), where j counts the number of mass eigenstates with eigenvalue M belonging to the representation H. This gives the final form for the mass term:

$$L_{E,M} = \gamma M \bar{\psi}_{nH M_j} S_{\bar{A}} \psi_{nH M_j} \quad (42)$$

Before discussing the mass spectrum in some more detail, let us see what happens if we add a bare d dimensional mass term to the action (8):

$$L_{d,M} = m_d \bar{\psi}(y,x) \psi(y,x) \quad (43)$$

This will also explain the appearance of $S_{\bar{A}}$ in the E dimensional mass term obtained from the d dimensional kinetic term. Decomposing $\psi(y,x)$ gives

$$L_{d,M} = m_d \psi_{nH M_j}^\dagger(y) S_{\bar{A}} \psi_{m'H'M'_j}(y) \bar{\psi}_{nH M_j}(x) \psi_{m'H'M'_j}(x) \quad (44)$$

The matrix $S_{\bar{A}}$ commutes with G transformations

$$S_{\bar{A}} \psi_{nH M_j} = \psi_{nH M'_j} S_{M'M_j} \quad (45)$$

*) The antihermitean part of $M^{(H)}$ is given by an integral over the total divergence $\partial_\alpha (e_D \bar{\psi}_k \Gamma^\alpha \psi_k)$. It vanishes if K^D has no boundaries.

The integration over the y coordinates gives us the contribution of this term to the E dimensional mass term:

$$L_{E,m_d} = m_d \bar{\varphi}_{nHM_j} S_{MM'jj'} \varphi_{nHM'j'} \quad (46)$$

First we consider the case $S_{\bar{A}} = 1$. The total mass term of a spinor ϕ_{nHMj} is given by

$$L_m = \bar{\varphi} (m_d + iMS_{\bar{A}}) \varphi \quad (47)$$

Using $\{S_{\bar{M}}, \Gamma^m\} = 0$, we see that the kinetic term is invariant under rotations

$$\varphi \rightarrow \varphi' = \exp i\alpha S_{\bar{A}} \quad (48)$$

(For E even, these are called chiral rotations.) Using such a redefinition of ϕ with $\tan 2\alpha = M/m_d$ we obtain the canonical form of the mass term:

$$L_m = \bar{\varphi}' \sqrt{m_d^2 + M^2} \varphi' \quad (49)$$

This has an important consequence: for $m_d \neq 0$ all E dimensional spinors must be massive*). No chiral E dimensional fermions can be found if the d dimensional action admits a bare mass term consistent with gen_d 7). To search for any realistic model out of dimensional reduction we have to start with $m_d = 0$.

The appearance of $S_{\bar{M}}$ anticommuting with Γ^μ is necessary for the consistency of the field equations:

$$\begin{aligned} (i\gamma^\mu D_\mu + i\gamma^\alpha D_\alpha + m_d)\psi &= (i\Gamma^\mu D_\mu + iMS_{\bar{A}} + m_d)\psi = 0 \\ [(\gamma^\mu D_\mu)^2 + (\gamma^\alpha D_\alpha)^2 + m_d^2]\psi &= [(\Gamma^\mu D_\mu)^2 + M^2 + m_d^2]\psi = 0 \end{aligned} \quad (50)$$

For $S_{\bar{A}} = \bar{\Gamma}_D$ the situation is very similar: we now have $S_{\bar{M}} = 1$, but the matrix $S_{MM'jj'}$ anticommutes with all $M^{(H)}$, again giving consistent field equations with canonical mass terms $\sqrt{m_d^2 + M^2}$. In our further discussion we neglect d dimensional mass terms and concentrate on the spectrum of $\Gamma^\alpha D_\alpha$ (or $i\bar{\Gamma}_D \Gamma^\alpha D_\alpha$ for $S_{\bar{A}} = \bar{\Gamma}_D$).

*) A similar discussion can be given for the case of a d-dimensional bare Majorana mass term.

First we note that all E dimensional mass terms obtained from dimensional reduction of a Dirac spinor are Dirac mass terms. No E dimensional Majorana mass terms appear. The massless modes, which are our main interest, are the zero eigenmodes of the D dimensional Dirac operator

$$\Gamma^\alpha D_\alpha \psi_{nH0j} = 0 \quad (51)$$

For every zero eigenmode, we obtain a massless ϕ_{nH0j} . For E even, it contains two massless Weyl spinors, for E odd and D even it is a massless Dirac spinor and for E and D odd ϕ decomposes into two massless Dirac spinors.

For D even, the eigenstates with non-vanishing mass are mixtures of representations contained in $\psi_+(y) = 1/2(1 + \bar{\Gamma}_D)\psi$ and in $\psi_-(y) = 1/2(1 - \bar{\Gamma}_D)\psi$. In a basis where $\bar{\Gamma}_D$ is diagonal, one has

$$\begin{aligned} \psi_+ &= \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix}, \quad \bar{\Gamma}_D = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ \Gamma^\alpha D_\alpha \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} &= M \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix}, \quad \Gamma^\alpha D_\alpha \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix} = M \begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} \end{aligned} \quad (52)$$

(and a corresponding equation for $S_{\bar{A}} = \bar{\Gamma}_D$). For $M \neq 0$, χ_1 and χ_2 must both be different from zero. If ψ_{nHMj} is an eigenmode with mass $M \neq 0$, the function $\bar{\Gamma}_D \psi_{nHMj}$ is linearly independent of ψ_{nHMj} with

$$\int d^D y \, e_D \, \psi_{nHMj}^\dagger (\bar{\Gamma}_D \psi_{nHMj}) = 0 \quad (53)$$

Since $\bar{\Gamma}_D$ anticommutes with the mass operator, the function $\bar{\Gamma}_D \psi_{nHMj}$ is also an eigenmode of the mass matrix with eigenvalue $-M$ *) and we denote

$$\bar{\Gamma}_D \psi_{nHMj} \equiv \psi_{nH,-M,j} \quad (54)$$

For D even, the massive spinors of every representation H appear in pairs with degenerate mass $|M|$. Massless modes, however, may entirely be contained in ψ_+ or ψ_- .

Let us now apply the charge conjugation operator $\mathcal{C}_D$ on the eigenvalue equations:

*) The sign of the mass term has no physical relevance. It can be rotated away by a transformation (48) (for $\{S_{\bar{M}}, \Gamma^{\bar{m}}\} = 0$).

$$\begin{aligned} \mathcal{C}_D (\Gamma^\alpha D_\alpha \psi_{mHMj}) &= \pm \Gamma^\alpha D_\alpha (\mathcal{C}_D \psi_{mHMj}) = M \mathcal{C}_D \psi_{mHMj} \\ &\quad (\text{for } S_{\bar{A}} = 1) \\ \mathcal{C}_D (i \bar{\Gamma}_D^\alpha \Gamma^\alpha D_\alpha \psi_{mHMj}) &= \pm (-1)^{\frac{D}{2}+1} i \bar{\Gamma}_D^\alpha \Gamma^\alpha D_\alpha (\mathcal{C}_D \psi_{mHMj}) = M \mathcal{C}_D \psi_{mHMj} \\ &\quad (\text{for } S_{\bar{A}} = \bar{\Gamma}_D) \end{aligned} \quad (55)$$

Here we used the fact that B_D is given either by B_{1D} or B_{2D} defined by

$$\begin{aligned} (\Gamma^\alpha)^* &= -B_{1D} \Gamma^\alpha B_{1D}^{-1} \\ (\Gamma^\alpha)^* &= B_{2D} \Gamma^\alpha B_{2D}^{-1} \end{aligned} \quad (56)$$

For D even, both B_{1D} and B_{2D} can be defined with $B_{2D} = B_{1D} \bar{\Gamma}_D$, whereas for D odd only one of them exists. (B_{1D} for $D = 5 \bmod 4$, B_{2D} for $D = 3 \bmod 4$.) The $+$ sign ($-$ sign) in Eq. (55) holds for $B_D \equiv B_{2D}$ ($B_D \equiv B_{1D}$).

The complex representations C can be divided into two classes, where the charge conjugation operator $\mathcal{C}_D$ maps every representation C of one class into a representation $\bar{C}$ of the other class:

$$\begin{aligned} \mathcal{C}_D \psi_{mCMj} &= \psi_{m\bar{C}, \pm M, j} \\ \mathcal{C}_D \psi_{m\bar{C}, \pm M, j} &= E_D \psi_{mCMj} \end{aligned} \quad (57)$$

To every spinor ϕ_C corresponds a spinor of the complex conjugate representation $\phi_{\bar{C}}$ with the same mass, up to a minus sign for the cases where $\mathcal{C}_D$ anticommutes with the mass operator. For complex representations and D even, we have at least four Dirac spinors for every eigenvalue $|M| \neq 0$. The massless modes are at least two-fold degenerate: one massless Dirac spinor ϕ_C and a corresponding massless Dirac spinor $\phi_{\bar{C}}$. If ψ_+ and ψ_- contain the same number of a given representation C , the massless modes belonging to C or $\bar{C}$ are also four-fold degenerate. For D odd, all eigenvalues $|M|$ are at least two-fold degenerate. For both D and E odd, we have again four Dirac spinors for every eigenvalue $|M|$.

For the discussion of real or pseudoreal representations we have to distinguish between different cases: for $D = 3 \bmod 4$, the charge conjugation operator commutes with the mass operator. Using the operator $\mathcal{C}_{DN}$ defined in Eq. (28), we obtain

$$C_{DN} \psi_{mHMj} = \psi_{mHMj'} (C_{DN})_{j'j} \quad (58)$$

with

$$\int d^D y \epsilon_D \psi_{mHMj} + C_{DN} \psi_{m'H'M'j'} = \delta_{mm'} \delta_{HH'} \delta_{MM'} (C_{DN})_{j'j} \quad (59)$$

$$C_{DN} C_{DN}^* = \alpha \epsilon_D \quad (60)$$

For $\alpha \epsilon_D = +1$, we can transform C_{DN} to the unit matrix by unitary transformations between the eigenmodes of mass M . For $\alpha \epsilon_D = -1$, this is not possible and a suitable form for C_{DN} is given by

$$C_{DN} = \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & 0 & 1 & & \\ & -1 & 0 & & \\ & & & \ddots & \\ 0 & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \quad (61)$$

For $D = 7 \bmod 8$ we have $\epsilon_D = 1$. The real representations are eigenstates of C_{DN} ($N = 1$ in this case)

$$B_D^{-1} \psi_{mRMj}^* = \psi_{mRMj} \quad (62)$$

The pseudoreal representations occur in pairs with equal masses

$$C_{DN} \psi_{mPM, 2\ell-1} = \psi_{mPM, 2\ell} \quad (63)$$

$$C_{DN} \psi_{mPM, 2\ell} = -\psi_{mPM, 2\ell-1}$$

and we have at least two Dirac spinors for every eigenvalue M . For $D = 3 \bmod 8$ we have $\epsilon_D = -1$ and the role of real and pseudoreal representations is inversed compared with $D = 7 \bmod 8$. For $D = 5 \bmod 4$ the massive modes cannot be eigenstates of C_{DN} . The anticommutation relation $\{C_{DN}, \Gamma_{\alpha}^{\alpha_D}\} = 0$ implies

$$C_{DN} \psi_{mHMj} = \psi_{mH, -M, j} \quad (64)$$

and all eigenmodes to $|M| \neq 0$ are at least two-fold degenerate. For massless modes, however, the same discussion as for $D = 3 \bmod 4$ applies: for $D = 9 \bmod 8$ the real

massless modes are eigenstates of $\mathcal{C}_D$ whereas the pseudoreal representations occur in pairs, and for $D = 5 \bmod 8$ we have massless self-conjugate pseudoreal functions ψ_{nPOk} .

For D even, $S_A^- = 1$, the charge conjugation operator commutes with the mass operator if we use $B_D \equiv B_{2D}$:

$$\int d^D y \mathcal{C}_D \psi_{MMj}^+ \mathcal{C}_{\text{DN2}} \psi_{M'H'M'j'} = \int d^D y \mathcal{C}_D \psi_{MMj}^+ B_{2D}^{-1} \psi_{M'H'M'j'}^* N_{jj'} \quad (65)$$

$$= \delta_{mm'} \delta_{HH'} \delta_{MM'} (C_{\text{DN2}})_{jj'}$$

$$C_{\text{DN2}} C_{\text{DN2}}^* = \alpha B_{2D} B_{2D}^* = \alpha \varepsilon_2 \quad (66)$$

For $\alpha \varepsilon_2 = 1$, C_{DN2} can be transformed into the unit matrix, whereas for $\alpha \varepsilon_2 = -1$ it has the form (61). We have $\varepsilon_2 = +1$ for $D = 6, 8 \bmod 8$. In these dimensions the massive real representations are eigenstates of $\mathcal{C}_{\text{DN2}}$ whereas there are four independent pseudoreal representations with the same value of $|M| \neq 0$. For $D = 2, 4 \bmod 8$ we have $\varepsilon_2 = -1$ and the role of real and pseudoreal representations is inversed. A similar discussion for the case $S_A^- = \bar{\Gamma}_D$ gives the same result. The massless modes can be taken as eigenstates of $\bar{\Gamma}_D$. For $D = 4 \bmod 4$ the charge conjugation $\mathcal{C}_{\text{DN2}}$ commutes with $\bar{\Gamma}_D$. For $D = 8 \bmod 8$ the real representations are eigenstates of both $\bar{\Gamma}_D$ and $\mathcal{C}_{\text{DN2}}$ and there is not necessarily a degeneracy. The pseudoreal representations cannot be eigenstates of $\mathcal{C}_{\text{DN2}}$ and are twofold degenerate. For $D = 4 \bmod 8$ the role of real and pseudoreal representations is inversed compared with $D = 8 \bmod 8$. For $D = 2 \bmod 4$ $\mathcal{C}_{\text{DN2}}$ anticommutes with $\bar{\Gamma}_D$ and to every massless spinor contained in ψ_+ corresponds a massless spinor contained in ψ_- , which can be obtained by applying $\mathcal{C}_{\text{DN2}}$ on ψ_+ *).

To resume, we find, with a few exceptions, two or four independent eigenmodes for every value of $|M|$. We list the number of these eigenmodes in Table 1. To every eigenmode corresponds one E dimensional Dirac spinor (except for D, E both odd where we have even two Dirac spinors for every eigenmode). Such a degeneracy of four-dimensional fermions with equal mass is certainly not compatible with observation: we do not see two or four different electrons with equal mass!

*) Note that the existence of a charge conjugation operator $\mathcal{C}_D$ with the property, that $\mathcal{C}_D \psi(y)$ is a spinor on a given space if $\psi(y)$ is a spinor on that space, implies a vanishing Atiyah-Singer index for all these spaces in $D = 2 \bmod 4$ dimensions.

However, we will see that this degeneracy is necessary for a consistent dimensional reduction of d dimensional Weyl and/or Majorana spinors. The Weyl/Majorana constraints will reduce this degeneracy. This reduction is not always complete, since the degeneracy only depends on D (up to a factor two for both D and E odd) whereas possible Majorana and Weyl constraints depend on d . For the interesting case of a Majorana-Weyl spinor for $d = 2 \bmod 8$ the degeneracy is completely absorbed.

3. WEYL AND MAJORANA CONSTRAINTS

d -dimensional Majorana and Weyl constraints

For vanishing d dimensional mass terms, the Dirac spinor often becomes reducible with respect to gen_d^*). The irreducible representations are then subject to Weyl and/or Majorana constraints. For d even, a Dirac spinor can be decomposed into two Weyl spinors

$$\begin{aligned}\psi_+(y, x) &= \frac{1}{2}(1 + \bar{\gamma}_d) \psi(y, x) \\ \psi_-(y, x) &= \frac{1}{2}(1 - \bar{\gamma}_d) \psi(y, x)\end{aligned}\tag{67}$$

$$\bar{\gamma}_d = \gamma_d \gamma^0 \gamma^1 \dots \gamma^{d-1}, \quad \gamma_d^2 = (-1)^{\frac{d}{2}-1}\tag{68}$$

and the irreducible spinor $\psi_+(y, x)$, for example, obeys the Weyl constraint

$$\bar{\gamma}_d \psi_+(y, x) = \psi_+(y, x)\tag{69}$$

A Majorana constraint

$$\mathcal{C}_d \psi_H(y, x) = B_d^{-1} \psi_H^*(y, x) = \psi_H^c(y, x) = \psi_H(y, x)\tag{70}$$

is possible for $B_d^* B_d = 1$, where B_d fulfils

$$(\Sigma^{\hat{m}\hat{n}})^* = B_d \Sigma^{\hat{m}\hat{n}} B_d^{-1}\tag{71}$$

and $\Sigma^{\hat{m}\hat{n}}$ are the generators of the d dimensional Lorentz group. The matrix B_d is given by one of the two matrices B_{1d}, B_{2d} fulfilling

$$\begin{aligned}(\gamma^{\hat{m}})^* &= -B_{1d} \gamma^{\hat{m}} B_{1d}^{-1} \\ (\gamma^{\hat{m}})^* &= B_{2d} \gamma^{\hat{m}} B_{2d}^{-1}\end{aligned}\tag{72}$$

*) For $d = 2, 3, 4 \bmod 8$, this also happens for non-vanishing bare mass terms.

For d even, both B_{1d} and B_{2d} can be defined and $B_{2d} = B_{1d} \bar{\gamma}_d$. Majorana spinors exist for $d = 2, 3, 4, 8, 9 \pmod{8}$. For $d = 2 \pmod{8}$, $\mathcal{C}_d$ commutes with $\bar{\gamma}_d$ and both Weyl and Majorana conditions can be applied. The Majorana-Weyl spinor ψ_{M+} obeys

$$\bar{\gamma}_d \psi_{M+}(y, x) = \mathcal{C}_d \psi_{M+}(y, x) = \psi_{M+}(y, x) \quad (73)$$

The kinetic term for a d dimensional Dirac spinor decomposes into a sum of independent kinetic terms for Weyl, Majorana or Majorana-Weyl spinors:

$$\begin{aligned} L_\psi &= \frac{i}{2} \bar{\psi} \gamma^{\hat{m}} D_{\hat{m}} \psi + h.c. \\ &= \frac{i}{2} \bar{\psi}_+ \gamma^{\hat{m}} D_{\hat{m}} \psi_+ + \frac{i}{2} \bar{\psi}_- \gamma^{\hat{m}} D_{\hat{m}} \psi_- + h.c. \quad \text{for } d \text{ even} \\ &= \frac{i}{2} \tilde{\psi}_1 \gamma^{\hat{m}} D_{\hat{m}} \psi_1 + \frac{i}{2} \tilde{\psi}_2 \gamma^{\hat{m}} D_{\hat{m}} \psi_2 \quad \text{for } d = 2, 3, 4, 8, 9 \pmod{8} \\ &= \frac{i}{2} \tilde{\psi}_{1+} \gamma^{\hat{m}} D_{\hat{m}} \psi_{1+} + \frac{i}{2} \tilde{\psi}_{2+} \gamma^{\hat{m}} D_{\hat{m}} \psi_{2+} \\ &\quad + \frac{i}{2} \tilde{\psi}_{1-} \gamma^{\hat{m}} D_{\hat{m}} \psi_{1-} + \frac{i}{2} \tilde{\psi}_{2-} \gamma^{\hat{m}} D_{\hat{m}} \psi_{2-} \quad \text{for } d = 2 \pmod{8} \end{aligned} \quad (74)$$

Here ψ_1 and ψ_2 are Majorana spinors ($\mathcal{C}_d \psi_i = \psi_i$) given by

$$\psi_1 = \frac{1}{\sqrt{2}} (\psi + \psi^c) \quad (75)$$

$$\psi_2 = -\frac{i}{\sqrt{2}} (\psi - \psi^c)$$

and $\tilde{\psi}$ is defined with

$$\tilde{\psi} = \psi^T B_d D_d \quad (76)$$

This has an important consequence for dimensional reduction: if there is no bare d dimensional mass term, the E dimensional mass matrices $M^{(H)}$ do not mix representations obtained from different d dimensional Weyl and/or Majorana spinors contained in the Dirac spinor. The spectrum of the E dimensional mass matrix for a Weyl, Majorana or Majorana-Weyl spinor is just a part of the spectrum for a Dirac

spinor. Especially, zero mass fermions in E dimensions still correspond to zero eigenmodes of the D dimensional Dirac operator $\Gamma^\alpha D_\alpha$. The spectrum for Weyl and/or Majorana spinors can be obtained by applying appropriate constraints on the E dimensional spinors $\phi_{nHM_j}(x)$. This is why the high degeneracy of the mass spectrum of a d dimensional Dirac spinor is needed for consistency: every Weyl or Majorana constraint reduces the number of independent spinors with a given mass $|M|$ by a factor of two, and applying a Majorana and a Weyl constraint for $d = 2 \bmod 8$ gives even a reduction by a factor of four.

Before discussing in some more detail the dimensional reduction of Weyl, Majorana and Majorana-Weyl spinors, we briefly describe what kind of massive or massless fermions can occur in E dimensions. All massive spinors belonging to complex representations must be Dirac spinors, since no Majorana mass terms are possible for complex spinors. For real representations a Majorana mass term

$$L_{Maj} = M \sum_n \varphi_{nRM_j}^T C_E \varphi_{nRM_j} \quad (77)$$

with C_E defined by

$$(\sum^{mn})^T = -C_E \sum^{mn} C_E^{-1} \quad (78)$$

is possible if C_E is antisymmetric. This is the case for $E = 2, 3, 4 \bmod 8$. The Majorana mass term for pseudoreal representations is given by

$$L_{Maj} = M \sum_{n,n'} \varphi_{nPM_j}^T C_E (N^{-1})_{nn'} \varphi_{n'PM_j} \quad (79)$$

Since N^{-1} is antisymmetric, C_E must now be symmetric and this is possible for $E = 6, 7, 8 \bmod 8$. All massless spinors for E even are Weyl spinors, except for $E = 2 \bmod 8$ where a real Majorana-Weyl spinor exists, and for $E = 6 \bmod 8$ with a pseudoreal Majorana-Weyl spinor. For E odd, we have massless real Majorana spinors for $E = 3, 9 \bmod 8$ and pseudoreal Majorana spinors for $E = 5, 7 \bmod 8$. All other massless spinors are Dirac spinors. We give the possible types of spinors for all E in Table 2. It is not difficult to convince oneself by counting the degrees of freedom that the degeneracy of eigenmodes listed in Table 1 is the minimal degeneracy required for consistently imposing Weyl, Majorana and Majorana-Weyl constraints for arbitrary d .

Dimensional reduction of Weyl spinors

We first discuss the dimensional reduction of a Weyl spinor $\psi_+ = \bar{\gamma}_d \psi_+$ for both E and D even. One has

$$\bar{\gamma}_d = (\bar{\Gamma}_E \otimes \bar{\Gamma}_D) \quad (80)$$

and the Weyl constraint implies for the E dimensional spinors

$$\bar{\Gamma}_E \varphi_{mHk}(x) = \Gamma_{kk'}^{(H)} \varphi_{mHk'}(x) \quad (81)$$

Here we used the fact that $\bar{\Gamma}_D$ commutes with G transformations:

$$\bar{\Gamma}_D \psi_{mHk}(y) = \psi_{mHk'}(y) \Gamma_{kk'}^{(H)} \quad (82)$$

$\Gamma_{kk'}^{(H)}$ has eigenvalues ± 1 , and we can classify the G representations ψ_{nHk} into representations

$$\begin{aligned} \psi_{mHk+} &= \bar{\Gamma}_D \psi_{mHk+} \\ \psi_{mHk-} &= -\bar{\Gamma}_D \psi_{mHk-} \end{aligned} \quad (83)$$

For the corresponding fields ϕ the Weyl constraint gives

$$\begin{aligned} \bar{\Gamma}_E \varphi_{mHk+} &= \varphi_{mHk+} \\ \bar{\Gamma}_E \varphi_{mHk-} &= -\varphi_{mHk-} \end{aligned} \quad (84)$$

In this basis the d dimensional Weyl constraint reduces each E dimensional Dirac spinor to a Weyl spinor.

Except for massless modes, the mass matrix $M^{(H)}$ mixes between spinors of different helicity. For the mass eigenstates with $|M| \neq 0$ one has

$$\Gamma_{kk'}^{(H)} \equiv \Gamma_{MM'jj'}^{(H)} = \delta_{M,-M} \delta_{j,j'} \quad (85)$$

and the Weyl condition reads

$$\bar{\Gamma}_E \begin{pmatrix} \varphi_{mHMj} \\ \varphi_{mH,-M,j} \end{pmatrix} = \begin{pmatrix} \varphi_{mH,-M,j} \\ \varphi_{mHMj} \end{pmatrix} \quad (86)$$

Splitting the spinors into eigenstates of $\bar{\Gamma}_E$, $\phi_{\pm} = (1 \pm \bar{\Gamma}_E)/2 \phi$, we have the constraint

$$\varphi_{nH,-M,j+} = \varphi_{nHMj+}, \quad \varphi_{nH,-M,j-} = -\varphi_{nHMj-} \quad (87)$$

which eliminates $\phi_{nH,-M,j}$ as an independent spinor. For eigenstates of $\mathcal{C}_{DN}$ we remain with one Dirac spinor ϕ_{nHMj} . Otherwise we have two spinors with equal mass. For H complex, they belong to complex conjugate representations. Note that we have to adapt the normalization of ϕ_{nHMj} , since the above identification (87) doubles the coefficient of the E dimensional kinetic term.

Massless spinors are eigenstates of the E dimensional helicity operator $\bar{\Gamma}_E$. The consequences of the Weyl constraint for the E dimensional spectrum depend crucially on the dimension D. For $D = 4 \bmod 4$, $\bar{\Gamma}_D$ commutes with the charge conjugation operator $\mathcal{C}_D$. This implies that to every complex G representation with a given E dimensional helicity, $\phi_{C+(-)}$, corresponds a complex conjugate representation $\phi_{\bar{C}+(-)}$ with the same helicity. The sum of all Weyl spinors ϕ_+ forms a self-conjugate reducible G representation.

For $E = 4 \bmod 4$ the E dimensional charge conjugation operator $\mathcal{C}_E$ ($\mathcal{C}_E \phi = B_E^{-1} \phi^*$) anticommutes with $\bar{\Gamma}_E$ and we can combine the two Weyl spinors into a massless Dirac spinor:

$$\varphi_{nCOj}^{(D)} = \varphi_{nCOj+} + B_E^{-1} \varphi_{n\bar{C}Oj+}^* \quad (88)$$

For $E = 2 \bmod 4$ we have $[\mathcal{C}_E, \bar{\Gamma}_E] = 0$ and we stay with two massless Weyl spinors belonging to complex conjugate G representations.

For $D = 4$ ($D = 8$) the pseudoreal (real) G representations obey $\psi_{nHOj}^{CN}(y) = \psi_{nHOj}(y)$. The massless modes are Weyl spinors. For the representations which are not eigenstates of $\mathcal{C}_{DN}$ Eq. (55) implies that to every massless Weyl spinor $\phi_{H,0,2\ell-1,+}$ corresponds a massless spinor $\phi_{H,0,2\ell,+}$. For $E = 4 \bmod 4$, they may be combined to a massless Dirac spinor:

$$\varphi_{n,H,0,2\ell-1}^{(D)} = \varphi_{n,H,0,2\ell-1,+} + B_E^{-1} (N_{nn'} \varphi_{n',H,0,2\ell,+})^* \quad (89)$$

The situation is different for $D = 2 \bmod 4$. Now $\mathcal{C}_D$ anticommutes with $\bar{\Gamma}_D$. For complex representations, this implies

$$\bar{\Gamma}_E \varphi_C = \delta \varphi_C \Rightarrow \bar{\Gamma}_E \varphi_{\bar{C}} = -\delta \varphi_{\bar{C}} \quad (90)$$

If $\psi_+(y) = \bar{\Gamma}_D \psi_+(y)$ contains n representations C , and $\psi_-(y) = -\bar{\Gamma}_D \psi_-(y)$ contains m of these representations, we are necessarily left with $n-m$ *) massless chiral spinors $\phi_{C+}(x)$ and with the same number of massless chiral spinors $\phi_{C-}(x)$! Note that for $E = 4 \bmod 4$, ϕ_{C+} and ϕ_{C-} have the same transformation properties with respect to $G \times \text{gen}_E$. If we have no further Majorana constraint, we have an even number of Weyl spinors for every complex C in this case. We also conclude, that if the D dimensional Dirac operator Γ_D^α has an odd number of zero mass eigenmodes belonging to a complex representation C , we necessarily will find chiral E dimensional spinors. All complex G representations ϕ_C not counting among the $n-m$ chiral fermions can be combined to massive or massless E dimensional Dirac spinors. If ϕ_{H+} is a massless real or pseudoreal Weyl spinor, $\{c_{DN}, \bar{\Gamma}_D\} = 0$ implies the existence of a corresponding massless spinor ϕ_{H-} with opposite eigenvalue of $\bar{\Gamma}_E$. Both can be combined into a massless Dirac spinor.

Finally, we have to investigate the Weyl constraint for both D and E odd. We find

$$\bar{\gamma}_d = (i)^{\frac{E+1}{2}} \Gamma^0 \Gamma^1 \dots \Gamma^{E-1} S_{\bar{M}} \equiv -i \Gamma^E \equiv T_{\bar{M}} \quad (91)$$

where Γ^E anticommutes with all Γ^m and with $S_{\bar{M}}$, $(\Gamma^E)^2 = -1$. The Weyl condition $\bar{\gamma}_d \psi(y, x) = \psi(y, x)$ reduces to the E dimensional condition

$$T_{\bar{M}} \phi_{nHMj}(x) = \phi_{nHMj}(x) \quad (92)$$

Since $T_{\bar{M}}$ commutes with the E dimensional Lorentz generators $\sum^{mn}$ and $T_{\bar{M}}^2 = 1$, the d dimensional Weyl condition reduces the two E dimensional Dirac spinors contained in ϕ_{nHMj} to one Dirac spinor.

Dimensional reduction of Majorana spinors

Majorana conditions can be imposed for $d = 2, 3, 4, 8, 9 \bmod 8$, where $B_d^* B_d = \epsilon_d = +1$. For D and E not both odd, the matrices B_{1d}, B_{2d} (72) can always be written as a direct product of $\bar{M} \times \bar{M}$ matrices B_{1E}, B_{2E} defined by

$$\begin{aligned} (\Gamma^m)^* &= -B_{1E} \Gamma^m B_{1E}^{-1} \\ (\Gamma^m)^* &= B_{2E} \Gamma^m B_{2E}^{-1} \end{aligned} \quad (93)$$

and the $\bar{A} \times \bar{A}$ matrices B_{1D}, B_{2D} defined in Eq. (56):

$$\begin{aligned} B_{1d} &= (B_{1E} \otimes B_{1D}), \quad B_{2d} = (B_{2E} \otimes B_{2D}) \quad \text{for } E = 2 \bmod 4 \\ B_{1d} &= (B_{1E} \otimes B_{2D}), \quad B_{2d} = (B_{2E} \otimes B_{1D}) \quad \text{for } E = 4 \bmod 4 \end{aligned}$$

*) For $m > n$, we have $m-n$ spinors ϕ_{C-} and ϕ_{C+} .

$$B_{1d} = (B_{1E} \otimes B_{1D}), \quad B_{2d} = (B_{2E} \otimes B_{2D}) \quad \text{for } E \text{ odd, } D = 4 \bmod 4 \quad (94)$$

$$B_{1d} = (B_{2E} \otimes B_{1D}), \quad B_{2d} = (B_{1E} \otimes B_{2D}) \quad \text{for } E \text{ odd, } D = 2 \bmod 4$$

For D and E even, both B_{1d} and B_{2d} can be defined. For $D(E)$ odd, only one of the B_{id} (B_{iE}) exists, and only the corresponding B_{id} can be defined in d dimensions. For D and E both odd, only one of the B_{id} can be defined by

$$B_{1d} = (B_{1,E+1} \otimes B_{1D}), \quad B_{2d} = (B_{2,E+1} \otimes B_{2D}) \quad \text{for } E = 1 \bmod 4 \quad (95)$$

$$B_{1d} = (B_{1,E+1} \otimes B_{2D}), \quad B_{2d} = (B_{2,E+1} \otimes B_{1D}) \quad \text{for } E = 3 \bmod 4$$

We use the relation $B_{2d} = B_{1d} \bar{\gamma}_d = B_{1d} T_M^-$ to define the other B_{id} . In Table 3, we list the decomposition of B_d for arbitrary D and E . For d even, we only list one B_{id} . The other can be obtained by multiplication with $\bar{\gamma}_d$, using

$$B_{2d} = B_{1d} \bar{\Gamma}_D \quad \text{for } D \text{ even} \quad (96)$$

$$B_{2E} = B_{1E} \bar{\Gamma}_E \quad \text{for } E \text{ even}$$

Dimensional reduction of a Majorana spinor $B_{id}^{-1} \psi(y,x)^* = \psi(y,x)$ leads to the E dimensional constraint

$$B_{m'mH'H'M'j'j'} U_M^- B_{dE}^{-1} \varphi_{m'H'M'j'}^*(x) = \varphi_{mH M_j}(x) \quad (97)$$

where

$$B_{dD}^{-1} \psi_{mH M_j}^*(y) = \psi_{m'H'M'j'}(y) B_{m'mH'H'M'j'j'} \quad (98)$$

One has $U_M^- = T_M^-$ for D and E odd if B_{id} has not the product form (92), otherwise $U_M^- = 1$. The matrix B_{id} is given by $B_d = B_{1d}$ for $d = 3, 4 \bmod 8$ and $B_d = B_{2d}$ for $d = 8, 9 \bmod 8$. For $d = 2 \bmod 8$ we may choose either of the B_{id} . The matrices B_{dD} , B_{kE} are then fixed by Table 3.

For complex representations C we have

$$B_{nn'CC'MM'jj'} = \beta \delta_{nn'} \delta_{CC'} \delta_{M, \pm M'} \delta_{jj'} \quad (99)$$

with $\beta = +1$ for $B_D^* B_D = 1$, whereas for $B_D^* B_D = -1$ it is $+1$ or -1 for two classes of representations which are complex conjugate to each other, according to $[\psi(y)^c]^c = -\psi(y)$. The choice of the sign in $\delta_{M, \pm M'}$ depends on the choice of B_D [see Eq. (55)]. The Majorana constraint identifies the charge conjugate of the spinor $\phi_{\bar{C}}$ with the spinor ϕ_C . Two Dirac spinors, one for C and one for $\bar{C}$, are thus reduced to one Dirac spinor of the representation C. Applying only a Majorana constraint in d dimensions can therefore not induce chiral E dimensional fermions!

For real and pseudoreal representations we first discuss the case where B_D is given by B_{2D} (see Table 3). (For E odd and $D = 4 \bmod 4$ we take the case $B_D = B_{1D}$ instead of $B_D = B_{2D}$.) For $B_D^* B_D = 1$ we have for real representations

$$B_{nn'RR'MM'jj'} = \delta_{nn'} \delta_{RR'} \delta_{MM'} \delta_{jj'} \quad (100)$$

and for pseudoreal representations

$$B_{nn'PP'MM'jj'} = \begin{cases} \delta_{nn'} \delta_{PP'} \delta_{MM'} \delta_{j,j'-1} & \text{for } j = 2\ell-1 \\ \delta_{nn'} \delta_{PP'} \delta_{MM'} \delta_{j,j'+1} & \text{for } j = 2\ell \end{cases} \quad (101)$$

Since $B_D^* B_D = 1$ implies $B_E^* B_E = 1$, the Majorana constraint reduces all real E dimensional Dirac spinors to Majorana spinors, except for the case D, E both odd, where two Dirac spinors are reduced to one Dirac spinor. Looking at Table 3, we see that massive real Majorana spinors only appear for $E = 2, 3, 4 \bmod 8$, consistent with Table 2. However, except for $D = 7 \bmod 8$, we have at least two Majorana spinors with equal $|M| \neq 0$ (see Table 1) and they can be recombined to a massive Dirac spinor. Unpaired massless real Majorana spinors can also exist for $D = 8 \bmod 8$. For $B_D^* B_D = -1$, the role of real and pseudoreal representations is inverted compared with the case $B_D^* B_D = 1$. For the pseudoreal representations we have $N \neq 1$ and

$$B_{nn'PP'MM'jj'} = \delta_{PP'} \delta_{MM'} \delta_{jj'} (N^{-1})_{nn'} \quad (102)$$

The constraint for pseudoreal Majorana spinors is

$$(N^{-1})_{nn'} B_E^{-1} \varphi_{n'PM_j}^*(x) = \varphi_{nPM_j}(x) \quad (103)$$

Let us look at the alternative case $B_D = B_{1D}$ (except for E odd, $D = 4 \bmod 4$ with $B_D = B_{2D}$). For the massive modes, we have

$$B_{nn'HH'MM'jj'} = \pm \delta_{HH'} \delta_{M,-M'} \delta_{jj'} (N^{-1})_{nn'} \quad (104)$$

The Majorana constraint combines the two Dirac spinors ϕ_{nHMj} and ϕ_{nH-Mj} into one Dirac spinor. Real (pseudoreal) massless spinors are again eigenstates of charge conjugation if $B_D^* B_D = 1$ (-1) and the discussion is similar as for the first case. Taking all this together, we find that dimensional reduction of a Majorana spinor (no further Weyl constraint!) can lead to a massive Majorana spinor only for $d = 3, 9 \bmod 8$ and $D = 7 \bmod 8$ (real Majorana spinor) or $D = 3 \bmod 8$ (pseudoreal Majorana spinor). Massless Majorana spinors can be obtained for $d = 3, 9 \bmod 8$ and $D = 7, 8, 9 \bmod 8$ (real) or $D = 3, 4, 5 \bmod 8$ (pseudoreal) and for $d = 2, 4, 8 \bmod 8$ and $D = 8 \bmod 8$ (real) or $D = 4 \bmod 8$ (pseudoreal).

Dimensional reduction of Majorana-Weyl spinors

Finally, there is the interesting case of a Majorana-Weyl spinor for $d = 2 \bmod 8$. We first discuss the massive spinors. Applying the Weyl constraint for $D = 6, 7, 8 \bmod 8$ ($D = 2, 3, 4 \bmod 8$) to real (pseudoreal) representations gives massive Dirac spinors, which are further reduced to massive Majorana spinors by using the Majorana constraint. For all other cases the Weyl constraint leaves us with two massive Dirac spinors which are related by the D dimensional charge conjugation operator. The Majorana constraint combines them into one Dirac spinor. Thus the degeneracy of the mass spectrum is completely absorbed by the Majorana and Weyl constraint. This is also true for the massless case:

Applying the Majorana constraint to a Weyl spinor for $D = 2 \bmod 4$ reduces two Weyl spinors which are charge conjugate to each other to one Weyl spinor. This can be done for complex, real or pseudoreal representations. Chiral $E = 4 \bmod 4$ dimensional fermions may be obtained from dimensional reduction of a Majorana-Weyl spinor. For $D = 4 \bmod 4$ and complex representations, the two Weyl spinors transforming as C and $\bar{C}$ are reduced to one Weyl spinor. Again, we may have chiral fermions. For $D = 4 (8) \bmod 8$, the pseudoreal (real) Weyl spinors are reduced to Majorana-Weyl spinors whereas two real (pseudoreal) Weyl spinors are reduced to one Weyl spinor. For D and E odd and C complex, the Majorana constraint reduces two Dirac spinors ϕ_C and $\phi_{\bar{C}}$ to one Dirac spinor. For $D = 3, 5 \bmod 8$ the pseudoreal Dirac spinors are reduced to Majorana spinors and two real Dirac spinors are reduced to one Dirac spinor. For $D = 7, 9 \bmod 8$, the role of real and pseudoreal representations is inversed compared with $D = 3, 5 \bmod 8$. In Table 4, we list the reduction of massless spinors by Majorana and Weyl constraints for $d = 2 \bmod 8$. For the interesting case of chiral four dimensional fermions the Majorana and Weyl constraints read explicitly, in a basis where $\bar{1}_E = \text{diag} (1, -1)$:

$$\begin{aligned}\bar{\Gamma}_E \varphi_{C+} &= \varphi_{C+} = \begin{pmatrix} \chi_{C+} \\ 0 \end{pmatrix} \\ \bar{\Gamma}_E \varphi_{\bar{C}-} &= -\varphi_{\bar{C}-} = -\begin{pmatrix} 0 \\ \chi_{\bar{C}-} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \chi_{\bar{C}-} \end{pmatrix} &= B_E^{-1} \begin{pmatrix} \chi_{C+}^* \\ 0 \end{pmatrix}\end{aligned}\tag{105}$$

The last constraint eliminates $\chi_{\bar{C}-}$ as an independent spinor and we are left with the chiral spinor ϕ_{C+} .

Conclusion

To conclude, we concentrate on $E = 4$. Even if we perform dimensional reduction by steps, we are finally interested in four-dimensional fermions. The possible massless fermions for this case are listed in Table 5. There are only two possibilities where chiral four-dimensional fermions may be obtained from dimensional reduction: we have to start with a Weyl spinor for $d = 6 \bmod 8$ or with a Majorana-Weyl spinor for $d = 2 \bmod 8$. For $d = 6 \bmod 8$, however, we have an unacceptable degeneracy of the mass spectrum. For real or complex representations we have two four-dimensional Dirac spinors for every eigenvalue $|M| \neq 0$. Similarly, we are left with two Weyl spinors for every representation H . Such a degeneracy is not observed in nature. In the context of our discussion, where the gauge group G is a subgroup of gen_D transformations, we necessarily have to start with a Majorana-Weyl spinor in $d = 2 \bmod 8$ dimensions!

It has been shown⁹⁾ that $D = 7$ is the minimal number of dimensions to incorporate a $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group. This leads us to $d = 18$ as the lowest dimension where one can hope to construct a realistic model out of pure d -dimensional gravity coupled to a spinorial matter field. It is interesting to note that the number of components of the 18 dimensional Majorana-Weyl spinor corresponds to the degrees of freedom needed for four generations of $SO(10)$ spinors. Is it possible to find a fourteen dimensional compact space admitting a gauge group $SO(10)$ [or its $SU(5)$ subgroup] and zero eigenmodes of the Dirac operator? Could the existence of several generations be naturally explained in such a context, may be even with a generation gauge group whose spontaneous symmetry breakdown could help to understand the generation mass pattern of observed fermions?

REFERENCES

- 1) T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin, Math. Phys. K1 (1921) 966.
O. Klein, Zeit. f. Physik 37 (1926) 895.
Y.M. Cho and P.G.O. Freund, Phys. Rev. D 12 (1975) 1711.
J.F. Luciani, Nucl. Phys. B135 (1978) 111.
J. Scherk and J.H. Schwarz, Nucl. Phys. B153 (1979) 61.
- 2) C. Wetterich, Phys. Lett. 110B (1982) 379.
- 3) C. Wetterich, Phys. Lett. 113B (1982) 377.
- 4) N. Hitchin, Adv. Math. 14 (1974) 1.
- 5) B. Biran, F. Englert, H. Nicolai and B. de Wit, CERN preprint TH-3489 (1983).
- 6) N.S. Manton, Nucl. Phys. B193 (1981) 391.
G. Chapline and R. Slansky, Los Alamos preprint (1982).
- 7) C. Wetterich, Nucl. Phys. B211 (1983) 177 (to appear).
- 8) A. Salam and J. Strathdee, Ann. Phys. 141 (1982) 316.
- 9) E. Witten, Nucl. Phys. B186 (1981) 412.

Table 1

Minimal number of independent eigenmodes of the D dimensional Dirac equation $\Gamma^\alpha D_\alpha \psi = M\psi$ with the same eigenvalue $|M|$. It is given in dependence of D for complex (C), real (R) and pseudoreal (P) representations for both cases $|M| \neq 0$ and $M = 0$.

	$ M \neq 0$			$M = 0$		
	C	R	P	C	R	P
$D = 2 \bmod 8$	4	4	2	2	2	2
$D = 3 \bmod 8$	2	2	1	2	2	1
$D = 4 \bmod 8$	4	4	2	2	2	1
$D = 5 \bmod 8$	2	2	2	2	2	1
$D = 6 \bmod 8$	4	2	4	2	2	2
$D = 7 \bmod 8$	2	1	2	2	1	2
$D = 8 \bmod 8$	4	2	4	2	1	2
$D = 9 \bmod 8$	2	2	2	2	1	2

Table 2

Possible massive and massless spinors for complex (C), real (R) and pseudo-real (P) representations. D: Dirac spinor, W: Weyl spinor, M: Majorana spinor, MW: Majorana-Weyl spinor.

E (mod 8)	$ M \neq 0$			M = 0		
	C	R	P	C	R	P
2	D	M	D	W	MW	W
3	D	M	D	D	M	D
4	D	M	D	W	W	W
5	D	D	D	D	D	M
6	D	D	M	W	W	MW
7	D	D	M	D	D	M
8	D	D	M	W	W	W
9	D	D	D	D	M	D

Table 3

Possible choice for B_{iE} , B_{iD} , B_{id} (in this order in the table) for arbitrary E and D (both mod 8). We also indicate the sign of $\epsilon = B^*B = \pm 1$.

	D = 2	D = 3	D = 4	D = 5	D = 6	D = 7	D = 8	D = 9
E = 2	$B_1^+ B_1^+ B_1^+$	$B_2^+ B_2^- B_2^-$	$B_1^+ B_1^- B_1^-$	$B_1^+ B_1^- B_1^-$	$B_2^+ B_2^+ B_2^+$	$B_2^+ B_2^+ B_2^+$	$B_2^+ B_2^+ B_2^+$	$B_1^+ B_1^+ B_1^+$
E = 3	$B_1^+ B_2^- B_2^-$	$B_1^{+(*)} B_2^- B_1^-$	$B_1^+ B_1^- B_1^-$	$B_2^{-(*)} B_1^- B_2^+$	$B_1^+ B_2^+ B_2^+$	$B_1^{+(*)} B_2^+ B_1^+$	$B_1^+ B_1^+ B_1^+$	$B_2^{-(*)} B_1^+ B_2^-$
E = 4	$B_2^- B_1^+ B_2^-$	$B_1^+ B_2^- B_1^-$	$B_2^- B_1^- B_2^+$	$B_2^- B_1^- B_2^+$	$B_1^+ B_2^+ B_1^+$	$B_1^+ B_2^+ B_1^+$	$B_1^+ B_2^+ B_1^+$	$B_2^- B_1^+ B_2^-$
E = 5	$B_2^- B_1^+ B_1^-$	$B_2^{-(*)} B_2^- B_2^+$	$B_2^- B_2^- B_2^+$	$B_1^{-(*)} B_1^- B_1^+$	$B_2^- B_1^- B_1^+$	$B_2^{-(*)} B_2^+ B_2^-$	$B_2^- B_2^+ B_2^-$	$B_1^{-(*)} B_1^+ B_1^-$
E = 6	$B_2^- B_2^- B_2^+$	$B_2^- B_2^- B_2^+$	$B_2^- B_2^- B_2^+$	$B_1^- B_1^- B_1^+$	$B_1^- B_1^- B_1^+$	$B_2^- B_2^+ B_2^-$	$B_1^- B_1^+ B_1^-$	$B_1^- B_1^+ B_1^-$
E = 7	$B_1^- B_2^- B_2^+$	$B_1^{-(*)} B_2^- B_1^+$	$B_1^- B_1^- B_1^+$	$B_2^{+(*)} B_1^- B_2^-$	$B_1^- B_2^+ B_2^-$	$B_1^{-(*)} B_2^+ B_1^-$	$B_1^- B_1^+ B_1^-$	$B_2^{+(*)} B_1^+ B_2^+$
E = 8	$B_2^+ B_1^+ B_2^+$	$B_1^- B_2^- B_1^+$	$B_1^- B_2^- B_1^+$	$B_2^+ B_1^- B_2^-$	$B_1^- B_2^+ B_1^-$	$B_1^- B_2^+ B_1^-$	$B_2^+ B_1^+ B_2^+$	$B_2^+ B_1^+ B_2^+$
E = 9	$B_2^+ B_1^+ B_1^+$	$B_2^{+(*)} B_2^- B_2^-$	$B_2^+ B_2^- B_2^-$	$B_1^{+(*)} B_1^- B_1^-$	$B_2^+ B_1^- B_1^-$	$B_2^{+(*)} B_2^+ B_2^+$	$B_2^+ B_2^+ B_2^+$	$B_1^{+(*)} B_1^+ B_1^+$

(*) The B matrix corresponds to the E + 1 dimensional B matrix.

Table 4

Weyl and Majorana constraints for $d = 2 \bmod 8$ for massless complex (C), real (R) or pseudo-real (P) spinors. D: Dirac spinor, M: Majorana spinor, W: Weyl spinor, MW: Majorana-Weyl spinor. The first symbol gives the result of the Weyl constraint, the second for the Majorana constraint and the third for both constraints combined.

	C	R	P
$D = 2 \bmod 8$	2W, D, W	2W, D, W	2W, D, W
$D = 3 \bmod 8$	2D, 2D, D	2D, 2D, D	D, D, M
$D = 4 \bmod 8$	2W, D, W	2W, D, W	W, M, MW
$D = 5 \bmod 8$	2D, 2D, D	2D, 2D, D	D, D, M
$D = 6 \bmod 8$	2W, D, W	2W, D, W	2W, D, W
$D = 7 \bmod 8$	2D, 2D, D	D, D, M	2D, 2D, D
$D = 8 \bmod 8$	2W, D, W	W, M, MW	2W, D, W
$D = 9 \bmod 8$	2D, 2D, D	D, D, M	2D, 2D, D

Table 5

Possible massless spinors from dimensional reduction to four dimensions. D: Dirac spinor, W: Weyl spinor, M: Majorana spinor, MW: Majorana-Weyl spinor. We give results for complex (C), real (R) and pseudoreal (P) representations.

d (mod 8)	Type of spinor	C	R	P
2	MW	W	W	W
3	M	D	M	D
4	W	D	W	D
5	D	2D	D	2D
6	W	2W	D	D
7	D	2D	2D	D
8	W	D	D	W
9	M	D	D	M