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ON THE CONSTRUCTION OF AXIALLY SYMMETRIC SU(N) MONOPOLES

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A B S T R A C T

For the axially symmetric SU(N) monopoles we show the existence of a superpotential determining their most relevant properties. We derive a systematic framework to generate these monopoles by applying a generalized inverse scattering method to their superpotential. To illustrate the method we construct in detail some SU(2) and SU(3) multimonopoles. In the SU(3) case we find a new six-parameter family of monopoles with $U(1) \times U(1)$ as an unbroken symmetry group and $n_1 = n_2 = 2$ as topological charges.

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1. - INTRODUCTION

Over the past two years there has been significant progress in the understanding of monopole solutions in the Bogomolny-Prasad-Sommerfield limit of spontaneously broken gauge theories. Soon after the first multimonopole solution - the axially symmetric two-monopole in $SU(2)$ - was constructed by two different methods [by the Atiyah-Ward (AW) ansätze¹⁾ and by Bäcklund transformations²⁾], it was successfully generalized by both methods³⁾ to monopoles with higher charges and with the same symmetry property. As axial symmetry in the $SU(2)$ case is rather restrictive⁴⁾, all these monopoles were located at a single point. Almost simultaneously with these developments, the separated two⁵⁾ and n monopole solutions⁶⁾ depending on the right number of parameters were also produced. Thus it seems that the most general static multimonopoles in $SU(2)$ are known; the main remaining difficulty lies in proving that the solutions are sufficiently smooth. An alternative method⁷⁾ - using an appropriate modification of the ADHM construction - overcomes this difficulty; however, its relation to the other methods is not yet fully understood.

Obviously the next step is to look for monopoles in more general groups, such as $SU(3)$. Both the AW ansätze⁸⁾⁻¹¹⁾ and the Bäcklund transformations¹²⁾ have been generalized for more complicated groups, however, explicit solutions have been derived so far only with the AW ansätze. The regular solutions with axial symmetry [which differ from the simple $SU(2)$ embeddings] known so far are the two six-parameter families of $SU(3)$ monopoles with unbroken groups $SU(2) \times U(1)$ ⁹⁾ and $U(1) \times U(1)$ ¹⁰⁾, respectively, as well as the deformed embedding of the single $SU(2)$ monopole into $SU(3)$ ¹¹⁾.

The aim of the present paper is to provide an alternative framework for constructing axially symmetric monopoles in $SU(N)$ gauge theories. Our method relies on two basic ingredients: on the existence of a superpotential for these monopoles, and on a successful application to this superpotential of our inverse scattering method worked out for the self-duality equations¹³⁾. The superpotential we define reduces for $N = 2$ to the one known in this case for some time¹⁴⁾, and even in the general case enables us to determine the most important physical quantities of the monopoles (e.g., length of the Higgs field, topological changes). Using the inverse scattering method, we can construct, by iteration, infinitely many new superpotentials - depending on certain free parameters - from a starting solution. The resulting expression is entirely explicit once the starting solution is specified. Thus, our idea for obtaining axially symmetric monopoles in $SU(N)$ is to choose an appropriate starting solution with a constant Higgs field and

vanishing energy density and topological changes (we call it a Higgs vacuum), and to fix the free parameters of the general expression in a way demanded by the topological charges and the regularity.

Having outlined the method in general, we use it to construct some SU(2) and SU(3) monopoles. In the SU(3) case, we find a new six-parameter family of monopoles with U(1)×U(1) as an unbroken group which reduces in an interesting way to that found by Ward⁹⁾ with unbroken SU(2)×U(1).

The paper is organized as follows. In Section 2 we derive the superpotential and discuss some of its properties. Section 3 contains a brief review of the inverse scattering method and a detailed application of it to the superpotential. The SU(2) monopoles are constructed in Section 4. In the three subsections of Section 5, we show how the method is applied to generate the SU(2) embeddings into SU(3), the spherically symmetric and the axially symmetric SU(3) monopoles, respectively. We make our conclusions in Section 6. Some of the technicalities we need in Section 3 are collected in Appendices A and B.

2. - THE SUPERPOTENTIAL FOR AXIALLY SYMMETRIC SU(N) MONOPOLES

We consider an SU(N) gauge theory coupled to a Higgs field in the adjoint representation in the limit of vanishing Higgs potential. The static field equations of this theory in the absence of electric fields ($A_i^a = 0$) are solved by configurations satisfying the Bogomolny equations¹⁵⁾:

$$F_{ij} = -\epsilon_{ijk} D_k \Phi \quad (1)$$

where

$$F_{ij} = F_{ij}^a \frac{t_a}{2} = \partial_i A_j - \partial_j A_i + i [A_i, A_j]$$

$$D_k \Phi = \partial_k \Phi + i [A_k, \Phi]; \quad \Phi = \Phi^a \frac{t_a}{2}, \quad A_i = A_i^a \frac{t_a}{2}$$

and t_a are the SU(N) generators. (We choose the coupling constant $g = 1$.)

The symmetry breaking is imposed on the solutions of (1) as a boundary condition at infinity ($r \rightarrow \infty$): $\sum_a \phi^{a^2} = 2\text{Tr}\phi^2 = p_1$. Furthermore, to obtain monopole solutions, we demand that

$$2 T r \phi^2 = p_1 - \frac{p_2}{r} + O(1/r^2) \quad r \rightarrow \infty \quad (2)$$

as $p_2 \neq 0$ guarantees non-zero topological charge(s).

For a (ϕ, A_i) solution of (1), (2), the energy

$$E = \int d^3x T r \left(\frac{1}{2} F_{ij}^2 + (D_i \phi)^2 \right) = 2 \int d^3x T r (D_i \phi)^2 = \int d^3x \Delta T r \phi^2 \quad (3)$$

is automatically finite, provided that $Tr \phi^2$ is free of singularities.

It is well known that the Bogomolny equations (1) are equivalent to the self-duality equations of a pure $SU(N)$ gauge theory in Euclidean space

$$F_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (4)$$

if we assume that all fields A_μ are independent of the Euclidean time x_4 , and reinterpret A_4 as the Higgs field ϕ .

As it was shown in Ref. 16), the self-duality equations (4) can be written in a convenient form

$$(g_{,y} g^{-1})_{,\bar{y}} + (g_{,z} g^{-1})_{,\bar{z}} = 0 \quad ; \quad g = \bar{D} D; \quad \bar{D}, D \in SL(N, \mathbb{C}) \quad (5)$$

where $y = x_1 + ix_2$, $\bar{y} = x_1 - ix_2$, $z = x_3 + ix_4$, $\bar{z} = x_3 - ix_4$ are complex co-ordinates and the vector components $A_y = \frac{1}{2}(A_1 - iA_2)$, etc., are expressed in terms of D ($\bar{D}$) as :

$$\begin{aligned} A_y &= -i D_{,y} D^{-1} & A_{\bar{y}} &= i \bar{D}^{-1} \bar{D}_{,\bar{y}} \\ A_z &= -i D_{,z} D^{-1} & A_{\bar{z}} &= i \bar{D}^{-1} \bar{D}_{,\bar{z}} \end{aligned} \quad (6)$$

In general, to have $SU(N)$ solutions, one must require $\bar{D} = D^+$ in which case g becomes Hermitian. Therefore, the Bogomolny equations (1) are equivalent to (5) for a Hermitian g that does not depend on x_4 . For monopoles, we still have to give the correct boundary conditions in terms of g .

A useful parametrization of g - that makes it possible to determine A_μ (in Yang's R gauge) from the elements of g - was given in Ref. 17). In this paper we will not use this parametrization as the most interesting quantity ; the length of the Higgs field $2\text{Tr}\phi^2 = 2\text{Tr}A_4^2$ is easily expressed in terms of g as :

$$2\text{Tr}\phi^2 = \frac{1}{2} \text{Tr}(g_{,x_3} g^{-1})^2 \quad (7)$$

when both D and g are independent of x_4 . This expression makes it possible to formulate the boundary condition (2) and the regularity requirement of the $SU(N)$ monopole problem for g .

From now on we will consider axially symmetric monopoles only. Axial symmetry is imposed in the previous formalism by assuming $D = D(\rho, x_3)$, $g = g(\rho, x_3)$ ($x_1 + ix_2 = \rho e^{i\varphi}$) which in turn simplifies (5) to

$$(g_{,s} g^{-1})_{,s} + (g_{,x_3} g^{-1})_{,x_3} = 0 \quad (8)$$

It is straightforward to prove that this equation makes it possible to define a function τ in the following way :

$$\begin{aligned} \tau_{,s} &= \frac{g}{4} \text{Tr}[(g_{,s} g^{-1})^2 - (g_{,x_3} g^{-1})^2] \\ \tau_{,x_3} &= \frac{g}{2} \text{Tr}[(g_{,s} g^{-1})(g_{,x_3} g^{-1})] \end{aligned} \quad (9)$$

It is also easy to see that τ has the interesting property :

$$\Delta\tau = -\frac{1}{2} \text{Tr}(g_{,x_3} g^{-1})^2 ; \quad \Delta = \partial_s^2 + \partial_{x_3}^2 + \frac{1}{g} \partial_s \quad (10)$$

In what follows we shall call τ the superpotential. To justify this name, we turn first to the axial (and mirror) symmetric Manton ansatz in $SU(2)$ ¹⁴⁾ ($N = 2$), and show that in this case τ as given by (9) is identical to the superpotential first introduced in Ref. 14) and later studied in Ref. 18).

The Manton ansatz is given in terms of six real functions $(\varphi_i, \eta_i, W_i, i = 1, 2)$ depending only on ρ and x_3 as :

$$\begin{aligned}\Phi &= \frac{1}{2} \begin{pmatrix} \varphi_2 & -i\varphi_1 \\ i\varphi_1 & -\varphi_2 \end{pmatrix} & A_\varphi &= \frac{\epsilon_{ij} x_i A_j}{g} = -\frac{1}{2} \begin{pmatrix} \eta_2 & -i\eta_1 \\ i\eta_1 & -\eta_2 \end{pmatrix} \\ A_{x_3} &= -\frac{1}{2} \begin{pmatrix} 0 & W_1 \\ W_1 & 0 \end{pmatrix} & A_g &= \frac{x_i A_i}{g} = -\frac{1}{2} \begin{pmatrix} 0 & W_2 \\ W_2 & 0 \end{pmatrix}\end{aligned}\quad (11)$$

When we proved the equivalence between the Bogomolny equations for this ansatz and the Ernst equation of general relativity, we pointed out ³⁾ the existence of a gauge within the ansatz (11), where all φ_i , η_i and W_i are expressed in terms of two real functions f and ψ :

$$\begin{aligned}W_1 &= -\varphi_1 = -\frac{\psi_{,x_3}}{f} & ; & & W_2 &= \frac{1}{g} \eta_1 = -\frac{\psi_{,g}}{f} & ; \\ \varphi_2 &= -\frac{f_{,x_3}}{f} & ; & & \frac{1}{g} \eta_2 &= \frac{f_{,g}}{f}\end{aligned}\quad (12)$$

In this case g has the form

$$g = \frac{1}{f} \begin{pmatrix} 1 & \psi \\ \psi & \psi^2 + f^2 \end{pmatrix}\quad (13)$$

and (8) reduces to the Ernst equation for f and ψ :

$$f \Delta f - (\nabla f)^2 + (\nabla \psi)^2 = 0 \quad ; \quad f \Delta \psi - 2 \nabla f \nabla \psi = 0\quad (14)$$

Now, using (13) and (12) in (9), we obtain:

$$\begin{aligned}\tau_{,g} &= \frac{g}{2f^2} \left(f_{,g}^2 + \psi_{,g}^2 - [f_{,x_3}^2 + \psi_{,x_3}^2] \right) = \frac{1}{2} \left(\left(\frac{\eta}{g} \right)^2 - g(\varphi)^2 \right) \\ \tau_{,x_3} &= \frac{g}{f^2} \left(f_{,g} f_{,x_3} + \psi_{,g} \psi_{,x_3} \right) = -(\varphi \eta)\end{aligned}\quad (15)$$

and indeed these equations are the same as the original ones introduced in (14). Finally, we mention two interesting properties of the $SU(2)$ superpotential that follow from (15), (12) and (10). The first is that all quantities invariant under the residual $U(1)$ gauge group of the ansatz (11); $(\phi)^2, (\eta)^2, (\phi\eta)$; can be determined from τ alone. The second property of τ is that it satisfies a very complicated, highly non-linear differential equation of fourth order (extensively studied in general relativity) obtained from (14) by eliminating f, ψ in favour of τ .

For $N > 2$, the superpotential as given by (9) will not, in general, determine all the quantities invariant under the residual, unbroken part of the gauge group; however, because of (10) and (7) it will still determine the length of the Higgs field. Thus, to describe monopoles, $-\Delta\tau$ must be free of singularities and its asymptotic behaviour must be consistent with (2).

In the next section - using an inverse scattering method - we show that one can construct a new τ with the aid of an initial g_0 solution of (8) and the associated τ_0 in the form :

$$\tau = \tau_0 + T \quad (16)$$

As this transformation can be iterated, it offers the possibility of generating infinitely many new solutions from an initial one. Therefore our strategy to generate $SU(N)$ multimonopoles is quite similar to what we followed in our earlier work using Bäcklund-transformations to construct the $SU(2)$ monopoles^{2),3)}; we start with the "Higgs vacuum" as initial state (g_0, τ_0) and iterate the transformation (16) on this solution.

The Higgs vacuum is a solution of (1) which has constant $\phi = \phi_0$ and vanishing A_i over the whole R^3 . Without loss of generality, we can assume that

$$\phi_0 = \text{diag}(\lambda_1, \dots, \lambda_N); \quad \sum_{i=1}^N \lambda_i = 0; \quad 2 \sum_{i=1}^N \lambda_i^2 = p_1 \quad (17)$$

The simplest g_0 for this solution is obtained by assuming that both D_0 and g_0 are diagonal and $D_0^+ = D_0$:

$$g_0 = \text{diag} \left(e^{-2\lambda_1 x_3}, \dots, e^{-2\lambda_N x_3} \right) \quad (18)$$

while the corresponding τ_0 solution of (9) is given as

$$\tau_0 = -\frac{g^2}{2} \sum_{i=1}^N \lambda_i^2 + c_0 \quad (19)$$

This is the g, τ_0 pair we shall use later. We mention here that in the applications we will sometimes change the sign of some of the terms in (18). Obviously, this change affects only the determinant of g_0 but not its hermiticity or the fact that it solves (8). It is also clear that this change does not modify the Higgs vacuum (17), it merely means that (17) is constructed with a $\bar{D}$ which is different from D^+ : $\bar{D} = ND^+$ where N is a suitable constant diagonal matrix.

Combining (16), (10) and (7) we see that the most natural assumption one can make is that the constant term in the asymptotics of $2\text{Tr}\phi^2$, (2), for the new solution is determined by τ_0 while the $1/r$ term is given by the asymptotic part of $-\Delta T$ in (16). As the $1/r$ term determines the topological charges, we discuss now what is known in general about these quantities.

When in (17) all the λ_i 's are different - the so-called "maximal breaking" - then the remaining gauge group is the direct product of $N-1$ $U(1)$'s. In this case it is known that there are just $N-1$ topological charges¹⁹⁾. If some of the λ_i 's coincide then the unbroken group contains non-Abelian parts as well. In this case the number of topological charges is less than $N-1$, as some of the charges of the previous case become magnetic weights²⁰⁾ in the limit of coinciding λ_i 's.

Let us explain all these in more detail for $SU(3)$, as our investigation will focus on this case. One can always choose a gauge so that asymptotically on the x_3 axis

$$\phi = \phi_0 - \frac{1}{2x_3} \text{diag}(n_1, n_2 - n_1, -n_2) + \mathcal{O}(1/z^2) \quad (20)$$

Topological arguments then guarantee that the numbers n_1 and n_2 are always integers¹¹⁾. If all the λ_i 's are different, then n_1 and n_2 are just the two topological charges. On the other hand, if $\lambda_1 = \lambda_2$ say, [when the unbroken group is $SU(2) \times U(1)$] then there is only one topological charge, n_2 ; as $\lambda_1 = \lambda_2$ means that we cannot distinguish in a gauge invariant way between n_1 and $n_2 - n_1$; and we say that n_1 is a magnetic weight. Now substituting (20) into (2) we obtain that the coefficient of the $1/r$ term of the length of the Higgs field is :

$$P_2 = 2 (\lambda_1 [n_1 + n_2] + \lambda_2 [2n_2 - n_1]) \quad (21)$$

[Here we used the fact that $\lambda_3 = -(\lambda_1 + \lambda_2)$.] Therefore we see that our new τ , (16), can describe monopoles with topological charges n_1, n_2 if asymptotically $-\Delta T$ falls off as $1/r$ with a coefficient given by (21).

3. - INVERSE SCATTERING METHOD FOR THE SUPERPOTENTIAL

In this section, we derive an expression yielding a new τ directly (i.e., without the explicit construction of the new g) in terms of a starting solution (g_0, τ_0) of (8) and (9). As the derivation is based on our inverse scattering method worked out to solve the full self-duality equations (5) ¹³⁾, we first summarize in a nutshell the essential points of this method as applied to the axially symmetric Bogomolny equations.

3.1 - Generalized inverse scattering method

The key to our solution-generating method for Eq. (5) is the following linear system :

$$\begin{aligned} (\hat{\lambda} \partial_{\bar{z}} + \partial_y) \psi &= g_{,y} g^{-1} \psi \\ (-\hat{\lambda} \partial_{\bar{y}} + \partial_z) \psi &= g_{,z} g^{-1} \psi \end{aligned} \quad (22)$$

where $\hat{\lambda}$ is an arbitrary complex parameter and ψ is an $N \times N$ matrix depending on $y, \bar{y}, z, \bar{z}$ and $\hat{\lambda}$. [We also demand that $\psi(\hat{\lambda}=0) = g$.] As the compatibility conditions of the equations (22) are just (5), we may say that (22) represents a linearization of (5).

Next we reduce the system (22) as applied to the axially symmetric Bogomolny equations. As we mentioned earlier in this case, g is independent of x_4 and φ and depends only on ρ and x_4 (where $x_1 + ix_2 = \rho e^{i\varphi}$). Therefore we require ψ to be independent of x_4 ; however, with the axial symmetry there is a subtlety, as ψ depends on φ as well, in the following way :

$$\psi(\hat{\lambda}, s, \varphi, x_3) = \psi(\hat{\lambda} e^{i\varphi}, s, x_3)$$

Now, introducing $\tilde{\lambda} = \hat{\lambda} e^{i\varphi}$ from (22) we obtain a linear eigenvalue problem for $\psi(\tilde{\lambda}, \rho, x_3)$ which, after changing $\tilde{\lambda} = -\lambda / \rho$, can be written for $\psi(\lambda, \rho, x_3)$ as

$$D_1 \psi = \left(\partial_{x_3} - \frac{2\lambda^2}{\lambda^2 + g^2} \partial_\lambda \right) \psi = \frac{s(s g_{,x_3} g^{-1} - \lambda g_{,s} g^{-1})}{\lambda^2 + g^2} \psi = \frac{s(V - \lambda U)}{\lambda^2 + g^2} \psi$$

$$D_2 \psi = \left(\partial_s + \frac{2\lambda g}{\lambda^2 + g^2} \partial_\lambda \right) \psi = \frac{s(s g_{,s} g^{-1} + \lambda g_{,x_3} g^{-1})}{\lambda^2 + g^2} \psi = \frac{s(U + \lambda V)}{\lambda^2 + g^2} \psi \quad (23)$$

When g is a real, symmetric 2×2 matrix then (23) is identical to the linear problem set up by Belinski and Zakharov for the Ernst equation ²¹⁾. For general g , to obtain new solutions of (23) [and thus also for (8)] we use an appropriate generalization of the method given in Ref. 22).

We assume that we know a $\psi_0(\lambda)$ solution of (23) for our initial g_0 with the correct boundary conditions $\psi_0(\lambda=0) = g_0$. Then we look for new solutions in the form $\psi(\lambda) = \chi(\lambda) \psi_0(\lambda)$, which in turn implies that the new solution of (8) is given as

$$g = \chi(0) g_0 \quad (24)$$

For $\chi(\lambda)$ we obtain from (23) :

$$D_1 \chi = \frac{s(V - \lambda U)}{\lambda^2 + g^2} \chi - \chi \frac{s(V_0 - \lambda U_0)}{\lambda^2 + g^2}$$

$$D_2 \chi = \frac{s(U + \lambda V)}{\lambda^2 + g^2} \chi - \chi \frac{s(U_0 + \lambda V_0)}{\lambda^2 + g^2} \quad (25)$$

The hermiticity of g and g_0 imposes a very important restriction on the analytic properties of χ in the complex λ plane ; $g \chi^{*-1} (-\rho^2 / \bar{\lambda}) g_0^{-1}$ satisfies the same equation as χ , thus

$$g \chi^{*-1} (-g^2 / \bar{\lambda}) g_0^{-1} = \chi(\lambda) \quad (26)$$

From (26), it immediately follows that $\chi(\infty) = I$.

As we explained elsewhere ²³⁾, the so-called "pure soliton" solutions of (25) correspond to assuming only simple poles in $\chi(\lambda)$ on the λ plane :

$$\chi(\lambda) = I + \sum_{k=1}^n \frac{R_k}{\lambda - \mu_k} \quad (27)$$

Here the residues R_k and the poles μ_k are independent of λ , and are obtained by substituting (27) into (25) and (26) and comparing the various residues of the two sides. In this way we find that μ_k satisfies

$$\mu_{k,x_3} + 2\mu_k^2 (\mu_k^2 + g^2)^{-1} = 0 ; \quad \mu_{k,g} - 2g\mu_k (\mu_k^2 + g^2)^{-1} = 0 \quad (28a)$$

whose solution is

$$\mu_k = w_k - x_3 + \sqrt{(w_k - x_3)^2 + g^2} \quad (28b)$$

with w_k being arbitrary constants. [In general, some of the poles μ_m may coincide with any of $\bar{\mu}_k$, $-\rho^2/\mu_k$, $-\rho^2/\bar{\mu}_k$ ($k \neq m$) as they also solve (28).] Furthermore, for each pole μ_k one associates an $s_k \leq N-1$ dimensional subspace and the R_k residues project on these subspaces :

$$(R_k)_{ab} = \sum_{j_k=1}^{s_k} n_a^{(k,j_k)} m_b^{(k,j_k)} \quad (29)$$

where

$$m_b^{(k,j_k)} = M_c^{(k,j_k)} \psi_0^{-1}(\mu_k)_{cb} \quad (30)$$

and $M^{(k,j_k)}$ form s_k linearly independent constant vectors. After expressing the $n_a^{(k,j_k)}$ vectors [using (26)] in terms of $m_b^{(k,j_k)}$, μ_k and g_0 , from (24) and (27) we obtain a rather complicated formula for the new g :

$$g_{ab} = (g_0)_{ab} - \sum_{i,j,t_i,t_j} (\mu_i \bar{\mu}_j)^{-1} (\Gamma^{-1})_{it_i, jt_j} (g_0)_{ac} \bar{m}_c^{(j,t_j)} m_d^{(i,t_i)} (g_0)_{db} \quad (31)$$

where

$$\Gamma_{it_i, jt_j} = \frac{m_a^{(i, t_i)} (g_0)_{ab} \bar{m}_b^{(j, t_j)}}{s^2 + \mu_i \bar{\mu}_j} \quad \begin{array}{l} a, b = 1, \dots, N \\ i, j = 1, \dots, n \\ t_i = 1, \dots, s_i \end{array}$$

The way we constructed this g automatically guarantees that it solves (8) and is Hermitian ; however, for our purposes we must also ensure that its determinant is one. To calculate this determinant, we use a different representation of g , based on the observation ²²⁾ that putting n poles at once in $\chi(\lambda)$, (27), is formally equivalent to putting the poles successively one by one. In this latter case, the first step is to go from the initial solution g_0 to g_1 obtained with the aid of a χ containing only one pole at $\lambda = \mu$ (this χ we call χ_1). When there is only one pole, the procedure we just outlined yields the following χ_1 : (see Appendix A)

$$\chi_1(\lambda) = I + \frac{\mu \bar{\mu} + s^2}{\bar{\mu}(\lambda - \mu)} P \quad (32)$$

where P is an s ($\leq N-1$) dimensional projector :

$$P_{ab} = \sum_{i,j=1}^s (g_0)_{ac} \bar{m}_c^j (\gamma^{-1})^{ji} m_b^i$$

$$\gamma^{ij} = m_b^i (g_0)_{bc} \bar{m}_c^j \quad (33)$$

and the m_b^i vectors are constructed using $\psi_0(\lambda)$ and s linearly independent constant vectors, M_c^i as in (30) : $m_b^i = M_c^i \psi_0^{-1}(\mu)_{cb}$.

Now if we combine (32) and (24), we see that adding one pole with an associated projector of dimension s to g_0 changes the determinant of g_1 to

$$\det g_1 = \left(-\frac{s^2}{\mu \bar{\mu}} \right)^s \det g_0 \quad (34)$$

As (34) depends only on the pole, μ , but not on the details of g_1 , one can easily iterate it and we find that adding n poles μ_k ($k = 1, \dots, n$) with associated projectors of dimension s_k modifies the determinant of the final g , (31), as

$$\det g = \det g_0 \prod_{k=1}^n \left(- \frac{g^2}{\mu_k \bar{\mu}_k} \right)^{s_k} \quad (35)$$

Therefore, to obtain a physically acceptable solution, $g_{(ph)}$ with $\det g_{(ph)} = 1$, we still have to "renormalize" g as given by (31). The simplest way to do this is to multiply g by $(\det g)^{-1/N} I$ [note that choosing g_0 appropriately we can always make (35) positive] :

$$g_{(ph)} = g (\det g)^{-1/N} \quad (36)$$

as Eq. (8) guarantees that $\det g$ in (35) satisfies :

$$(s(\ln \det g)_{,s})_{,s} + s(\ln \det g)_{,x_3 x_3} = 0$$

However, for g 's having block diagonal forms, the normalization matrix need not be proportional to the unit matrix but can have a similar block diagonal form. To illustrate this, consider the $N = 3$ case assuming that g has the form

$$g = \begin{pmatrix} G & 0 \\ 0 & g_{33} \end{pmatrix} \quad (37)$$

where G is a 2×2 matrix. Then a possible physical solution, $g_{(ph)}$ is obtained in the form

$$g_{(ph)} = g M \quad ; \quad M = \begin{pmatrix} (\det g)^{-1/2} I_2 & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

We shall use this possibility to construct some $SU(3)$ monopoles in Section 5.

3.2 - Constructing the superpotential

Now we are equipped with everything we need to obtain the new τ without explicitly determining the new g in (9). We derive the appropriate expression for τ in two steps ; in the first one we get a τ satisfying (9) with the "unphysical" g [given by (31)] on the right-hand side, and then in the second step we use a simple trick to determine from this τ the physical one, $\tau_{(ph)}$, that satisfies (9) with $g_{(ph)}$ on the right-hand side.

To integrate Eq. (9) we rewrite it in a more convenient compact form

$$(\partial_s + i\partial_{x_3})\tau = \frac{1}{4s} \text{Tr} (U + iV)^2 \quad (39)$$

where $U = \rho g_{,\rho} g^{-1}$ and $V = \rho g_{,x_3} g^{-1}$ as in (23). Clearly, in order to carry out the integration explicitly we must determine $U + iV$ in terms of some known quantity. Surprisingly, this is possible : the equality of the residues at $\lambda = i\rho$ of the second equation in (25) gives $U + iV$ in terms of $U_0 + iV_0$ and $\chi(i\rho)$. Furthermore, as a first step, we calculate τ for the simple case when, on the right-hand side of (39) g is g_1 , obtained from g_0 by adding one pole at $\lambda = \mu$. [See (24) and (32).] In this case, substituting (32) into (25) at $\lambda = i\rho$ we find that

$$U_1 + iV_1 = 2i \frac{s}{\bar{\mu}} \frac{s^2 + \bar{\mu}\mu}{(s + i\mu)^2} P \chi_1^{-1}(is) + \chi_1(is) (U_0 + iV_0) \chi_1^{-1}(is) \quad (40)$$

where P is the projector given in (33) and $\chi_1(i\rho) = \chi_1(\lambda=i\rho)$ of (32).

Now, the fact that the m_b^i vectors in P are constructed with the aid of a $\psi_0(\lambda)$ solution of (23) implies certain equations for $(\partial_\rho + i\partial_{x_3})m_b^i$; using these and (28) we were able to integrate (39) with (40) on the right-hand side (see Appendix B), yielding

$$\tau_1 = \tau_0 + \ln \left\{ c \det \gamma \left(\frac{s \mu \bar{\mu}}{(s^2 + \mu \bar{\mu}) |s^2 + \mu^2|} \right)^s \right\} \quad (41a)$$

Here s is the dimension of the projector P and γ is the $s \times s$ matrix defined in (33), while c is an arbitrary constant.

The great virtue of (41a) is that it offers the possibility of iterations. Indeed we can take g_1 [as given by (24) and (32)] and τ_1 as the starting solution of (8) and (9) and add to g_1 one more pole at $\lambda = \mu_2$, with an associated projector of dimension s_2 to obtain a new g_2 . To do this we need the starting solution of (23) for g_1 ; it is given as $\psi_1(\lambda) = \chi_1(\lambda)\psi_0(\lambda)$. Repeating the whole procedure with these new ingredients we trivially get the τ_2 satisfying (9) with g_2 on the right-hand side as

$$\tau_2 = \tau_1 + \ln \left\{ c' \det \gamma_1 \left(\frac{s \mu_2 \bar{\mu}_2}{(s^2 + \mu_2 \bar{\mu}_2) |s^2 + \mu_2^2|} \right)^{s_2} \right\} \quad (41b)$$

Here τ_1 is given by (41a) and γ_1 is an $s_2 \times s_2$ matrix, the analogue of γ :

$$\gamma_1^{ij} = n_a^i (g_1)_{ab} \bar{n}_b^j ; \quad n_a^i = N_b^i \Psi_1^{-1}(\mu_2)_{ba} ; \quad i, j = 1, \dots, s_2 \quad (33b)$$

and N_b^i are s_2 linearly independent constant vectors. Obviously, if we repeat this construction several times, finally we obtain the τ that satisfies (9) with g of (31) on the right-hand side.

It is really remarkable that the addition of poles to g_0 via (27) in terms of the superpotential corresponds to a superposition law (41a) and (41b). This superposition property is similar to that of some solitons in 2D theories, thus it emphasizes the solitonic nature of the solutions we generate.

Even more remarkably - in two special cases - we were able to express the result of iterations in a single formula written directly in terms of the original g_0 and $\psi_0(\lambda)$. Indeed, when all the projectors are one-dimensional having calculated τ_2 in (41b) explicitly, by induction we could prove that after n iterations

$$\tau_n = \tau_0 + \ell_n \left\{ c \frac{s^n \prod_{k=1}^n (\mu_k \bar{\mu}_k)}{\prod_{k=1}^n |s^2 + \mu_k^2|} \det \Gamma \right\} \quad (42)$$

Here

$$\Gamma^{ij} = \frac{m_a^i (g_0)_{ab} \bar{m}_b^j}{s^2 + \mu_i \bar{\mu}_j}$$

[with $m_a^i = M_b^i \psi_0^{-1}(\mu_1)_{ba}$] is obtained from (31) by substituting $t_1 = 1$. The other case in which we were successful was when we added two poles with projectors of dimensions 2 and 1 respectively. This time we found that

$$\tau_2 = \tau_0 + \ell_n \left\{ c \frac{s^3 (\mu_1 \bar{\mu}_1)^2 (\mu_2 \bar{\mu}_2)}{|s^2 + \mu_1^2|^2 |s^2 + \mu_2^2|} \det \Gamma \right\} \quad (43a)$$

Here Γ^{ij} is again given by (31) in the following form :

$$\Gamma^{ij} = \frac{m_a^i (g_0)_{ab} \bar{m}_b^j}{s^2 + \mu_1 \bar{\mu}_1} \quad i, j = 1, 2 \quad \Gamma^{33} = \frac{n_a (g_0)_{ab} \bar{n}_b}{s^2 + \mu_2 \bar{\mu}_2} \quad (43b)$$

$$\Gamma^{is} = \overline{\Gamma}^{3i} = \frac{m_a^i (g_0)_{ab} \bar{n}_b}{s^2 + \mu_1 \bar{\mu}_2} \quad i = 1, 2 \quad (43b) \quad \text{Contd.}$$

with $m_a^i = M_b^i \psi_0^{-1}(\mu_1)_{ba}$, $n_a = N_b \psi_0^{-1}(\mu_2)_{ba}$ and M_b^i, N_b being constant (i.e., ρ, x_3 independent) vectors ($M_b^i, i = 1, 2$ must also be linearly independent).

Equations (42) and (43) suggest an obvious generalization for the more general cases, however, for our purposes (42) and (43) are sufficient therefore we do not consider the general formula for τ_n any longer.

We now turn to the determination of the "physical" superpotential, $\tau_{(ph)}$, that satisfies (9) and (39) with $g_{(ph)}$ on the right-hand side. The key point is to observe that replacing g by $g_{(ph)} = g^M$ [where M is either $(\det g)^{-1/N} I_N$ or the block diagonal matrix in Eq. (38)] on the right-hand side of (39) leads to additional terms only :

$$\begin{aligned} \frac{1}{4s} \text{Tr} (U_{(ph)} + i V_{(ph)})^2 &= \frac{1}{4s} \text{Tr} (U + i V)^2 + \frac{s}{4} \text{Tr} (M_{,s} \bar{M}^{-1} + i M_{,x_3} \bar{M}^{-1})^2 + \\ &+ \frac{1}{2} \text{Tr} (U_g M_{,s} \bar{M}^{-1} g^{-1} - V_g M_{,x_3} \bar{M}^{-1} g^{-1}) + \\ &+ \frac{i}{2} \text{Tr} (U_g M_{,x_3} \bar{M}^{-1} g^{-1} + V_g M_{,s} \bar{M}^{-1} g^{-1}) \end{aligned} \quad (44)$$

Therefore we can try to look for $\tau_{(ph)}$ in the form

$$\tau_{(ph)} = \tau + F \quad (45)$$

When $M = (\det g)^{-1/N} I_N$ from (45), (44) and (39) we get for F :

$$\begin{aligned} (\partial_s + i \partial_{x_3}) F &= - \frac{1}{N} \left\{ \frac{s}{4} \left[(\ln \det g)_{,s}^2 - (\ln \det g)_{,x_3}^2 \right] + \right. \\ &\left. + i \frac{s}{2} (\ln \det g)_{,s} (\ln \det g)_{,x_3} \right\} \end{aligned} \quad (46)$$

where $\det g$ is given in (35). [The block diagonal M of (38) will be discussed together with the applications in Section 5.1.]

The integration of (46) is based on the repeated application of (28a). Furthermore, in $\ln \det g$ we treat the coefficients of $\ln \mu_k$ and $\ln \bar{\mu}_k$ as independent ones

$$\begin{aligned} \ln \det g &= C + 2 \sum_k s_k \ln s - \sum_k s_k \ln \mu_k - \sum_k s_k \ln \bar{\mu}_k \rightarrow \\ &\rightarrow C + \sum_k (s_k + t_k) \ln s - \sum_k s_k \ln \mu_k - \sum_k t_k \ln \bar{\mu}_k \end{aligned} \quad (47)$$

to avoid apparent singularities when in the applications we consider such μ_1 's that are somehow related to each other (e.g., $\mu_2 = \bar{\mu}_2$, etc.). [Equation (47) is used in the following sense : when $\mu_k = \bar{\mu}_k = \mu'_k$, say, then $t_k = 0$ and $s_k = 2s'_k$; if $\mu_2 = \bar{\mu}_1$ and $s_1 = 2, s_2 = 1$ then we substitute $s'_1 = 1+2 = t'_1, s'_2 = 0 = t'_2$, etc.] Substituting (47) into (46) after a painful amount of algebra we get

$$F = \ln \left\{ \tilde{c} \frac{\prod_k \mu_k^{\frac{s_k}{N}(\frac{1}{2}Z - s_k)} \bar{\mu}_k^{\frac{t_k}{N}(\frac{1}{2}Z - t_k)} (\mu_k^2 + s^2)^{\frac{s_k^2}{2N}} (\bar{\mu}_k^2 + s^2)^{\frac{t_k^2}{2N}} s^{-\frac{\sum^2}{4N}}}{\prod_k |\mu_k - \bar{\mu}_k|^{\frac{s_k t_k}{N}} \prod_{k>l} (\mu_k - \mu_l)^{\frac{s_k s_l}{N}} (\bar{\mu}_k - \bar{\mu}_l)^{\frac{t_k t_l}{N}} (\mu_k - \bar{\mu}_l)^{\frac{s_k t_l}{N}} (\bar{\mu}_k - \mu_l)^{\frac{t_k s_l}{N}}} \right\} \quad (48)$$

where $\Sigma = \sum_l (s_l + t_l)$ and $\tilde{c}$ is a constant. To illustrate the usefulness of introducing an independent t_k we show what F looks like - as obtained from (48) - in the previously mentioned two examples :

a) n real poles $\mu_k = \bar{\mu}_k$ with dimensions s'_k

$$F = \ln \left\{ \tilde{c} \frac{\prod_k (\mu_k)^{\frac{2s'_k}{N}(\sum_l s'_l - 2s'_k)} (\mu_k^2 + s^2)^{\frac{2s'^2_k}{N}} s^{-\frac{1}{N}(\sum_l s'_l)^2}}{\prod_{k>l} (\mu_k - \mu_l)^{\frac{4s'_k s'_l}{N}}} \right\} \quad (49)$$

b) $\mu_2 = \bar{\mu}_1$ with $s_1 = 2, s_2 = 1$

$$F = \ln \left\{ \tilde{c} \frac{|\mu_1^2 + s^2|^{\frac{9}{N}} s^{-\frac{9}{N}}}{|\mu_1 - \bar{\mu}_1|^{\frac{9}{N}}} \right\} \quad (50)$$

Now if we put together (45), (48) and (41a), (41b) and compare them to (16), we see that T in (16) is given by the sum of several logarithms, coming from both the normalization factor F and from the "unphysical" τ . We note that the resulting formula - especially in the two cases (42), (43) when we explicitly carried out the iterations on τ - is entirely explicit; and once g_0 and $\psi_0(\lambda)$ are determined, it contains as free parameters the number, associated dimension (s_k) and w_k parameter (28b) of the poles μ_k as well as the ρ and x_3 independent vectors (M_b^i , etc.). The $\psi_0(\lambda)$ solution of (23) for the g_0 (18) of our "Higgs vacuum" is

$$\Psi_0(\lambda) = \text{diag} \left(e^{-2\lambda_1(x_3 + \frac{\lambda}{2})}, \dots, e^{-2\lambda_N(x_3 + \frac{\lambda}{2})} \right) \quad (51)$$

Therefore in the applications we determine those values of the free parameters which guarantee that $-\Delta\tau_{(ph)}$ is regular everywhere and has the expected behaviour at $r \rightarrow \infty$.

It is interesting to point out that in all cases investigated so far, the method based on the AW ansätze and its generalization yielded also a function S that determined $\Sigma \phi_i^2$ in precisely the same way as T in $\tau_{(ph)}$: $\Sigma \phi_i^2 = \text{const} -\Delta \ln S$. Therefore these two functions must be closely related, and in fact we shall find that up to terms giving no contribution to $\Sigma \phi_i^2$ this is indeed the case. Furthermore the superposition law (41a), (41b) we derived here for any N can be viewed also as a generalization - at least in the axially symmetric case - of the result so far known only for $N = 2$ (24).

4. - APPLICATION TO SU(2) MONOPOLES

In this Section, as a first illustration of the method, we generate the axially symmetric SU(2) multimonopoles. In this case the general expressions of the previous sections simplify greatly as all projectors must be one-dimensional and in the (g_0, τ_0) of the Higgs vacuum (18) and (19), we have only one independent parameter, λ_1 (that we shall assume to be positive):

$$g_0 = \text{diag} \left(e^{-2\lambda_1 x_3}, e^{2\lambda_1 x_3} \right), \quad \tau_0 = -2g^2\lambda_1^2 + c_0 \quad (52)$$

First we proceed to determine the connection between the number of poles assumed (n), and the magnetic charge of the solution, i.e., we estimate the asymptotic behaviour of $-\Delta\tau_{(ph)}$. Computing (51) and m_b^i for (52) we see that for large r $m^i g_0 \bar{m}^j$ increases exponentially :

$$\begin{aligned} m_a^i (g_0)_{ab} \bar{m}_b^j &\underset{r \rightarrow \infty}{\sim} \exp \left[2\lambda_1 \left(\chi_3 + \frac{1}{2} [\mu_i + \bar{\mu}_j] \right) \right] = \\ &= \exp \left[\lambda_1 (R(w_i) + \bar{R}(\bar{w}_j)) + w_i + \bar{w}_j \right] \sim \exp [2\lambda_1 r] ; R(w) = \sqrt{(w - \chi_3)^2 + g^2} \end{aligned}$$

Therefore, we expect that with n poles as in (42) $\det \Gamma \underset{r \rightarrow \infty}{\sim} \exp [2\lambda_1 n r]$. On the other hand, by inspecting F in (48) (with $N = 2$, $s_k = 1$) we conclude that in $-\Delta\tau_{(ph)}$ it gives rise to terms vanishing faster than $1/r$ for large r . Thus

$$\begin{aligned} -\Delta\tau_{(ph)} &\underset{r \rightarrow \infty}{\sim} -\Delta\tau_0 - \Delta \ln e^{2\lambda_1 n r} = \\ &= 4\lambda_1^2 - 4\lambda_1 \frac{n}{r} \sim \left(2\lambda_1 - \frac{n}{r} \right)^2 + O\left(\frac{1}{r^2}\right) \end{aligned} \quad (53)$$

i.e., the number of poles, n , just gives the magnetic charge.

To obtain the n monopole solutions, we have to fix the remaining free parameters (w_k 's, M_a^k , $k = 1, \dots, n$) to guarantee regularity. Furthermore, from topological and symmetry⁴⁾ considerations we know that the length of the Higgs field, $-\Delta\tau_{(ph)}$, must vanish somewhere on the symmetry axis ; the location of this zero giving the place of the superimposed (for $n > 1$) magnetic charges. As we are going to indicate, these two requirements are so strong that they completely fix the values of w_k and M_b^k . Before entering into the details, we remark that both the ρ terms and the pure μ_k products in (42), (48) and (49) are irrelevant from the point of view of regularity as $\Delta \ln \rho = 0 = \Delta \ln \mu_k$.

For $n = 1$ we find that the normalization factor F does not contain singularities if μ is real. With a real μ from (49), (42) and (45), we get

$$\tau_{(ph)} = \tau_0 + \ln \left\{ \frac{m g_0 \bar{m}}{g^2 + \mu^2} \mu g^{1/2} \tilde{c} \right\} \quad (54)$$

Now recalling that for the present case, to obtain a positive expression for $\det g$ in (35), we must change the sign of one of the terms in (52) [which of course affects the corresponding $\psi_0(\lambda)$ in a similar way], we have

$$m g_0 \bar{m} = |M_1|^2 \exp [2\lambda_1 (w + R(w))] - |M_2|^2 \exp [-2\lambda_1 (w + R(w))] \quad (55)$$

Using (55) and the identity $\mu^2 + \rho^2 = 2R(w)$ in (54), we conclude that the relevant term in (54), $\ln(mg_0 \bar{m}/2R(w))$, is positive definite and hence gives rise to no singularities in the length of the Higgs field if $|M_1|^2 = \exp[-2\lambda_1 w]A$, $|M_2|^2 = \exp[2\lambda_1 w]A$, i.e., if

$$\tau_{(ph)}^{1MP} = -2g^2 \lambda_1^2 + \ln \left\{ C A g^{1/2} \frac{\sinh(2\lambda_1 R(w))}{R(w)} \right\} \quad (56)$$

It is easy to check that $-\Delta \tau_{(ph)}^{1MP}$ has exactly one zero at $\rho = 0$ $x_3 = w$, thus it really describes the BPS one-monopole located on the x_3 axis.

In the $n = 2$ case, the use of real μ_k 's ($k = 1, 2$) would imply two different zeros of $\sum \phi_1^2$ on the x_3 axis at $x_3 = w_k$. As the singularities we found in this case survive the $w_2 \rightarrow w_1$ limit, we must consider complex μ_k 's instead. With complex μ 's, an inspection of (48) shows that the only hope to avoid singularities is to assume $\mu_2 = \bar{\mu}_1$; in this case from (48), (42) and (45) we obtain

$$\tau_{(ph)} = \tau_0 + \ln \left\{ \det \Gamma \frac{(\mu_1 \bar{\mu}_1)^2}{|\mu_1 - \bar{\mu}_1|^2} C \right\} = \tau_0 + \ln S \quad (57)$$

(57) is most conveniently analyzed when written in terms of the oblate spheroidal co-ordinates

$$\frac{\xi + i\chi_3}{\alpha} = \sqrt{(1-\xi^2)(1+\eta^2)} + i\eta; \quad 0 \leq \eta \leq \infty; \quad -1 \leq \xi \leq 1 \quad (58)$$

where $i\alpha$ gives the constant in μ_1 : $\mu_1 = i\alpha - x_3 + \sqrt{(i\alpha - x_3)^2 + \rho^2}$. Using (58) and noting that in this case g_0 is given by (52), we find that S has the form :

$$S = \frac{\tilde{c}}{\xi^2 + \eta^2} \left\{ (\xi^2 + \eta^2) \prod_{k=1}^2 (A_{kk} e^{2\lambda_k \alpha \eta} + B_{kk} e^{-2\lambda_k \alpha \eta}) - \right. \\ \left. - (1 + \eta^2) \left| A_{12} e^{2\lambda_1 \alpha (\eta + i[1-\xi])} + B_{12} e^{-2\lambda_1 \alpha (\eta + i[1-\xi])} \right|^2 \right\} \quad (59)$$

where $A_{ij} = M_1^{(i)} \bar{M}_1^j$, $B_{ij} = M_2^{(i)} \bar{M}_2^j$, $i, j = 1, 2$. To guarantee that S has no zero on the ring $\xi = \eta = 0$ (i.e., $x_3 = 0$, $\rho = \alpha$) the constants A_{ik} , B_{ik} must satisfy

$$A_{12} e^{i 2\lambda_1 \alpha} + B_{12} e^{-i 2\lambda_1 \alpha} = 0 \quad (60)$$

[we note that demanding the regularity of the elements of Γ_{ij} in (57) gives the same condition]. Substituting (60) into (59) we finally get

$$S = \frac{\tilde{c} 2 |A_{12}|^2}{\xi^2 + \eta^2} (1 - \xi^2)(1 + \eta^2) \left\{ \frac{\cos [4\lambda_1 \alpha \xi] + \frac{1}{2}(t^2 + t^{-2})}{1 - \xi^2} - \right. \\ \left. - \frac{\cosh [4\lambda_1 \alpha \eta] + \frac{1}{2}(t^2 + t^{-2})}{1 + \eta^2} \right\} \quad (61)$$

where $t^2 = |M_1^{(2)} / M_1^{(1)}|^2 > 0$. By inspecting (61) one finds that $\sum_1 \phi_1^2$ has the correct asymptotics on the x_3 axis ($\xi = \pm 1$), as well as having no singularity here if α and t^2 are related by

$$\cos [4\lambda_1 \alpha] + \frac{1}{2}(t^2 + t^{-2}) = 0 \quad (62)$$

The only solution of this equation is $t = 1$, $\alpha = \pi / (4\lambda_1)$; and in this case from (57), (52), (61), (62) we obtain

$$\tau_{(ph)}^{2MP} = -\frac{\pi^2}{8} (1 - \xi^2)(1 + \eta^2) + \ln \left\{ c' \frac{(1 + \eta^2)(1 - \xi^2)}{\xi^2 + \eta^2} \left[\frac{\cos^2(\frac{\pi}{2} \xi)}{1 - \xi^2} - \frac{\cosh^2(\frac{\pi}{2} \eta)}{1 + \eta^2} \right] \right\} \quad (63)$$

Using monotonicity arguments it is straightforward to prove that

$$\frac{1}{(\xi^2 + \eta^2)} \left[\frac{\cos^2(\frac{\pi}{2} \xi)}{1 - \xi^2} - \frac{\cosh^2(\frac{\pi}{2} \eta)}{1 + \eta^2} \right]$$

is always negative, thus $-\Delta\tau_{(ph)}^{2MP}$ is regular everywhere. Furthermore - as one expects - $(-\Delta\tau_{(ph)}^{2MP})$ as follows from (63) is identical to the expression of $\sum_i \phi_i^2$ for the 2MP first derived by Bäcklund transformations ²⁾, where the only zero of $\sum_i \phi_i^2$ at the origin ($\xi = \pm 1, \eta = 0$) was also exhibited.

It is remarkable that for this solution the only allowed value of w in μ_1 , $w = i(\pi/4\lambda_1)$ is identical to (after the rescaling $\lambda_1 \rightarrow \frac{1}{2}$) the only possible value of the Bäcklund parameter found in Ref. 2). The reason behind this is an established equivalence in the case of Ernst equations between the Bäcklund transformations of Harrison and Neugebauer ²⁵⁾ and the inverse scattering method of Belinski and Zakharov ²⁶⁾. Using this equivalence and the results of Ref. 3) the pattern to construct the monopoles with $n > 2$ is clear : when $n = 2k$ we must take g_0 as in (52) and $\mu_{2j} = \bar{\mu}_{2j-1}$ ($j = 1, \dots, k$) with $w_{2j-1} = i(2j-1)(\pi/4\lambda_1)$, while for $n = 2k+1$ we must take (52) with a sign changed and one real μ (with $w = 0$), and $2k$ complex μ 's with $w_{2j} = w_{2j+1}^* = ij(\pi/2\lambda_1)$ ($j = 1, \dots, k$). The constant vectors $M_a^{(k)}$ are then fully determined by demanding the regularity of Γ_{ij} in (42) on the "rings" $\rho = (2j-1)\pi/(4\lambda_1)$ ($\rho = j\pi/(2\lambda_1)$) and that of $-\Delta\tau_{(ph)}$ on the symmetry axis.

5. - APPLICATION TO SU(3) MONOPOLES

In this section, to demonstrate the power and usefulness of our method, we generate some known ^{9), 27)} as well as some new SU(3) monopole solutions. In this case we have to face the full complexity of the procedure outlined in Section 3. The most interesting novel features are that the dimensions of the projectors can now vary between one and two, and that - at least naively - the exponents in the normalization factor F are no longer integers. The only gain compared to the SU(2) case is the fact - as was pointed out by Ward - that we do not have to worry about the zero in $\sum_i \phi_i^2$ as it is no longer necessarily there.

The starting Higgs vacuum solution (g_0, τ_0) and the corresponding $\psi_0(\lambda)$ solution of (23) are given by (18), (19) and (51) with $N = 3$. Using them we again derive - as a first step - the connection between the number of poles, as well as the dimension of their associated projectors in the new solution and the topological charges n_1, n_2 by estimating the asymptotic behaviour of $-\Delta\tau_{(ph)}$. As even in this case the normalization factor F , (48), gives no contribution to the $1/r$ terms, it is sufficient to consider the "unphysical" τ as given by (41a)

and (41b), etc. If we have only one pole, (28b), with $s = 1$ then $\det \gamma$ in (42) is $m g_0 \bar{m}$ and (assuming $\lambda_1 \geq \lambda_2 \geq \lambda_3$) asymptotically it increases as

$$\det \gamma \underset{r \rightarrow \infty}{\sim} e^{2\lambda_1 r} \quad (64)$$

Substituting this and τ_0 into (41) we obtain

$$-4\tau \underset{r \rightarrow \infty}{\sim} 2 \sum_{i=1}^3 \lambda_i^2 - 4\lambda_1/r \quad (65)$$

Comparing this to (2) and (21), we conclude that adding one pole with a one-dimensional projector to g_0 gives fractional values for the topological charges $n_1 = \frac{4}{3}$, $n_2 = \frac{2}{3}$. If we assume that the projector associated to the pole is a two-dimensional one, then $\det \gamma \sim e^{2(\lambda_1 + \lambda_2)r}$ giving

$$-4\tau \underset{r \rightarrow \infty}{\sim} 2 \sum_{i=1}^3 \lambda_i^2 - 4(\lambda_1 + \lambda_2)/r \quad (66)$$

which in turn means that in this case $n_1 = \frac{2}{3}$, $n_2 = \frac{4}{3}$, i.e., they are again fractionals. Clearly, to obtain regular monopole solutions we have to iterate the above transformations until they yield integers n_1 and n_2 , as we know from general principles that for smooth, monopole-like solutions, both n_1 and n_2 must be integers.

The simplest possibility is to add two independent μ_k poles (28b) with projectors of dimension two and one, respectively, to g_0 which leads one via (43), (65) and (66) to expect that in this case both n_1 and n_2 will be two.

Surprisingly we found that the only way we can have n_1 and n_2 less than this value is to assume a block diagonal g (after adding to g_0 one pole with $s = 1$) normalized with a similar block diagonal M [see (37), (38)]. As we shall show, the point in this case is that F gets an extra term contributing to the asymptotic $1/r$. These solutions will turn out to be the so-called minimal embeddings of the $SU(2)$ monopoles into $SU(3)$.

As an alternative possibility we also investigated what happens if we use $\tilde{\mu} = -\rho^2/\mu$ where μ is given by (28b)

$$\tilde{\mu} = -\frac{\tilde{s}^2}{\mu} = w - x_3 - R(w) \quad (67)$$

for the pole position. [We note here that as long as the asymptotic behaviour of $-\Delta\tau$ is concerned, using μ or $\tilde{\mu}$ of (28a) makes no difference.] Proceeding in a similar way as before we obtain that when the projector associated to $\tilde{\mu}$ is one-dimensional, then the contribution to the topological charges is $n_1 = \frac{2}{3}$, $n_2 = \frac{4}{3}$ while in the case of the 2D projector it is $n_1 = \frac{4}{3}$, $n_2 = \frac{2}{3}$. Comparing these to (65) and (66), we find yet another way to obtain integers n_1 and n_2 ($n_1 = n_2 = 2$), namely assuming two poles with one-dimensional projectors, say, at $\mu_1 = \mu$ and $\mu_2 = \tilde{\mu}$, respectively. Of course, in this case, we must cope with the apparent singularity in Γ , (42), resulting from the vanishing of $\rho^2 + \mu_1\mu_2$.

Now we turn to the implications of having non-integer exponents (for $N = 3$, $s_k, t_k = 1$ or 2) in the normalization factor F , (48). As we mentioned earlier, the ρ term and μ_k products are irrelevant ; however, the $\mu_k - \mu_l$ type differences and the products of $\mu_k^2 + \rho^2 = 2\mu_k R(w_k)$ can produce singularities if their exponent is not an integer. To show this, we note that $\Delta \ln R(w_k)$ has a singularity at $R(w_k) = 0$, thus around $R(w_k) \sim 0$ we must somehow compensate the $R(w_k)$ originating from F . However, from τ as given by (41a), (41b) and (43) - at least with the g_0 and $\psi_0(\lambda)$ we use - we can at best obtain $R(w_k)$ raised to an integer power. Thus to avoid singularities the exponent of $R(w_k)$ in F must also be an integer. For a low number of poles we found three ways to achieve this : $\mu_2 = \tilde{\mu}_1$ with $s_1 = 2, s_2 = 1$, when in (50) $N = 3$ gives really integer exponents : $\mu_2 = -\rho^2/\mu_1$ with $s_1 = s_2 = 1$ and real w_1 in which case F in (49) becomes zero (with $c = 1$) ; and the block diagonal normalization of the block diagonal g mentioned earlier.

Thus we see that the particular choices of poles and projector dimensions that give a reasonable asymptotic behaviour are the same as the ones selected by requiring regularity. Encouraged by this we turn to the detailed discussion of these special cases.

5.1 - Block diagonal g - SU(2) embeddings

It is easy to see from (31) that using our g_0 (18) one can achieve the block diagonal g , (37) (in general, i.e., not only with one pole), by choosing one-dimensional projectors and $M_a^{(k)}$ vectors with vanishing third

component : $M_3^{(k)} = 0$ ($k = 1, \dots, n$) ^{*)}. As all these solutions effectively use only a 2×2 submatrix of g , it is natural to expect that they give the embeddings of $SU(2)$ monopoles into $SU(3)$.

Making the same ansatz for $\tau_{(ph)}$ as before, (45), we obtain that the block diagonal normalization of $g_{(ph)}$, (38), when substituted into (44) changes the equation for F as

$$\begin{aligned} (\partial_s + i\partial_{x_3})F = & -\frac{1}{2} \left\{ \frac{g}{4} (\ln \det g_{,s}^2 - \ln \det g_{,x_3}^2) + i\frac{g}{2} (\ln \det g)_{,x_3} (\ln \det g)_{,s} \right\} \\ & + g \frac{\lambda_3}{2} (\ln \det g)_{,x_3} - i g \frac{\lambda_3}{2} (\ln \det g)_{,s} \end{aligned} \quad (68)$$

Equation (68) suggests to look for F in the form $F = F_0 + F_1$ where F_0 satisfies (46) with $N = 2$ [i.e., is given by (48) with $N = 2$] while for F_1 we have

$$(\partial_s + i\partial_{x_3})F_1 = \frac{\lambda_3}{2} (g (\ln \det g)_{,x_3} - i g (\ln \det g)_{,s}) \quad (69)$$

Using (35) and (28) it is straightforward to integrate (69); in particular when we add only one pole at $\lambda = \mu$, F_1 becomes

$$F_1 = \ln \left\{ e^{\frac{\lambda_3}{2} (\mu + \frac{g^2}{\mu})} \right\} - \ln \left\{ e^{\lambda_3 R(w)} \right\} \quad (70)$$

It is the F_1 part of F that modifies the asymptotic behaviour of $-\Delta\tau_{(ph)} = -\Delta(\tau + F_0 + F_1)$ since

$$-\Delta F_1 = -2\lambda_3 / R(w) \sim_{\tau \rightarrow \infty} 2(\lambda_1 + \lambda_2) / \tau \quad (71)$$

Comparing (71) to (2) and (21) we see that F_1 contributes $-\frac{1}{3}$ and $-\frac{2}{3}$ to n_1 and n_2 , respectively. Therefore, the net topological charges of the solution obtained by this normalization as derived from the sum of (71) and (65) are $n_1 = 1$, $n_2 = 0$. To investigate the regularity of the solution we construct $\tau_{(ph)}$ from (42), (49) (with $N = 2$, $n = 1$, $s = 1$) and (70) :

^{*)} Furthermore one finds that $g_{33} = (g_0)_{33} = \exp[-2\lambda_3 x_3]$.

$$\tau_{(ph)} = -\frac{g^2}{2} \sum_{i=1}^3 \lambda_i^2 + \ln \left\{ e^{\lambda_3 R(w)} \frac{m g_0 \bar{m}}{2 R(w)} s^{1/2} c \right\} = -\frac{g^2}{2} \sum_{i=1}^3 \lambda_i^2 + \ln S_0 \quad (72)$$

As even in this case we must change one sign in ε_0 , S_0 has the form (remember that $M_3 = 0$) :

$$S_0 = \frac{s^{1/2} c}{2 R(w)} \left[|M_1|^2 e^{2\lambda_1 w + (\lambda_1 - \lambda_2) R(w)} - |M_2|^2 e^{2\lambda_2 w - (\lambda_1 - \lambda_2) R(w)} \right] \quad (73)$$

For regularity S_0 must not have zero ; this is guaranteed if $|M_1|^2 = A e^{-2\lambda_1 w}$ $|M_2|^2 = A e^{-2\lambda_2 w}$ and in this case $\tau_{(ph)}$ assumes the form

$$\tau_{(ph)} = -\frac{g^2}{2} \sum_{i=1}^3 \lambda_i^2 + \ln \left\{ C A s^{1/2} \frac{\sinh [(\lambda_1 - \lambda_2) R(w)]}{R(w)} \right\} \quad (74)$$

Comparing (74) to (56) as well as recalling that the topological charges are $n_1 = 1, n_2 = 0$, we see that this spherically symmetric solution is really the embedding of the BPS one-monopole into $SU(3)$. It is interesting to point out that, unlike in the $SU(2)$ case, $-\Delta\tau_{(ph)}$ as follows from (74) has no zero at all.

Of course, the procedure we carried out here for embedding the LMP can also be applied to block diagonal g -s having more than one pole. Indeed in the case of n poles (with one-dimensional projectors) the solution of (69) is

$$F_1 = \ln \left\{ e^{\lambda_3 \sum_{k=1}^n R(w_k)} \right\} \quad (75)$$

therefore on the basis of (65) and (42) we expect that the topological charges of the solution are $n_1 = n(\frac{4}{3} - \frac{1}{3}) = n_1, n_2 = n(\frac{2}{3} - \frac{2}{3}) = 0$. Furthermore the regularity analysis would lead to $M_a^{(k)} = s$ ($a = 1, 2$) being slightly different from their $SU(2)$ counterparts, but the final expression of $\tau_{(ph)}$ would be quite similar to the one obtained for $SU(2)$. (In fact with two poles - just like for the LMP - we found that the only change is in τ_0 and in replacing $2\lambda_1$ by $\lambda_1 - \lambda_2$.) Therefore we do not expect these solutions to contain any free parameter, thus they are really the embeddings of the corresponding $SU(2)$ monopoles into $SU(3)$.

Of course, using the same g , (18), but keeping the first or second components of the $M_a^{(k)}$ vectors in (31) ($k = 1, \dots, n$) zero one can have g 's in forms different from (37)

$$\begin{pmatrix} g_{11} & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad (76a) \quad \text{or} \quad \begin{pmatrix} * & 0 & * \\ 0 & g_{22} & 0 \\ * & 0 & * \end{pmatrix} \quad (76b)$$

where $g_{11} = \exp(-2\lambda_1 x_3)$, $g_{22} = \exp(-2\lambda_2 x_3)$ and $*$ stands for a non-zero entry. These choices correspond to using different subspaces for the embedding. $g_{(ph)}$ in these cases is still defined by $g_{(ph)} = gM$ but the M 's now have the form :

$$M = \text{diag} (1, \det g^{-1/2}, \det g^{-1/2}) \quad (77a)$$

$$M = \text{diag} (\det g^{-1/2}, 1, \det g^{-1/2}) \quad (77b)$$

Along the same lines as before, we constructed the solutions for (76a), (76b), (77a) and (77b), assuming one pole only and verified that they are indeed embeddings of the LMP into $SU(3)$ with topological charges $n_1 = 2$, $n_2 = 1$ and $n_1 = 1$, $n_2 = 1$, respectively,

Now we turn to the other, more exciting possibilities found at the beginning of Section 5 ; they will turn out to be specific to $SU(3)$ having no $SU(2)$ analogue.

5.2 - Poles at μ_1 and $-\rho^2/\mu_1$ - spherically symmetric $SU(3)$ monopole

In this subsection, as the next step, we investigate the solution obtained by adding two real poles (with one-dimensional projectors) at $\lambda = \mu_1$ and $\lambda = -\rho^2/\mu_1$ to g_0 . As we already pointed out in this case F , (48), vanishes, thus $\tau_{(ph)}$ is given by (42) with $n = 2$, $N = 3$; however, the off-diagonal elements of the Γ matrix

$$\Gamma_{ij} = \frac{m_a^i (g_0)_{ab} m_b^j}{g^2 + \mu_i \mu_j} \quad \begin{matrix} i, j = 1, 2 \\ a, b = 1, 2, 3 \end{matrix} \quad m_a^i = M_b^i \psi_0^{-1}(\mu)_{ba} \quad (42)$$

seem to develop a singularity as with this μ_1, μ_2 pair $\rho^2 + \mu_1 \mu_2 \equiv 0$. Clearly to circumvent this difficulty one must resort to some limiting procedure. Our limiting procedure assumes that

$$\mu_1 = w - \lambda_3 + R(w), \quad \mu_2 = \beta - \lambda_3 - R(\beta); \quad \beta = w + \epsilon \quad (78a)$$

and

$$M_a^i = \hat{M}_a^i + \epsilon N_a^i \quad (78b)$$

where $\epsilon \rightarrow 0$ and $\hat{M}_a^i, N_a^i$ are constant vectors. Substituting (78a,b) into (42b) [remember that $g_0, \psi_0(\lambda)$ are given by (18) and (51)] one derives that the limit avoids the apparent singularity if

$$\sum_j' e^{2\lambda_j w} \hat{M}_j^{(1)} \hat{M}_j^{(2)} = 0 \quad (79)$$

[Here - and in the following - as a remnant of the sign change in g_0 , $\sum_j'$ is defined as $\sum_j' a_j = \sum_{j=1}^3 (-1)^{j-1} a_j$.] Assuming (79) is satisfied, $\det \Gamma$ acquires the form :

$$\begin{aligned} \det \Gamma = & \frac{1}{4 \epsilon^2 R^2(w)} \left[\sum_i' e^{2\lambda_i(w+R(w))} \hat{M}_i^{(1)2} \right] \left[\sum_i' e^{2\lambda_i(w-R(w))} \hat{M}_i^{(2)2} \right] - \\ & - \left(\frac{A \mu_1^2 R(w) + \mu_1 \epsilon^2 \sum_i' \hat{M}_i^{(1)} \hat{M}_i^{(2)} e^{2\lambda_i w}}{\epsilon^2} \right)^2 \\ A = & \sum_i' (\hat{M}_i^{(1)} N_i^{(2)} + \hat{M}_i^{(2)} N_i^{(1)}) e^{2\lambda_i w} \end{aligned} \quad (80)$$

The only way we could guarantee the regularity of τ [(42) with (80)] was to demand that in addition to (79) $\hat{M}_a^{(i)}$ and $N_a^{(i)}$ should also satisfy

$$\sum_i' \hat{M}_i^{(1)} \hat{M}_i^{(2)} \lambda_i e^{2\lambda_i w} = 0; \quad \sum_j' \hat{M}_j^{(1)2} \lambda_j^2 e^{2\lambda_j w} = 1 \quad i=1,2; \quad A=0 \quad (81)$$

The solution of (79) and (81) is (recalling the assumption $\lambda_1 \geq \lambda_2 \geq \lambda_3$)

$$N_a^i = 0; \quad \hat{M}_j^{(1)} = \hat{M}_j^{(2)} = e^{-\lambda_j w} \left[(-1)^{j-1} \prod_{k \neq j} (\lambda_j - \lambda_k) \right]^{-\frac{1}{2}} \quad (82)$$

Using these values for the vectors M_a^i finally (80) and (42) yield for the superpotential

$$\tau_{(ph)} = -\frac{g^2}{2} \sum_{j=1}^3 \lambda_j^2 + \ln \left\{ \tilde{c} \frac{g^2}{16 R^4(w)} \left(\sum_{i=1}^3 A_i e^{2\lambda_i R(w)} \right) \left(\sum_{i=1}^3 A_i e^{-2\lambda_i R(w)} \right) \right\} \quad (83)$$

with $A_i = \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}$. This solution is clearly spherically symmetric (with respect to $x_3 = w, \rho = 0$) and realizing that the brackets in the argument of the logarithm in (83) are the functions Q_1 and Q_2 introduced in Ref. 27), it is straightforward to prove that (83) gives the spherically symmetric monopole of Bais and Wilkinson ²⁷⁾. We can now understand why it was not possible to obtain this type of solution in the $SU(2)$ case : having only two component vectors (79) and (81) cannot be satisfied at once.

Surprisingly the $\hat{M}_a^k$ vectors are singular in the $\lambda_2 \rightarrow \lambda_1$ limit that - as we explained in Section 2 - corresponds to having $SU(2) \times U(1)$ rather than $U(1) \times U(1)$ as an unbroken symmetry group. However, the superpotential (83) as a whole has a sensible limit with $\lambda_2 = \lambda_1 - \delta, \delta \rightarrow 0$:

$$\tilde{\tau}_{(ph)} = -3g^2 \lambda_1^2 + \ln \left\{ \frac{\tilde{c} g^2}{(6\lambda_1)^4 R^4(w)} \left(e^{2\lambda_1 R} (6\lambda_1 R - 1) + e^{-4\lambda_1 R} \right) \left(e^{-2\lambda_1 R} (-6\lambda_1 R - 1) + e^{4\lambda_1 R} \right) \right\} \quad (84)$$

which is, of course, nothing but the superpotential for the spherically symmetric Bais-Weldon ²⁸⁾ monopole.

Based on this observation it is natural to expect that something similar happens for the axially symmetric monopoles as well, i.e., to assume that the solution with $\lambda_1 = \lambda_2$ is obtained from the solutions with $\lambda_1 > \lambda_2$ in a limit which is singular for the M_a^k vectors but keeps the superpotential finite. In the next subsection we shall use this principle to construct some axially symmetric $SU(3)$ monopoles that in the $\lambda_2 \rightarrow \lambda_1$ limit reduce to those found by Ward ¹⁰⁾.

5.3 - Axially symmetric $SU(3)$ monopoles

Next we investigate the last and most interesting possibility found earlier : the case when we have two complex poles related by $\mu_2 = \bar{\mu}_1$ with $s_1 = 2$ and $s_2 = 1$, respectively. This choice enables us to construct a new family of axially symmetric $SU(3)$ monopoles containing a free parameter.

The physical superpotential in this case is given by (45), (43) and (50) with $N = 3$:

$$\tau_{(ph)} = \tau_0 + \ln \left\{ c \frac{(\mu_1 \bar{\mu}_1)^3}{|\mu_1 - \bar{\mu}_1|^3} \det \Gamma \right\}$$

where

$$\Gamma_{ij} = \frac{m^i g_0 \bar{m}^j}{s^2 + |\mu_1|^2} \quad i, j = 1, 2 ; \quad \Gamma_{i3} = \bar{\Gamma}_{3i} = \frac{m^i g_0 \bar{n}}{s^2 + \mu_1^2} ; \quad (85)$$

$$\Gamma_{33} = \frac{n g_0 \bar{n}}{s^2 + |\mu_1|^2} ; \quad m_a^i = \tilde{M}_b^i \Psi_0^{-1}(\mu_1)_{ba} , \quad n_a = \tilde{N}_b \Psi_0^{-1}(\bar{\mu}_1)_{ba}$$

and $\tilde{M}_a^i, \tilde{N}_b$ are independent of ρ and x_3 . We again found it useful to introduce the oblate spheroidal co-ordinates (58) ; $\rho + ix_3 = \alpha [\sqrt{(1-\xi^2)(1+\eta^2)} + i\xi\eta]$. As a necessary condition for regularity, we require the Γ_{i3} elements to be finite on the "ring" $\xi = \eta = 0$ (i.e., $\rho = \alpha, x_3 = 0$), this implies for the $\tilde{M}_a^i \tilde{N}_a$ vectors [remember the sign change in g_0 (18)] :

$$\sum_k' \tilde{M}_k^{(j)} \bar{\tilde{N}}_k e^{2\lambda_k i} = 0 \quad , \quad j = 1, 2 \quad (86a)$$

For later convenience we absorb the exponentials into the vectors by redefining them as $\tilde{M}_k^{(j)} = M_k^{(j)} e^{-\lambda_k i}, \tilde{N}_k = N_k e^{i\lambda_k}$ ($k = 1, \dots, 3 ; j = 1, 2$) which simplifies (86a) to

$$\sum_k' M_k^{(j)} \bar{N}_k = 0 \quad j = 1, 2 \quad (86b)$$

Surprisingly, after a fair amount of algebra during which we several times used (86b) and the fact that $\sum_{i=1}^3 \lambda_i = 0$ it was possible to cast $\tau_{(ph)}$, (85), in a remarkably simple form :

$$\tau_{(ph)} = \tau_0 + \ln \left\{ \frac{\tilde{c} (1-\xi^2)(1+\eta^2)}{\alpha^3 (\xi^2 + \eta^2)} [F_1(\eta) - F_2(\xi)] \right\} \quad (87a)$$

where

$$F_1(\eta) = (1+\eta^2)^{-1} \left\{ \sum_i' |N_i|^2 (\Sigma \Delta) + \right.$$

$$+ 4 \sum_{k>l, j+k, l}^{} \sinh^2 [\alpha \eta (\lambda_k - \lambda_l)] |N_k|^2 (\Delta(k, j) + \Delta(j, k)) \}$$

$$F_2(\xi) = F_1(i\xi) \quad (87b)$$

and

$$\Delta(k, j) = (-1)^{k+j-2} \begin{vmatrix} M_k^{(1)} \bar{M}_k^{(1)} & M_k^{(1)} \bar{M}_k^{(2)} \\ \bar{M}_j^{(1)} M_j^{(2)} & \bar{M}_j^{(2)} M_j^{(2)} \end{vmatrix}$$

while $(\Sigma\Delta)$ stands for $\sum_{k>j} (\Delta(k, j) + \Delta(j, k))$.

To guarantee the regularity of (87), (86b) is not sufficient since - as we explained in Section 4 - for acceptable behaviour on the symmetry axis ($\xi = \pm 1$) the numerator of the dangerous term, $1 - \xi^2$, must vanish there :

$$-\sum_i |N_i|^2 (\Sigma\Delta) + 4 \sum_{k>l, j+k, l}^{} \sinh^2 [\alpha (\lambda_k - \lambda_l)] |N_k|^2 (\Delta(k, j) + \Delta(j, k)) = 0 \quad (88)$$

Therefore we first seek a $F_1(\eta)$ in the form

$$F_1(\eta) = \frac{D + \sum_{k>l} \sinh^2 [\alpha \eta (\lambda_k - \lambda_l)] A_{kl}}{\lambda + \eta^2} \quad (89)$$

where D, A_{kl} are required to satisfy (88) and $\tau_{(ph)}$ is regular everywhere.

Having determined D and A_k this way, we of course require (89) to have the form of (87b) ; this together with (86b) constitutes a system of equations for $M_k^{(1)}, N_k$. Then we solve this system to obtain the correct values of the vectors.

Proceeding this way after some trial and error we finally came to

$$F_1(\eta) = - \frac{Q(\eta)Q(-\eta)}{\lambda + \eta^2}, \quad Q(\eta) = \frac{1}{2} \sum_{j=1}^3 \frac{e^{-2\alpha \lambda_j \eta}}{\prod_{k \neq j} \sinh [2\alpha (\lambda_j - \lambda_k)]} \quad (90)$$

This corresponds to the following constants in (89) :

$$A_{kl} = \{ \sinh^2 [2\alpha (\lambda_k - \lambda_l)] \prod_{k \neq l} \pi(k, l) \}^{-1}$$

$$\pi(k, \ell) = \sin[2\alpha(\lambda_j - \lambda_k)] \sin[2\alpha(\lambda_j - \lambda_\ell)] \quad j \neq k, \ell$$

$$D = \sum_{k > \ell} \sin^2[\alpha(\lambda_k - \lambda_\ell)] A_{k\ell} \quad (91)$$

It is easy to see that this choice of parameters $D, A_{k\ell}$ guarantees that when $\lambda_2 = \lambda_1 - \delta$ and $\delta \rightarrow 0$, then $Q(\eta)$ already tends to a finite function $Q_0(\eta)$:

$$Q_0(\eta) = \frac{1}{2 \sin^2(6\lambda_1\alpha)} \left\{ e^{4\lambda_1\alpha\eta} - e^{-2\lambda_1\alpha\eta} [\cos(6\lambda_1\alpha) + \eta \sin(6\lambda_1\alpha)] \right\} \quad (92)$$

Substituting (92) into (90) and (87) and making the identifications :

$a = \lambda_1\alpha\eta$, $b = \lambda_1\alpha\xi$, $\psi = 6\lambda_1\alpha$, one obtains (up to an irrelevant $\ln \rho^2$ term) the function constructed by Ward ¹⁰⁾.

Now we come to the construction of the vectors $M_k^{(1)}, N_k$. First we parametrize them with the aid of three real variables a_i (satisfying $a_1 \geq a_2 \geq a_3$) in such a way that Eqs. (86b) hold automatically :

$$M_k^{(1)} = \left[(-1)^{k-1} \prod_{j \neq k} (a_k - a_j) \right]^{-1/2}, \quad M_k^{(2)} = a_k^{-1} \left[(-1)^{k-1} \prod_{j \neq k} (a_k - a_j) \right]^{-1/2}$$

$$N_k = \sqrt{2} a_k \left[(-1)^{k-1} \prod_{j \neq k} (a_k - a_j) \right]^{-1/2} \quad (93)$$

Plugging (93) into (87b) and comparing to (89) with $D, A_{k\ell}$ given by (91) we obtain a complicated system of algebraic equations ; its solution is given in the following form $a_2 = za_3, a_1 = xa_2$ where

$$x = \frac{\cos \alpha (\lambda_2 - \lambda_3) \cos \alpha (\lambda_1 - \lambda_3) + \sin \alpha (\lambda_1 - \lambda_2) K}{\cos \alpha 3\lambda_1 \cos \alpha (\lambda_1 - \lambda_3)}$$

$$z = \frac{\cos \alpha (\lambda_1 - \lambda_3) \cos \alpha (\lambda_1 - \lambda_2) + K \sin \alpha (\lambda_2 - \lambda_3)}{\cos \alpha 3\lambda_2 \cos \alpha (\lambda_1 - \lambda_2)}$$

$$K = (2 \cos \alpha (\lambda_1 - \lambda_2) \cos \alpha (\lambda_2 - \lambda_3) \cos \alpha (\lambda_1 - \lambda_3))^{1/2}$$

and a_3 is the real root of

$$2^3 K^4 = (\lambda z^2)^2 a_3^6 \quad (95)$$

For consistency a_k must be real and must satisfy $a_1 \geq a_2 \geq a_3$. We checked that this happens for $|\alpha| < \pi/6\lambda_1$. This then completes the construction of a solution..

This way we found a new six-parameter family of monopoles with $U(1) \times U(1)$ as an unbroken symmetry group and $n_1 = n_2 = 2$ as topological charges. [The six parameters are : the location of the origin (3), the orientation of the symmetry axis (2) and α .]

The $\alpha \rightarrow 0$ limit corresponds to spherical symmetry and to implement it requires the adjustment of the α dependence of the integration constant $\tilde{C}$ in (87). In the oblate spheroidal co-ordinate system, (58), in the $\alpha \rightarrow 0$ limit one has to make the replacements $\alpha\sqrt{1+\eta^2} \rightarrow r$, $\xi \rightarrow \cos\theta$ where θ is the usual polar angle. Thus in this limit from (90) and (87) we obtain :

$$\tau_{(ph)} \xrightarrow{\alpha \rightarrow 0} \tau_0 + \ln \left\{ \frac{\tilde{C} s^2}{8^2 \alpha^5 r^2} \left[-\frac{1}{r^2} \tilde{Q}(r) \tilde{Q}(-r) + 4\alpha^2 \frac{\cos^4 \theta}{\sin^2 \theta} \right] \right\}$$

$$\tilde{Q}(r) = \sum_{j=1}^3 \frac{e^{-2\lambda_j r}}{\pi_{\lambda_j} (\lambda_j - \lambda_k)} \quad (96)$$

Clearly (96) has a meaningful limit if $\tilde{C} \xrightarrow{\alpha \rightarrow 0} \alpha^5 \tilde{C}$ and in this case it yields (83), the superpotential of the spherically symmetric Bais-Wilkinson monopole.

We must still show that the solution (87) with (90) and $\tilde{C} = \alpha^5 \tilde{C}$ is regular, i.e. that

$$\frac{1}{\xi^2 + \eta^2} \left\{ \frac{Q(i\xi)Q(-i\xi)}{1 - \xi^2} - \frac{Q(\eta)Q(-\eta)}{1 + \eta^2} \right\} \quad (97)$$

has no zero in R^3 . Obviously the first term in the bracket is never negative and can be shown to vanish at $\xi = \pm 1$. On the other hand the second term grows exponentially for large η , and on the ring ($\xi = \eta = 0$) the two terms are equal. Therefore the inequality

$$\frac{Q(\eta)Q(-\eta)}{1 + \eta^2} > Q^2(0) > \frac{Q(i\xi)Q(-i\xi)}{1 - \xi^2} \quad \xi, \eta \neq 0 \quad (98)$$

would guarantee the regularity of (87). Using monotonicity and convexity considerations we indeed proved that (98) is true for $|\alpha| < \pi/2(\lambda_1 - \lambda_3)$.

Our solution can be interpreted as if the two "constituent" monopoles were separated by an "imaginary distance" $i\alpha$. They share this property with the SU(2) 2MP, but unlike in the SU(2) case this separation parameter is not fixed by regularity.

It is interesting to compare our solution (87) with (90) to that found by Athorne¹⁰⁾ which also describes a family of axially symmetric SU(3) monopoles with an unbroken group $U(1) \otimes (1)$ and topological charges $n_1 = n_2 = 2$. However, in his case the constituent monopoles are separated by a real distance. In fact the relation between the two families is even more intimate : if we make the substitutions $\alpha \rightarrow i\delta/2$, $\eta \rightarrow iy$, $\xi \rightarrow x$ in (90) and (87) and interpret δ, y, x as prolate spheroidal co-ordinates :

$$s + ix_3 = \frac{\delta}{2} \left(\sqrt{(1-x^2)(y^2-1)} + ix y \right); \quad -1 \leq x \leq 1; \quad 1 \leq y \leq \infty \quad (99)$$

Then we immediately obtain (again up to irrelevant terms) the function generating Athorne's solution.

It is interesting to observe that the spherically symmetric Bais-Wilkinson monopole can be deformed in two independent ways into an axially symmetric one, as both our and Athorne's solution reduce to this monopole in the spherically symmetric limit. This strongly suggests that the most general axially symmetric SU(3) monopoles with topological charges $n_1 = n_2 = 2$ form a family with at least seven parameters.

6. - CONCLUSION

In this paper we gave a general framework for constructing axially symmetric monopoles solving the Bogomolny equations of an SU(N) gauge theory coupled to an adjoint Higgs field. Rewriting the Bogomolny equations in an appropriate form, we proved the existence of a superpotential that determines the length of the Higgs field $\sum_1 \phi_1^2$ and showed that in the $N = 2$ case our definition coincides with that of Ref. 14). Using an inverse scattering method worked out for the self-duality

equations, we derived a transformation that enables one to construct a new superpotential - containing some free parameters - from an old solution. We pointed out that by iterating this transformation it is possible to obtain infinitely many new solutions of the Bogomolny equations. Therefore, to construct the $SU(N)$ monopoles we start from a "Higgs vacuum" (i.e., from a solution of the Bogomolny equations with constant Higgs field and identically vanishing gauge fields over the whole R^3), and iterate our superpotential changing transformation on the superpotential of this state, until a judicious choice of the free parameters guarantees the correct asymptotic behaviour and the regularity of the new solution. It is important to stress that once the Higgs vacuum is fixed the rest of the construction requires pure algebra only.

We applied this framework to obtain axially symmetric $SU(2)$ and $SU(3)$ monopoles. In both cases we established the relation between the topological charges and the number and properties of the transformations required for the construction of these monopoles. In the $SU(2)$ case we constructed the one and two monopoles in detail and, based on this, sketched the way to obtain the n monopole. In the $SU(3)$ case, we investigated in detail the simplest three possibilities consistent with the regularity and integer topological charges: the $SU(2)$ embeddings, the spherically symmetric monopoles with $n_1 = n_2 = 2$ and - most importantly - the axially symmetric monopoles with $n_1 = n_2 = 2$. In this last case we found a new six-parameter family of $SU(3)$ monopoles with $U(1) \times U(1)$ as an unbroken group that - by construction - reduce in an interesting way to Ward's solution⁹⁾, when the asymptotic Higgs field has coinciding eigenvalues. As this family is different from the one found by Athorne¹⁰⁾ we conjecture that the most general axially symmetric $SU(3)$ monopoles with $n_1 = n_2 = 2$ form a wider, at least seven-parameter family.

We think these achievements make the method rather attractive. However, its usefulness would be enhanced by a better understanding of its relation to the method using the Atiyah-Ward ansätze. In particular, it would be important to establish how the choice of parameters in the A-W method is related to the choice of our parameters, as in all cases investigated so far the A-W method has also produced a function that is practically identical to our final superpotential. This connection between the parameters would simplify both methods, since in the A-W method the parameters are chosen at an early stage and the function is obtained at the end of a long procedure, while, in our case, to obtain the correct parameters at the end is a rather complicated task. From the point of view of regularity it would be desirable to connect our framework with that of Nahm⁷⁾ which is

capable of guaranteeing the regularity by construction. Then it would be interesting to see how the connection between the Bäcklund transformation method and the superpotential method that exists for $SU(2)$ generalizes for any N .

Finally we mention that the connection between monopoles and instantons ¹⁴⁾ was elaborated by Chakrabarti and his collaborators ²⁹⁾ using methods similar to ours. In a recent paper ³⁰⁾, he produced a framework capable of generating "instanton chains" that give the most general $SU(2)$ monopole solution in the limit when the number of instantons goes to infinity. It would be useful to understand the relation between instantons and monopoles in the case of more complicated groups as well.

APPENDIX A

In this Appendix we give the details of the construction of $\chi_1(\lambda)$ having the form

$$\chi_1(\lambda) = I + \frac{R}{\lambda - \mu} \quad (A.1)$$

with R and μ being independent of λ ; in fact μ is given by (28b).

First we note that the $\chi_1 \chi_1^{-1} = I$ identity taken at $\lambda = \mu$ implies that R_{ab} and $\chi_1^{-1}(\mu)_{ab}$ are degenerate matrices :

$$R_{ab} = \sum_{j=1}^s n_a^j m_b^j ; \quad \chi_1^{-1}(\mu)_{ab} = \sum_{k=s+1}^N q_a^k p_b^k ; \quad \sum_{a=1}^N m_a^j q_a^k = 0 \quad (A.2)$$

Next we slightly rewrite Eqs. (25) :

$$\begin{aligned} D_1 \chi_1 \chi_1^{-1} + \chi_1 \frac{s V_0 - \lambda U_0}{\lambda^2 + s^2} \chi_1^{-1} &= \frac{s V - \lambda U}{\lambda^2 + s^2} \\ D_2 \chi_1 \chi_1^{-1} + \chi_1 \frac{s U_0 + \lambda V_0}{\lambda^2 + s^2} \chi_1^{-1} &= \frac{s U + \lambda V}{\lambda^2 + s^2} \end{aligned} \quad (A.3)$$

On the right-hand side of these equations there is no pole at $\lambda = \mu$, thus the residues of the poles on the left-hand side must vanish. The residue of the second-order pole on the left-hand side of (A.3) vanishes because of (28a) while the absence of first-order poles yields for m_a^j :

$$\begin{aligned} D_1 m_a^j \Big|_{\lambda=\mu} + m_b^j \left(\frac{s V_0 - \mu U_0}{\mu^2 + s^2} \right)_{ba} &= 0 \\ D_2 m_a^j \Big|_{\lambda=\mu} + m_b^j \left(\frac{s U_0 + \mu V_0}{\mu^2 + s^2} \right)_{ba} &= 0 \end{aligned} \quad (A.4)$$

Using the $\psi_0(\lambda)$ solution of (23) we can solve (A.4) in the following form :

$$m_a^j = M_b^j \psi_0^{-1}(\mu)_{ba} \quad j = 1, \dots, s \quad (A.5)$$

where M_b^j are constant vectors. The n_a^j part of R_{ab} is determined from (26)

$$g = \chi_1(\lambda) g_0 \chi_1^+ \left(-\frac{s^2}{\lambda} \right) = \left(I + \frac{R}{\lambda - \mu} \right) g_0 \left(I - \frac{\lambda R^+}{s^2 + \lambda \bar{\mu}} \right) \quad (A.6)$$

by demanding the vanishing of the residue of the apparent pole at $\lambda = -\rho^2/\bar{\mu}$ on the left-hand side :

$$(g_0)_{ac} \bar{m}_c^j - \frac{\bar{\mu}}{s^2 + \mu \bar{\mu}} \sum_{i=1}^s n_a^i m_b^i (g_0)_{bc} \bar{m}_c^j = 0 \quad (A.7)$$

The solution of (A.7) is :

$$n_a^i = \frac{s^2 + \mu \bar{\mu}}{\bar{\mu}} \sum_{j=1}^s (\gamma^{-1})^{ij} (g_0)_{ac} \bar{m}_c^j ; \quad \gamma^{ij} = m_b^i (g_0)_{bc} \bar{m}_c^j \quad (A.8)$$

Finally substituting (A.8) into (A.2) we see that

$$R_{ab} = \frac{s^2 + \mu \bar{\mu}}{\bar{\mu}} \sum_{k,j=1}^s (\gamma^{-1})^{kj} (g_0)_{ac} \bar{m}_c^j m_b^k = \frac{s^2 + \mu \bar{\mu}}{\bar{\mu}} P_{ab} \quad (A.9)$$

It is easy to check that P_{ab} is really a projector of dimension s , thus plugging (A.9) into (A.1) we really obtain (32).

APPENDIX B

In this Appendix we integrate Eq. (39) with (40) on the right-hand side :

$$(\partial_s + i\partial_{x_3})\tau = \frac{1}{4s} \text{Tr} (U_1 + iV_1)^2 \quad (\text{B.1})$$

where $U_1 + iV_1 =$

$$= 2i \frac{s}{\bar{\mu}} \frac{s^2 + \mu\bar{\mu}}{(s+i\mu)^2} P \chi_1^{-1}(is) + \chi_1(is) (U_0 + iV_0) \chi_1^{-1}(is) = A + B \quad (\text{B.2})$$

Using this notation

$$\frac{1}{4s} \text{Tr} (U_1 + iV_1)^2 = \frac{1}{4s} (\text{Tr} A^2 + 2 \text{Tr} AB + \text{Tr} B^2)$$

It is obvious that

$$\frac{1}{4s} \text{Tr} B^2 = \frac{1}{4s} \text{Tr} (U_0 + iV_0)^2 = (\partial_s + i\partial_{x_3})\tau_0 \quad (\text{B.3})$$

On the other hand using the form of χ_1 , (32), the $PP = P$ property of any projector and the fact that for an s dimensional projector $\text{Tr} P = s$ we obtain

$$\begin{aligned} \frac{1}{4s} \text{Tr} A^2 &= \frac{1}{4s} (-4) \frac{s^2}{\bar{\mu}^2} \frac{(s^2 + \mu\bar{\mu})^2}{(s+i\mu)^4} \text{Tr} P \chi_1^{-1}(is) P \chi_1^{-1}(is) = \\ &= \frac{1}{4s} (-4) \frac{s^2}{\bar{\mu}^2} \frac{(s^2 + \mu\bar{\mu})^2}{(s+i\mu)^4} (-s) \frac{\bar{\mu}^2}{s^2} \left(\frac{s+i\mu}{s+i\bar{\mu}} \right)^2 = \frac{s}{s} \left[\frac{s^2 + \mu\bar{\mu}}{(s+i\mu)(s+i\bar{\mu})} \right]^2 \end{aligned} \quad (\text{B.4})$$

Finally, the third term, $1/2s \text{Tr} AB$, has the form :

$$\frac{1}{2s} \text{Tr} AB = \frac{i}{\bar{\mu}} \frac{s^2 + \mu\bar{\mu}}{(s+i\mu)^2} \text{Tr} (U_0 + iV_0) \chi_1^{-1}(is) P \quad (\text{B.5})$$

Exploiting again (32) and the $PP = P$ property we can write

$$Tr(U_0 + iV_0) \chi^{-1}_\lambda(i\zeta) P = i \frac{\bar{\mu}}{\zeta} \frac{\zeta + i\mu}{\zeta + i\bar{\mu}} \zeta \sum_{i,j=1}^S [g_{0,\zeta} + i g_{0,x_3}]_{ab} \bar{m}_b^j (\gamma^{-1})^{ji} m_a^i \quad (B.6)$$

The crucial step in the integration of (B.1) is to find an appropriate expression for (B.6). As (B.6) has no free indices i, j and contains the inverse of γ^{ij} [defined in (A.8) or (33)] one naturally thinks about the possibility of using the ρ and x_3 derivatives of $\ln \det \gamma$. To try this possibility we need an expression for the ρ and x_3 derivatives of $m_a^j, \bar{m}_a^j$; this can be derived by exploiting the fact that $\psi_0(\lambda)$ [that determines m_a^j according to (A.5)] satisfies (23) as well :

$$\begin{aligned} (\partial_\zeta + i\partial_{x_3}) \bar{m}_b^i &= - \frac{\zeta [g_{0,\zeta}^{-1} g_{0,\zeta} + i g_{0,\zeta}^{-1} g_{0,x_3}]_{bc} \bar{m}_c^i}{\zeta + i\bar{\mu}} \\ (\partial_\zeta + i\partial_{x_3}) m_b^i &= - \frac{\zeta m_c^i [g_{0,\zeta} g_{0,\zeta}^{-1} + i g_{0,\zeta} g_{0,x_3}^{-1}]_{cb}}{\zeta + i\mu} \end{aligned} \quad (B.7)$$

Using (B.7) it is straightforward to prove that

$$\begin{aligned} (\partial_\zeta + i\partial_{x_3}) \ln \det \gamma &= Tr \{ (\partial_\zeta + i\partial_{x_3}) \gamma \gamma^{-1} \} = \\ &= - \frac{\zeta^2 + \mu\bar{\mu}}{(\zeta + i\mu)(\zeta + i\bar{\mu})} \sum_{i,j=1}^S m_b^i [g_{0,\zeta} + i g_{0,x_3}]_{bc} \bar{m}_c^j (\gamma^{-1})^{ji} \end{aligned} \quad (B.8)$$

If we combine (B.8), (B.6), (B.5) and (B.3) and (B.4), we finally obtain

$$(\partial_\zeta + i\partial_{x_3}) \tau = \frac{\zeta}{\zeta} \left(\frac{\zeta^2 + \mu\bar{\mu}}{(\zeta + i\mu)(\zeta + i\bar{\mu})} \right)^2 + (\partial_\zeta + i\partial_{x_3}) \ln \det \gamma + (\partial_\zeta + i\partial_{x_3}) \tau_0 \quad (B.9)$$

Therefore the only task remaining is to integrate the first term on the right-hand side of (B.9) :

$$(\partial_s + i\partial_{t_3})Q = \frac{s}{s} \left(\frac{s^2 + \mu\bar{\mu}}{(s+i\mu)(s+i\bar{\mu})} \right)^2 \quad (\text{B.10})$$

We solve (B.10) by making an ansatz for Q in the form

$$Q = a_1 \ln s + a_2 \ln \mu\bar{\mu} + a_3 \ln(s^2 + \mu\bar{\mu}) + a_4 \ln(s^2 + \mu^2)(s^2 + \bar{\mu}^2)$$

Then using (28a) we eliminate the μ derivatives from $(\partial_\rho + i\partial_{x_3})Q$ and determine a_i from requiring the sum to be the right-hand side of (B.10). This way we found

$$Q = \ln \left\{ \frac{s \mu\bar{\mu}}{(s^2 + \mu\bar{\mu}) |s^2 + \mu^2|} \right\}^s \quad (\text{B.11})$$

Finally, substituting (B.11) into (B.9) we obtain the new superpotential :

$$\tau = \tau_0 + \ln \left\{ c \det \gamma \left(\frac{s \mu\bar{\mu}}{(s^2 + \mu\bar{\mu}) |s^2 + \mu^2|} \right)^s \right\}$$

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