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THE QUARK TRIANGLE : APPLICATION TO PION AND ETA DECAYS

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A B S T R A C T

A general expression for the two-photon form factor of neutral pseudoscalar mesons is given in a quark triangle model. Various electromagnetic decay amplitudes are calculated, including Dalitz pair decays and the rare decay into a lepton pair, where an old calculation in a similar (nucleon loop) model is corrected. The results are in agreement with other calculations based on vector meson dominance, giving a new example of Q^2 duality. Some remarks concerning the soft limit in the processes are given.

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1. INTRODUCTION

The two-photon mediated electromagnetic decays of the neutral pseudoscalar mesons (P-mesons), $\pi^0 \rightarrow \gamma\gamma$, $\pi^0 \rightarrow e^+e^-\gamma$, $\eta \rightarrow \gamma\gamma$, $\eta \rightarrow \mu^+\mu^-\gamma$, ... etc. are reasonably well understood theoretically. One exception is, however, the case of the rare pion decay $\pi^0 \rightarrow e^+e^-$, where experiments^{1,2)} persist to give a much higher value than expected from most theoretical models³⁻⁶⁾. The only plausible theoretical model giving a rather high probability for this decay has been the quark loop model (or rather nucleon loop model at that time) proposed by Pratap and Smith some ten years ago⁷⁾.

Since the appearance of their paper, much more information concerning the two-photon coupling of P-mesons has been obtained, both experimentally and theoretically. In particular, it has been realized that in the low-momentum region, the behaviour of the form factor should be dominated by the vector mesons coupling to the respective P-mesons. Likewise, one believes that for large momentum transfers the form factor should behave as the QCD perturbative predictions indicate⁸⁾.

Another important discovery has been that of Q^2 -duality, namely, that if one chooses constituent quark masses appropriately the vector meson dominance (VMD) behaviour of the form factor can be mimicked by the quark loop (i.e. triangle) diagram⁹⁾. In the case of the rare lepton pair decay $P \rightarrow l^+l^-$, this duality should be expected to be even more successful, since there is an integration over the two-photon loop, causing an effective smearing of the form factor.

Since the results of Ref. (7) do not fit well into these ideas, we have found it appropriate to reanalyze the quark loop diagram for this process. Following the dispersion relation method of Ref. (7), we have indeed discovered some calculational errors in their analysis. We have also been able to check our corrected results using a direct Feynman parametrization of the amplitude and performing the integration using adaptive Monte Carlo numerical integration methods. In addition, performing the soft pseudoscalar limit (letting the mass m_P tend to zero), there are considerable simplifications and the two-loop diagram can be

calculated directly. However, as we shall see, in this process the soft limit is not a sensible approximation (especially for the pions), since the amplitude has an imaginary part from the two-photon real intermediate state starting at $t = m_P^2 = 0$. This means that the real part of the amplitude has a "cusp" at $t = 0$ (its derivatives are infinite), and the extrapolation from $t = 0$ to $t = m_P^2$ is not at all smooth.

On the other hand, our final results for the physical P masses are in very good agreement with the duality ideas mentioned above, bringing agreement between various theoretical models. However, there is still the disturbing [but therefore also interesting⁵⁾] disagreement with the experimental rate for the $\pi^0 \rightarrow e^+e^-$ decay.

The remainder of this work is organized as follows. In Section 2, the general two-photon form factor is discussed and calculated in the quark loop model. Applications are given for $P \rightarrow \gamma\gamma$, $P \rightarrow \ell^+\ell^-\gamma$ and $e^\pm e^\pm \rightarrow e^\pm e^\pm P$. In Section 3, the process $P \rightarrow l^+l^-$ is calculated using the dispersive method of Ref. (7). In Section 4, the direct perturbative evaluation of the two-loop diagram is treated, together with a discussion concerning the soft limit. Section 5 contains the conclusions and a discussion of the results.

2. THE TWO-PHOTON FORM FACTOR IN THE QUARK LOOP MODEL

We first calculate the Feynman amplitude for $P \rightarrow \gamma^* \gamma^*$ using the notation in Fig. 1, where we parametrize the $Pq\bar{q}$ coupling by $g\bar{u}(p_q)\gamma^5 v(p_{\bar{q}})$. Introducing $P^2 = (k_1 + k_2)^2 = s_0$ and the quark mass M , we have (with $s_1 = k_1^2$ and $s_2 = k_2^2$ the invariant mass squared of the photons)

$$\begin{aligned} A(s_0, s_1, s_2; M) &\equiv ie^2 \langle e_q^2 \rangle f_{\gamma\gamma} \epsilon_{\nu\mu\rho\sigma} k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma F_P(s_1, s_2; M) = \\ &= 8g e^2 \langle e_q^2 \rangle M \epsilon_{\nu\mu\rho\sigma} k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma \times C, \end{aligned} \quad (1)$$

where ϵ_1 and ϵ_2 are the photon polarizations and where the form factor $F_P(s_1, s_2; M)$ is normalized to the real two-photon coupling $F_P(0, 0; M) = 1$.

Our results will mostly be for only one quark species. In the general case, a summation over different quarks should be performed but in most of our applications the u and d quarks give the main contribution and we put $M_u = M_d = M$. Our results can be trivially modified to the general case. In Eq. (1), $\langle e_q^2 \rangle$ is the average charge squared, taking into account a colour factor and the quark composition of the various P-mesons.

The integral in Eq. (1) has the appearance of a scalar three-point function with internal masses M and external (mass)² s_0, s_1 and s_2 . Indeed, one has

$$C = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[q^2 - M^2 + i\epsilon][q^2 + k_1^2 - M^2 + i\epsilon][(q+k_1+k_2)^2 - M^2 + i\epsilon]} \quad (2)$$

This enables us to employ the very useful techniques devised by 't Hooft and Veltman¹⁰⁾ for evaluating three-point functions. Working through the algebra one may obtain a simple expression displaying the symmetry in s_0, s_1 and s_2 :

$$\frac{(2\pi)^4 C}{i\pi^2} = \frac{1}{\lambda^{\frac{1}{2}}(s_0, s_1, s_2)} \sum_{i=0,1,2} \left\{ \text{Li}_2\left(\frac{\gamma_i}{\gamma_i - \gamma_{-i}}\right) + \text{Li}_2\left(\frac{\gamma_i}{\gamma_i - \gamma_{-i}}\right) - \text{Li}_2\left(\frac{\gamma_i - 1}{\gamma_i - \gamma_{-i}}\right) - \text{Li}_2\left(\frac{\gamma_i - 1}{\gamma_i - \gamma_{-i}}\right) \right\}, \quad (3)$$

plus additional terms containing logarithms which never enter for real (positive or negative) values of s_i and M^2 . Here λ is the usual triangle function,

$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$, and

$$\gamma_i = \frac{1}{2} \left\{ 1 + \frac{s_j + s_k - s_i}{\lambda^{\frac{1}{2}}(s_0, s_1, s_2)} \right\}, \quad i \neq j \neq k; \quad i \neq k \quad (4)$$

$$\gamma_{\pm i} = \frac{1}{2} \left\{ 1 \pm \sqrt{1 - (4M^2 - i\epsilon)/s_i} \right\} \quad (5)$$

Our definition of the Spence function $\text{Li}_2(x)$ coincides with that of Lewin¹¹⁾:

$$\text{Li}_2(x) = - \int_0^x \frac{\ln(1-t)}{t} dt \quad (6)$$

We can now study a few special cases of the general expression (3).

i) $s_0 = m_P^2$, $s_1 = s_2 = 0$. This gives the decay constant $f_{\gamma\gamma}$ into real photons

$$f_{\gamma\gamma} = -\frac{gM}{\pi^2 m_P^2} \arcsin^2\left(\frac{m_P}{2M}\right), \quad (7)$$

where for simplicity here and in the following we assume $m_P < 2M$. In order to check this result we can take the soft limit, $m_P \rightarrow 0$, thus obtaining $f_{\gamma\gamma}(0) = -g/(4\pi^2 M)$ which is, of course, the well-known formula for the triangle anomaly.

ii) $s_0 = m_P^2$, $s_1 = s$, $s_2 = 0$. For $4m_1^2 < s < m_P^2$, this corresponds to the physical process of single Dalitz pair decay, $P \rightarrow l^+ l^- \gamma$. Using the properties of the Spence functions¹¹⁾ and keeping track of the $i\epsilon$ prescription of Eq. (5), we find from Eqs. (3) and (7)

$$F_P(s, 0; M) = \frac{m_P^2}{m_P^2 - s} \left\{ 1 - \frac{\arcsin^2\left(\frac{\sqrt{s}}{2M}\right)}{\arcsin^2\left(\frac{m_P}{2M}\right)} \right\}. \quad (8)$$

The results for $m_P^2 > 4M^2$ and $s > 4M^2$ are easily obtained by analytical continuation of Eq. (8) (the latter case may be relevant for, for example, the process $e^+e^- \rightarrow P + \gamma$).

Taking the soft limit of the Dalitz form factor (8) leads to the well-known result⁹⁾

$$F_P^{\text{soft}}(s, 0; M) = \frac{4M^2}{s} \arcsin^2\left(\frac{\sqrt{s}}{2M}\right). \quad (9)$$

Expanding Eqs. (8) and (9) close to $s = 0$ gives

$$F_P(s, 0; M) \approx 1 + \frac{s}{m_P^2} \left\{ 1 - \frac{m_P^2/4M^2}{\arcsin^2\left(\frac{m_P}{2M}\right)} \right\} + \dots \quad (10)$$

and

$$F_P^{\text{soft}}(s, 0; M) \approx 1 + \frac{1}{12M^2} s + \dots, \quad (11)$$

giving explicit expressions for the experimentally interesting slope at $s = 0$, $F'_p(s, 0; M)_{s=0}$. For the decay $\pi^0 \rightarrow e^+e^-\gamma$ one has $s < m_\pi^2 \ll M^2$, and in this case also Eq. (10) effectively reduces to the expression (11). Unfortunately, the data on the pion form factor are as yet not conclusive, with different experiments not even agreeing on the sign of the slope. One can only compare our result with the well-known VMD form factor for the Dalitz pairs

$$F_\pi^{\text{VMD}}(s, 0; M_{s,w}) = \frac{M_{s,w}^2}{M_{s,w}^2 - s} \simeq 1 + \frac{1}{M_{s,w}^2} s + \dots \quad (12)$$

From Eqs. (11) and (12) it is clear that the perturbative calculation presented here and the non-perturbative result (12) can be matched if one chooses $M = M_{u,d} \sim 200\text{--}300$ MeV. This indicates that in these applications of Q^2 -duality, the relevant quantity is the constituent quark mass and not the current mass.

To investigate more closely how Q^2 -duality works, not only for the slope near the origin but for the whole Dalitz form factor, one may consider the decay $\eta \rightarrow \mu^+\mu^-\gamma$, which covers a larger range of s . In this case the analysis is slightly more complicated due to the appearance of strange quarks in the η meson [for definiteness, we take $\eta \sim (u\bar{u} + d\bar{d} - s\bar{s})/\sqrt{3}$, as in Ref. (9)]. The generalization of Eqs. (8) to (10) and the use of constituent masses $M_{u,d} = 300$ MeV, $M_s = 400$ MeV lead to the slope parameter $F'_\eta(s, 0) = 1.3 \text{ GeV}^{-2}$ which is not too far off from the experimental value⁽¹²⁾ $1.9 \pm 0.4 \text{ GeV}^{-2}$. The curves for $F_\eta(s, 0)$ and $F_\eta^{\text{soft}}(s, 0)$ are shown in Fig. 2, together with the experimental data⁽¹²⁾ and the simple and successful VMD prediction. The shape of these curves is seen to be quite similar and reproduces rather well the general trend of the data. In fact, if one wants to fine-tune parameters (which is not our intent) one may get an almost perfect Q^2 -duality by reducing the quark masses introduced in F_η (F_η^{soft}) by $\sim 10\%$ (15%). We note, however, that a nucleon loop as originally considered by Pratap and Smith⁽⁷⁾ gives a much too high characteristic mass for the form factor and has been shown to be ruled out by the data⁽¹³⁾. One may point out that for all applications in this section, taking the soft limit is a rather good approximation, even for the

relatively heavy η meson. This can be traced to the fact that the amplitude for $P \rightarrow \gamma^* \gamma^*$ does not become singular (i.e. develops an imaginary part) until m_P^2 , s_1 or s_2 exceeds $4M^2$. The real part of the amplitude is quite smooth for invariant mass values away from these singularities (in particular, for $m_P^2 \sim 0$; s_1 and s_2 small). This is in contrast to the situation in the decay $P \rightarrow l^+ l^-$ (to be treated in the next section), where a branch cut starts for the 2γ intermediate state at $m_P^2 = 0$.

iii) For $s_1 = -Q^2 < 0$ and $s_2 = 0$, which is the kinematical situation for exclusive single-tagged two-photon production of P-mesons (in the equivalent photon approximation) in e^+e^- machines, the corresponding evaluation of Eq. (3) gives

$$F_P(-Q^2, 0; M) = \frac{-gM}{2\pi^2 f_{\gamma\gamma}} \frac{1}{m_P^2 + Q^2} \left\{ \frac{1}{2} \ln^2 \left(\frac{\beta_Q + 1}{\beta_Q - 1} \right) + 2 \arcsin^2 \left(\frac{m_P}{2M} \right) \right\}, \quad (13)$$

where $\beta_Q = \sqrt{1 + 4M^2/Q^2}$.

We finally state the result for the soft limit. From Eq. (3) one easily finds

$$F_P^{\text{soft}}(-Q^2, -Q^2; M) = \frac{2M^2}{\beta_Q Q^2} \ln \left(\frac{\beta_Q + 1}{\beta_Q - 1} \right). \quad (14)$$

It is quite interesting to note that in all the special cases considered here, i.e. when at least one of the s_i equals zero, the expression (3) reduces to elementary functions. However, in the general case with unequal $s_i \neq 0$, Spence functions appear explicitly.

3. THE DECAY $P \rightarrow l^+ l^-$ IN A QUARK LOOP MODEL, DISPERSIVE CALCULATION

We now consider the rare decay $P \rightarrow l^+ l^-$ as calculated through the two-photon loop diagram in Fig. 3. In principle, we could now take the expression (3) for the general two-photon coupling and insert that form factor in the photon loop of Fig. 3. However, this is very inconvenient in practice, since one encounters integrations over branch cuts which are difficult to parametrize. A direct calculation of the diagram using Feynman parametrization and adaptive numerical integration

routines is also difficult, for two reasons. First, for the physical processes $\pi^0 \rightarrow e^+e^-$, $\eta \rightarrow \mu^+\mu^-$, etc., there is an imaginary (absorptive) part of the diagram corresponding to the two-photon real intermediate state. The corresponding singularity in Feynman parameter space has to be isolated before numerical integration can be performed. Secondly, even if one manages that, the actual computation using, for example, Monte Carlo techniques is very slowly converging, especially for $\pi^0 \rightarrow e^+e^-$ where m_p/m_1 is large. However, to check our calculations, we will use the technique in Section 4 for special cases where these two difficulties do not arise.

For the moment, we will choose instead the elegant unitary method of Ref. (14), which was adopted by Pratap and Smith (PS) in their calculation⁷⁾. Since we disagree with these authors in many intermediate steps and in the final results, we present those parts of the calculation in some detail.

The idea is to first use CP invariance to write the Feynman amplitude as

$$A(P \rightarrow l^+l^-) = i \bar{u}(p_-) \gamma^5 F(t) v(p_+), \quad (15)$$

where $t = m_p^2$. This gives the decay rate

$$\Gamma(P \rightarrow l^+l^-) = \frac{m_p}{8\pi} \beta_0 |F(m_p^2)|^2, \quad (16)$$

where $\beta_0 = \sqrt{1 - 4m_l^2/m_p^2}$. Projecting out the 1S_0 state of the lepton pair and using the unitarity condition for the amplitude, one finds

$$\begin{aligned} \text{Abs } F(t) = & \frac{-i}{(2t)^{\frac{1}{2}}} \frac{(2\pi)^4}{2} \int d\mu_n \delta^{(4)}(p_+ + p_- - \sum p_n) \times \\ & \times \langle n | T | (l^+l^-)_{^1S_0} \rangle^* A(P \rightarrow n), \end{aligned} \quad (17)$$

where the integration is over the phase space volume of the possible intermediate states n . Because of CP conservation, the absorptive part of $F(t)$ is equal to the imaginary part of $F(t)$ after one has added the contributions from all three intermediate states displayed in Fig. 4. The real part for the physical amplitude is then obtained by using an unsubtracted dispersion relation:

$$\operatorname{Re} F(m_\pi^2) = \frac{P}{\pi} \int_0^\infty \frac{\operatorname{Abs} F(t)}{t - m_\pi^2} dt. \quad (18)$$

As we will verify later by the direct calculation of the diagram, no subtractions are needed here.

a) Contribution from the two-photon state

For the two-photon cut (Fig. 3a) we need, according to Eq. (17) $A(P \rightarrow \gamma\gamma)$ and $\langle \gamma\gamma | T | (\ell^+ \ell^-)_{S_0} \rangle^*$. Using Eq. (8), we find for $0 < t < 4M^2$

$$A(P \rightarrow \gamma\gamma) = \frac{i\alpha M g}{\pi t} \left[4 \arcsin^2\left(\frac{\sqrt{t}}{2M}\right) \right] \epsilon_{\nu\mu\sigma} k_1^\nu k_2^\mu \epsilon_1^\sigma \epsilon_2^\sigma \quad (19)$$

where the expression in square brackets is analytically continued to $\pi^2 - \ln^2[(1+\beta_q)/(1-\beta_q)]$ for $t > 4M^2$, where $\beta_q = \sqrt{1 - 4M^2/t}$. The imaginary part of this analytical continuation is cancelled later by a corresponding contribution from the two-quark intermediate state.

It is now a straightforward task to calculate the transition amplitude for the singlet S lepton pair into two photons and perform the integration over the two-photon phase space according to Eq. (17). We find

$$\operatorname{Abs} F^{\gamma\gamma}(t) = \frac{g\alpha^2 m_\ell M}{4\pi\beta_\ell t} \ln\left(\frac{1+\beta_\ell}{1-\beta_\ell}\right) \left\{ \pi^2 - \ln^2\left(\frac{1+\beta_q}{1-\beta_q}\right) \right\} \theta(t - 4M^2), \quad (20)$$

where $\beta_1 = \sqrt{1 - 4m_1^2/t}$. For $4m_1^2 \leq t \leq 4M^2$, this becomes

$$\operatorname{Abs} F^{\gamma\gamma}(t) = \frac{g\alpha^2 m_\ell M}{\pi\beta_\ell t} \ln\left(\frac{1+\beta_\ell}{1-\beta_\ell}\right) \left\{ \arcsin^2\left(\frac{\sqrt{t}}{2M}\right) \right\}, \quad (21)$$

and for $0 \leq t < 4M_1^2$ the expression is

$$\operatorname{Abs} F^{\gamma\gamma}(t) = \frac{2g\alpha^2 m_\ell M}{\pi t \sqrt{4m_\ell^2/t - 1}} \arctan\left(\sqrt{4m_\ell^2/t - 1}\right) \left\{ \arcsin^2\left(\frac{\sqrt{t}}{2M}\right) \right\}. \quad (22)$$

This last formula is given incorrectly in PS⁷⁾.

b) Contribution from the $q\bar{q}$ intermediate state

This is the technically most complicated part of the calculation. The amplitude for $l^+l^- \rightarrow q\bar{q}$ to be evaluated is the two-photon box diagram. Fortunately, since we only need the singlet S projection there are no infrared divergences. Summing over the quark spins and taking the 1S_0 projection of the leptons, one finds

$$\text{Abs } F^{q\bar{q}}(t) = \frac{i \beta_q \alpha^2 m_l M q}{\pi^3 t} \frac{1}{2} \int_{-1}^1 d\cos\theta I(t, \cos\theta), \quad (23)$$

where $\beta_q = \sqrt{1 - 4M^2/t}$ and $I(t, \cos\theta)$ is the basic two-photon loop integral

$$I(t, \cos\theta) = \int d^4k \frac{2 \{P^2 k^2 - (P \cdot k)^2\}}{k^2 (P-k)^2 [(P-k)^2 - m_l^2] [(q-k)^2 - M^2]}, \quad (24)$$

where q is the quark momentum. In PS the integral was evaluated using the Mandelstam representation. We prefer to use a direct evaluation employing the methods of Behrends et al.¹⁵⁾ for the two-photon loop integral. Whichever method one chooses, the calculation is algebraically tedious. We find for the imaginary part of the integral

$$\begin{aligned} \text{Im } I(t, \cos\theta) = & - \frac{\pi^2 t}{2 \lambda^{\frac{1}{2}}(s, m_l^2, M^2)} \left\{ \left(\ln\left(-\frac{s}{t}\right) + 2 \right) \ln \left| \frac{s - M^2 - m_l^2 - \sqrt{\lambda}}{s - M^2 - m_l^2 + \sqrt{\lambda}} \right| + \right. \\ & + 2 \text{Li}_2(-r_1(s)) - 2 \text{Li}_2(-r_2(s)) + \frac{1}{2} \ln^2(r_1(s)) - \frac{1}{2} \ln^2(r_2(s)) + \\ & \left. + 2 \ln\left(\frac{\sqrt{\lambda}}{-s}\right) \ln\left(\frac{r_1(s)}{r_2(s)}\right) \right\} \times \theta(t - 4M^2), \end{aligned} \quad (25)$$

where

$$\begin{aligned} S &= m_l^2 + M^2 - \frac{1}{2} t (1 - \beta_l \beta_q \cos\theta), \\ r_1(s) &= (\sqrt{\lambda} - s - M^2 + m_l^2) / (\sqrt{\lambda} + s + M^2 - m_l^2), \\ r_2(s) &= (\sqrt{\lambda} + s + m_l^2 - M^2) / (\sqrt{\lambda} - s - m_l^2 + M^2), \\ \sqrt{\lambda} &\equiv \lambda^{\frac{1}{2}}(s, m_l^2, M^2). \end{aligned}$$

This expression also disagrees with that of PS^{*}). Equation (25) can now be inserted into Eq. (23), where for practical reasons the angular integration has to be performed numerically.

c) Contribution from the $q\bar{q}\gamma$ intermediate state

This part consists of evaluating the amplitudes for $P \rightarrow q\bar{q}\gamma$ (bremsstrahlung radiation from one of the quarks) and $(l^+l^-)_{1S_0} \rightarrow q\bar{q}\gamma$ (Dalitz-pair-like decay of the lepton system into quark pair plus photon). We disagree with PS in intermediate results, e.g. we find

$$\begin{aligned} \sum_{\text{spins}} A(P \rightarrow q\bar{q}\gamma) A^*((l^+l^-)_{1S_0} \rightarrow q\bar{q}\gamma) = \\ = \frac{2^{11} i g \pi^2 \alpha^2 m_l M}{s \sqrt{2t}} \frac{1}{(1 - \beta_l^2 \cos^2 \theta)} \frac{1}{(1 - \beta_q'^2 \cos^2 \theta')} , \end{aligned} \quad (26)$$

where s is the invariant mass of the quark pair, $\beta_q'^2 = 1 - 4M^2/s$, and θ and θ' are the polar angles in the lepton pair and quark pair c.m.s., respectively. [This differs by a factor $2/s$ from the PS equation (4.29).] However, we agree on the final expression for this contribution

$$\text{Abs } F^{q\bar{q}\gamma}(t) = \frac{g \alpha^2 m_l M}{\pi t \beta_l} \ln \left(\frac{1 + \beta_l}{1 - \beta_l} \right) \mathcal{J}(\beta_q) \theta(t - 4M^2) , \quad (27)$$

where

$$\mathcal{J}(\beta_q) = 2L \ln \left(\frac{2}{1 - \beta_q} \right) + 2L_{12} \left(-\frac{1 + \beta_q}{1 - \beta_q} \right) + \frac{\pi^2}{6} - \frac{L^2}{2} - \frac{1 + \beta_q^2}{2} L + \beta_q , \quad (28)$$

where $L = \ln [(1 + \beta_q)/(1 - \beta_q)]$.

*) Our result, Eq. (25), has been checked in a few special cases already studied in Ref. (5). On the other hand, we have been informed by J. Smith (private communication) that the error in Eq. (4.19) in PS seems to amount to an incorrect sign of the third term in that expression. Correcting that, our expressions agree numerically.

d) Numerical results for dispersive calculation

Using Eqs. (8) and (21), we of course recover the "unitarity lower bound" for the branching ratio $B_P \equiv \Gamma(P \rightarrow l^+ l^-) / \Gamma(P \rightarrow \gamma\gamma)$ obtained from only the absorptive part of $F^{\gamma\gamma}(m_P^2)$ [Eq. (21)]:

$$B_P^{\text{unitary}} = \alpha^2 \left(\frac{m_l}{m_P} \right)^2 \frac{1}{2\beta_l} \ln^2 \left(\frac{1+\beta_l}{1-\beta_l} \right). \quad (29)$$

To this must be added the contribution from the dispersive part of $F(m_P^2)$ given by Eq. (18), where $\text{Abs } F(t)$ is given by Eqs. (20), (21), (22), (23), and (27). In Tables 1 and 2 we present our results for the dispersive part for $\pi^0 \rightarrow e^+ e^-$ and $\eta \rightarrow \mu^+ \mu^-$, respectively. We present results both for the "realistic" value 300 MeV for the quark mass M and a nucleon value $M = 939$ MeV; the latter to compare with the PS calculation and to show the effect of a large effective cut-off mass for the form factor. Also shown are the corresponding branching ratios $B_P^{l^+ l^-}$ in units of the unitarity lower bound (29). It is interesting to note that the values of $B_\pi^{e^+ e^-}$ and $B_\eta^{\mu^+ \mu^-}$ obtained in this model for $m_q = 300$ MeV are virtually identical to those obtained in various VMD models^{4,16)} and in the QCD-inspired model of Ref. (5). This indicates that there is a "duality" at work also in this decay (i.e. that the bare loop of quarks calculated with constituent masses gives the same result as a form factor saturated by vector mesons, i.e. $q\bar{q}$ bound states).

This reinforces the statement made in Ref. (5) that if one wants agreement with the known properties of the electromagnetic decays $\pi^0 \rightarrow \gamma\gamma$, $\pi^0 \rightarrow e^+ e^- \gamma$, $\eta \rightarrow \gamma\gamma$, and $\eta \rightarrow \mu^+ \mu^- \gamma$, it is very difficult to obtain values for $B_\pi^{e^+ e^-}$ and $B_\eta^{\mu^+ \mu^-}$ significantly different from the unitarity lower bound. At present, the experimental value for $\eta \rightarrow \mu^+ \mu^-$ ¹⁷⁾ is in agreement with this, whereas two recent experiments on $\pi^0 \rightarrow e^+ e^-$ claim values^{1,2)} of ~ 4 times the unitarity lower bound, which is difficult to reconcile with conventional theoretical ideas [for a recent attempt, see Ref. (18) but also Ref. (19)].

We note that our results in Tables 1 and 2 for $M = 939$ MeV are quite different from those of PS, apart from the $q\bar{q}\gamma$ contribution.

4. PERTURBATIVE CALCULATION OF $P \rightarrow l^+ l^-$

The Feynman amplitude in Fig. 2 can be written

$$A(P \rightarrow l^+ l^-) = \frac{32 \langle e_q^2 \rangle e^4 g m_l M}{\sqrt{2} m_p (2\pi)^8} \times \int \int d^4 q d^4 k \frac{2 \{ P^2 k^2 - (P \cdot k)^2 \}}{[q^2 - M^2][(\bar{q} - k)^2 - M^2][(P - q)^2 - M^2] k^2 [P - k]^2 [(P - k)^2 - m_l^2]} \quad (30)$$

The Feynman parametrization of the two-loop integral can be performed in many ways, and it is important to find one which is simple and numerically tractable. Using the general method of Bjorken and Drell²⁰⁾ gives, for instance, a very complicated expression that we found difficult to treat with our numerical routines. Introducing again the function $F(t)$ defined in Eq. (15), the simplest parametrization we could find is

$$F(t) = \frac{2\alpha^2 m_l M g}{\pi^2} \times \mathcal{J} \quad (31)$$

where

$$\mathcal{J} = \int_0^1 dx \int_0^x dy \int_0^1 dz \int_0^z du \int_0^u dv \frac{a^3 v^2 \{m_l^2 - t/4\} - \frac{3}{2} a \mathcal{R}}{\mathcal{R}^2}, \quad (32)$$

where

$$a = (x - y)(x - y - 1)$$

$$b = y(x - y)$$

$$c = ty(y - 1) + M^2,$$

and

$$\mathcal{R} = a^2 m_l^2 v^2 - ac(1-z) + t \{ b^2(1-z)^2 + a^2 v - a^2 u(1+v-u) + - 2ab(1-z)(u-v/2) \}. \quad (33)$$

To check the results obtained in Section 3, we calculated Eq. (32) using the adaptive Monte Carlo integration program VEGAS, written by G.P. Lepage for QED applications²¹⁾. The calculations were performed for negative values of t and for

several different values for M and m_1 . The results obtained agreed with those using Eq. (18) within the statistical errors of the Monte Carlo computation. One typical result: for $t = -m_\eta^2$, $M = 300$ MeV, $m_1 = 105.6$ MeV, we obtained $F(-m_\eta^2) = 1.8 \times 10^{-6}$ g using Eq. (18) and $F(-m_\eta^2) = (1.80 \pm 0.01) \times 10^{-6}$ g when using VEGAS to evaluate Eq. (32) with 10 iterations and using 10^4 evaluations of the integrand per iteration. The χ^2 per iteration was around 0.8, indicating a healthy behaviour of the numerical integral.

We have also checked Eq. (30) in the soft limit, $t \rightarrow 0$. In this case, the integral in Eq. (27) simplifies considerably. A direct calculation gives (at $\vec{P} = 0$)

$$\text{Re } F(0) = \frac{2i\alpha^2 g m_\ell}{\pi^4 M} \int d^4 k \frac{\vec{k}^2}{(k^2)^2 (k^2 - 2p_{\ell,k})} \int_0^1 dx \int_0^{1-x} d\gamma \frac{-M^2}{k^2(x+\gamma)(1-x-\gamma) - M^2}, \quad (34)$$

which can be evaluated to be

$$\text{Re } F(0) \simeq \frac{3m_\ell \alpha^2 g}{2\pi^2 M} \left\{ \ln\left(\frac{M}{m_\ell}\right) + \frac{17}{12} + \mathcal{O}\left(\frac{m_\ell^2}{M^2} \ln\left(\frac{M}{m_\ell}\right)\right) \right\}. \quad (35)$$

Also this result coincides with what we get from the dispersive calculation in this limit. However, it is clear from Eq. (22) for the absorptive (imaginary) part of F near $t = 0$ that forming derivatives of Eq. (18) will give divergent results at that point. Consequently, one expects the real part of F to show a cusp at $t = 0$, therefore invalidating the smoothness assumption necessary for the soft limit. Indeed, as is shown in Fig. 5 (for $M = 300$ MeV; $m_1 = 511$ keV), the point $t = 0$ is very peculiar and $\text{Re } F(0)$ differs strongly both in sign and in magnitude from the value at the physical mass, $\text{Re } F(m_\pi^2)$. Of course the rapid change of $\text{Re } F(t)$ near $t = 0$ is connected to the rapid change of $\text{Im } F(t)$ [Eqs. (21) and (22)] close to the threshold at $t = 0$ for real photons in the intermediate state. Indeed, the quantity $|F(t)|^2$ relevant for the rate of $P \rightarrow l^+ l^-$ has a somewhat smoother behaviour, at least for $t > 0$. However, $|F(0)|^2$ still overestimates the physical value $|F(m_\pi^2)|^2$ for $\pi^0 \rightarrow e^+ e^-$ by around 40%, so the conclusion must be that the soft limit is a bad approximation in this process.

On the other hand, varying the physical pion mass by, say, $\pm 20\%$ has very little influence on $|F(m_\pi^2)|^2$, so in this sense our prediction for $\pi^0 \rightarrow e^+e^-$ is stable.

5. CONCLUSIONS

In this paper, we have concerned ourselves with the question of electromagnetic decay and production of neutral pseudoscalar mesons, e.g. $P \rightarrow \gamma\gamma$, $P \rightarrow \ell^+\ell^-\gamma$, $e^+e^- \rightarrow P\gamma$, $e^+e^- \rightarrow e^+e^-P$, $P \rightarrow l^+l^-$, etc. This has been done in the framework of a simple quark triangle diagram with a phenomenological "hard" (pointlike) coupling $g\gamma^5$ between the meson and the constituent quarks coupled to it. Since this diagram has been remarkably successful in calculating the $\pi^0 \rightarrow \gamma\gamma$ width (effectively corresponding to all external masses = 0 in the triangle), the idea has been that the same diagram should give sensible results also for the other electromagnetic processes. In those cases, one makes an off-shell continuation of one or more of the external, initially massless legs. Using recently developed, powerful techniques¹⁰⁾, we have been able to express the general triangle amplitude $P \rightarrow \gamma^*\gamma^*$ in a simple, symmetric and closed form involving the Spence function [Eq. (3)]. When at least one of the external masses is zero, the expression reduces to elementary functions, for which we have also given the results.

We have given particular attention to the single Dalitz pair decays, $P \rightarrow \gamma l^+l^-$, which are experimentally interesting for the moment, with a good statistics experiment existing for $\eta \rightarrow \mu^+\mu^-\gamma$ ¹²⁾ and with experiments currently being planned to resolve the question of the pion form factor. For this Dalitz form factor, we have found that Q^2 -duality may be fulfilled in a quite local sense, at least for low $|Q^2|$, provided that one chooses constituent quark masses appropriately. In this case, the shape of the experimentally measured $\eta \rightarrow \mu^+\mu^-\gamma$ form factor is successfully reproduced. The soft pseudoscalar limit was found to be an excellent approximation for $\pi^0 \rightarrow e^+e^-\gamma$ and reasonable also for $\eta \rightarrow \mu^+\mu^-\gamma$.

The main part of this paper has been devoted to the technically complicated calculation of $P \rightarrow l^+l^-$ proceeding through both a quark loop and a photon loop.

This has been done by using a dispersion relation [correcting an earlier similar calculation⁷⁾], by using direct numerical Monte Carlo integration and, in the soft-P limit, by an explicit analytical calculation. The agreement obtained with these different methods gives confidence to our results. Our main conclusions for $P \rightarrow 1^+ 1^-$ are the following:

- i) The effective smearing of the two-photon form factor achieved by integrating over the loop makes the Q^2 -duality very successful, as one would expect. That is, models based on coupling to bound states (vector mesons) give essentially identical predictions for the branching ratios $\Gamma(P \rightarrow 1^+ 1^-)/\Gamma(P \rightarrow \gamma\gamma)$ as the perturbative triangle diagram model predictions, if one identifies the form factor cut-off m_V (mass of the vector meson) with $2M$ (twice the quark mass).
- ii) The soft pseudoscalar limit is not smooth in this process, due to the two-photon cut starting at $m_P^2 = 0$. The effect is particularly dramatic for $\pi^0 \rightarrow e^+ e^-$ (Fig. 5), where the reduced amplitude is different both in sign and in magnitude at $t = 0$ compared to the physical point $t = m_\pi^2$. Although the calculations simplify considerably, we must thus warn against employing the soft limit in this process.
- iii) With duality having been restored through our correction of the work⁷⁾, virtually model-independent predictions emerge for the electromagnetic decays of neutral pseudoscalar mesons, in particular $\pi^0 \rightarrow e^+ e^-$ and $\eta \rightarrow \mu^+ \mu^-$. We predict branching ratios close to the unitarity lower bound [Eq. (29)] for both these decays. We wish to end this paper by appealing to experimentalists to try to clear up the enigmatic situation concerning the π^0 decays, especially $\pi^0 \rightarrow e^+ e^- \gamma$ and $\pi^0 \rightarrow e^+ e^-$. Either the experimental value for $\pi^0 \rightarrow e^+ e^-$ has been considerably overestimated [this is, in fact, what happened with the $\eta \rightarrow \mu^+ \mu^-$ decay¹⁷⁾], or current theoretical ideas concerning the $P\gamma^*\gamma^*$ coupling need a thorough revision. We also remind the reader of the remote but exciting possibility of exotic, non-electromagnetic contributions to the decay $\pi^0 \rightarrow e^+ e^-$ [see Ref. (5)].

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REFERENCES

- 1) J. Fischer et al., Phys. Lett. 73B (1978) 364; (E) 76B (1978) 663.
- 2) R.E. Mischke et al., Phys. Rev. Lett. 48 (1982) 1153.
J.S. Frank et al., Los Alamos preprint LA-UR 82-1954 (1982).
- 3) S.M. Berman and D.A. Geffen, Nuovo Cimento 18 (1960) 1192.
- 4) C. Quigg and J.D. Jackson, UCRL report 18487 (1968), unpublished.
- 5) L. Bergström, Z. Phys. C14 (1982) 129.
- 6) K. Babu and E. Ma, Phys. Lett. 119B (1982) 449.
- 7) M. Prataap and J. Smith, Phys. Rev. D 5 (1972) 2020.
- 8) G.P. Lepage and S.J. Brodsky, Phys. Rev. D 22 (1980) 2157.
- 9) A. Bramon and E. Massó, Phys. Lett. 104B (1981) 311.
- 10) G. 't Hooft and M. Veltman, Nucl. Phys. B153 (1979) 365.
- 11) L. Lewin, Polylogarithms and associated functions, North-Holland (New York), 1981.
- 12) R.I. Dzhelyadin et al., Phys. Lett. 94B (1980) 548.
- 13) L. Bergström and H. Snellman, Z. Phys. C8 (1981) 363.
- 14) B.R. Martin, E. de Rafael and J. Smith, Phys. Rev. D 2 (1970) 1979;
(E) Phys. Rev. D 4 (1971) 272.
- 15) F.A. Behrends, K.J.F. Gaemers and R. Gastman, Nucl. Phys. B63 (1973) 381.
- 16) Ll. Ametller, L. Bergström, A. Bramon and E. Massó, to appear.
- 17) R.I. Dzhelyadin et al., Phys. Lett. 97B (1980) 471.
- 18) G.B. Tupper and M.A. Samuel, Phys. Rev. D 26 (1982) 3302.
- 19) L. Bergström and E. Ma, Univ. of Hawaii at Manoa preprint UH-511-485-82
(November, 1982).
- 20) J.D. Bjorken and S.D. Drell, Relativistic Quantum Fields, McGraw-Hill (New York), 1965.
- 21) G.P. Lepage, J. Comp. Phys. 27 (1978) 192.

Table 1

Summary of results for $\pi^0 \rightarrow e^+e^-$, where the amplitude is defined as $i\bar{u}(p_-)F(m_\pi^2)\gamma^5 v(p_+)$ [Eq. (15)]. The various contributions from the diagrams in Figs. 4a-c to $\text{Re } F(m_\pi^2)$ are shown, together with the total real and imaginary parts of $F(m_\pi^2)$. In the last column is shown the branching ratio $\Gamma(\pi^0 \rightarrow e^+e^-)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ in units of the unitary lower bound [Eq. (29)], which in this case is $B^{\text{unit}} = 4.75 \times 10^{-8}$. All values of F are in units of $g \times 10^{-8}$, where g is the coupling constant at the $\pi q\bar{q}$ vertex.

M (MeV)	Re $F_{q\bar{q}}(m_\pi^2)$	Re $F_{q\bar{q}\gamma}(m_\pi^2)$	Re $F^{\gamma\gamma}(m_\pi^2)$	Tot Re F	Im F	<u>branch.ratio</u> <u>unitary limit</u>
939	- 9.60	1.43	7.45	-0.72	2.58	1.08
300	-22.6	3.98	14.7	-3.95	8.19	1.23

Table 2

Same as Table 1, but for the decay $\eta \rightarrow \mu^+\mu^-$. In this case, $B^{\text{unit}} = 1.07 \times 10^{-5}$. The values of F are in units of $g \times 10^{-6}$.

M (MeV)	Re $F_{q\bar{q}}(m_\eta^2)$	Re $F_{q\bar{q}\gamma}(m_\eta^2)$	Re $F^{\gamma\gamma}(m_\eta^2)$	Tot Re F	Im F	<u>branch.ratio</u> <u>unitary limit</u>
939	-2.61	1.36	3.46	2.21	1.71	2.67
300	-4.00	3.43	4.89	4.32	8.28	1.27

Figure captions

- Fig. 1 : Diagrams for $P \rightarrow \gamma^* \gamma^*$ in a quark loop model. The four-momenta and polarizations of the photons are k_1, k_2 and ϵ_1, ϵ_2 , respectively.
- Fig. 2 : The square of the Dalitz form factor for $\eta \rightarrow \mu^+ \mu^- \gamma$ as a function of s , the invariant mass squared of the muon pair. Calculations are for the vector meson dominance (VMD) model (solid line), quark loop model with $M_u = M_d = 300$ MeV, $M_s = 400$ MeV (dashed line) and the quark loop model in the soft limit ($m_\eta = 0$) (dotted line). Experimental points are from Ref. (12).
- Fig. 3 : Amplitude for the lowest-order electromagnetic decay process $P \rightarrow l^+ l^-$ in the quark loop model. A diagram with the two photons crossed should also be included.
- Fig. 4 : (a-c): The three possible unitary cuts contributing to the absorptive part of the amplitude according to Eq. (17).
- Fig. 5 : The real part of F [Eq. (18)] as a function of $t = m_P^2$. Here the case of $\pi^0 \rightarrow e^+ e^-$ is considered, with $m_e = 511$ keV and $m_q = 300$ MeV. The physical value of $\text{Re } F$ and the soft limit are indicated. The values of $\text{Re } F$ are in units of $g \times 10^{-8}$.

Fig. 1

Fig. 2

Fig. 3

(a)

(b)

(c)

Fig. 4

Fig. 5