Electrodynamics of the planar model: Its phase diagram, continuum limit, and mass spectrum

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We consider the lattice version of the Abelian Higgs model in $1+1$ dimensions with the Higgs field constrained to have a constant length. In statistical-mechanics terminology this is the planar Heisenberg model coupled to the electromagnetic field. Both the Euclidean Lagrangian and Minkowski Hamiltonian formulations of this lattice gauge theory are used to study its phase diagram, continuum limit, and mass spectrum. The theory contains two dimensionless coupling constants: κ , the length of the local spin variable, and $\chi = (2g^2 a^2)^{-1}$, where g is the electrodynamic coupling and a is the lattice spacing. We find that the theory confines charges for all κ and fixed, finite χ . However, in the continuum limit ($a \rightarrow 0$) the theory loses its confinement property and exhibits the Higgs mechanism for $\kappa^2 > \kappa_c^2$. The critical point κ_c^2 is estimated to lie between 0.6 and 0.8. If $\kappa^2 < \kappa_c^2$, the continuum limit of the theory retains its confining property and appears to be simply the free electromagnetic field (a confining linear potential).

I. INTRODUCTION

The lattice version of quantum chromodynamics (QCD)¹⁻³ is a formulation of the theory which confines quarks for sufficiently large coupling. It is hoped that the continuum limit of the lattice theory also leads to confinement, but this is a delicate question to study since the renormalization group indicates that the coupling vanishes in this regime.⁴ It is certainly possible that the lattice theory passes to an unconfined phase in the limit, although analyses of non-Abelian theories in $3+1$ dimensions happily suggest the contrary.⁵

In this article we study a simple model field theory whose lattice version confines but whose continuum version does so only for a certain range of bare parameters. We will identify the physical origin of the confinement property and understand why it can be lost in the continuum limit. The theory is one whose lattice version has been studied by a host of authors⁶: the Abelian Higgs model in $1+1$ dimensions with the Higgs field constrained to have a constant length. The lattice version of the model consists of plane rotors at sites coupled together in a gauge-invariant fashion via an electromagnetic field. We therefore refer to it as the "electrodynamics of the planar model." Since the phase diagram of the planar model has been understood in the last few years, we can borrow methods of analysis and physical intuition from statistical mechanics.

We find that the model has the following features. On the lattice it can be rewritten as an ensemble

of vortices of finite size interacting through a short-range potential. These vortices are the same as those of the planar model, the coupling to the electromagnetic field having given them finite sizes and short-ranged interactions. These vortices disorder the system destroying the Higgs mechanism and lead to confinement. We make this point by computing the Wilson loop integral in the Euclidean formulation of the model. The theory contains two dimensionless parameters, κ the length of the spin variable, and $\chi = (2g^2 a^2)^{-1}$, where g is the electrodynamic coupling and a is the lattice spacing. The computation of the Wilson loop integral shows that for $\kappa^2 \gtrsim 1/\pi$ the vortex gas attains a vanishing density in the continuum limit ($ga \rightarrow 0$) and the confining potential vanishes there. This indicates that the theory exhibits the Higgs phenomenon in the continuum limit. In fact the theory reduces to that of a free massive vector meson in this limit.

The continuum limit for $\kappa^2 \lesssim 1/\pi$ appears to be completely different. In this case the Euclidean formulation of the theory indicates that the vortices do not become dilute as $a \rightarrow 0$. This suggests that the continuum limit of the theory should continue to confine. To study this possibility we turn to the Hamiltonian formulation of the theory and compute the interquark potential and low-lying mass spectrum of the theory in strong-coupling expansions through eighth order in χ . We find that the interquark potential vanishes as $a \rightarrow 0$ for $\kappa^2 > \kappa_c^2 \approx 0.6-0.8$ by extrapolating the strong-coupling expansions to the continuum ($\chi \rightarrow \infty$) limit. We find

no evidence for any phase transitions at finite x . However, for $\kappa^2 < \kappa_c^2$ the series calculations indicate that the interquark potential *survives* in the continuum limit. The series expansions for the lowest-lying vector and scalar states of the lattice theory also indicate that there is a critical point at $\kappa_c^2 \approx 0.6-0.8$. For κ^2 well above the critical point, the ratio of the masses of the scalar and vector states is found to be about 1.84 from the strong-coupling expansions. This is reasonably close to the value 2 expected in the classical continuum limit. However, for $\kappa^2 < \kappa_c^2$ the masses themselves appear to grow large as $a \rightarrow 0$ suggesting that in an exact calculation they become infinitely massive and disappear from the spectrum. In summary, for $\kappa^2 < \kappa_c^2$ the continuum limit of the lattice theory confines and has no nontrivial spectrum: it appears to be just the free electromagnetic field. Additional evidence for these results is obtained from a very simple implementation of the renormalization-group idea to the Euclidean formulation of the model.

We also study local states in the theory which in strong coupling consist of a fractional charge of magnitude α at one site, a charge of magnitude $-\alpha$ at a neighboring site, and a fractional flux tube connecting them in accordance with Gauss's law. We argue that the parameter α labels *bands* in the lattice theory which are simply related to various discrete states of the continuum limit of the theory. The strong-coupling expansions for these bands are studied in detail.

The theory is also briefly studied in a background field generated by half a unit of charge at positive spatial infinity and minus half a unit of charge at negative spatial infinity. For κ^2 sufficiently large its vacuum state is a condensate of half-asymptotic particles.⁷

II. EUCLIDEAN FORMULATION OF THE ELECTRODYNAMICS OF THE PLANAR MODEL

As discussed in the Introduction, we will construct a tractable lattice model of quark confinement which has the "pathological" property of passing to an unconfined phase at $a=0$ for a range of values of its bare parameters. We will see that for this range of parameters the lattice theory has field configurations which cause confinement but these configurations disappear in the continuum limit. As discussed by Polyakov,⁸ compact quantum electrodynamics in 2+1 dimensions has this property. The theory we study here contains matter fields and has a more interesting phase diagram. It is a lattice formulation of the Abelian Higgs model in 1+1 dimensions,

$$\mathcal{L} = |(\partial_\mu - igA_\mu)\phi|^2 - \frac{1}{4}F_{\mu\nu}^2, \quad (2.1)$$

where $\phi^*\phi = \kappa^2 = \text{const.}$

As a first step towards an understanding of this theory delete the electromagnetic field from Eq. (2.1) and write the theory on a Euclidean two-dimensional lattice. The constraint $\phi^*\phi = \kappa^2$ is solved by writing ϕ as $\kappa \exp(i\theta)$. A possible two-dimensional lattice action is then

$$A = J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j), \quad (2.2)$$

where $\langle ij \rangle$ denotes a sum over nearest neighbors. Equation (2.2) defines the planar model which is the subject of much interest in statistical mechanics. That the naive continuum limit of Eq. (2.2) is the $A_\mu = 0$ form of Eq. (2.1) is easily verified by considering slowly varying fields, replacing discrete differences by derivatives, and accounting for factors of the lattice spacing a appropriately.

It is useful to review briefly some facts and history concerning the planar model. Old high-temperature expansions⁹ indicated the existence of a continuous phase transition at a critical temperature T_c . The character of the phase transition was conceptually quite puzzling. The low-temperature phase could not be characterized by a spontaneous magnetization [i.e., $\langle \exp(i\theta_i) \rangle \neq 0$] because a continuous symmetry cannot be spontaneously broken in a two-dimensional system with short-range forces.¹⁰ This puzzle was solved by the work of Kosterlitz and Thouless¹¹ who pointed out that there are two distinct types of stationary spin configurations of the action Eq. (2.2). A constant-spin field is one case. Another is a vortex configuration shown in Fig. 1. Vortices have several distinctive properties. They are quantized. Their action varies as $\ln(R/a)$ where R is the linear size of the spin system and a is the lattice spacing. To

FIG. 1. The spin configuration making up a vortex.

see this approximate Eq. (2.2) by the continuum expression, $J \sum_r [\nabla \theta(r)]^2$, and evaluate it for the vortex of Fig. 1,

$$\text{action (vortex)} \approx \pi J \ln(R/a). \quad (2.3)$$

The singular character of this spin configuration is reflected in the fact that Eq. (2.3) diverges as $a \rightarrow 0$. As will be discussed further below, the fact that Eq. (2.3) has a large volume divergence means that vortices are important in the model only for sufficiently large temperatures. In fact, Kosterlitz and Thouless¹¹ suggested that it is the presence or absence of vortices which distinguish the low- and high-temperature phases of the model. They suggested that below T_c the only significant excitations are conventional spin waves, but as T is increased above T_c bound pairs of a vortex and an antivortex can dissociate and influence the long-distance behavior of the system. A renormalization-group analysis¹² of the model indicates that below T_c the spin-spin correlation functions are power-behaved in the distance between spins while above T_c , where vortices are relevant, the correlation function falls exponentially. Kosterlitz and Thouless¹¹ presented a very informal but instructive determination of T_c . They argued that free vortices will appear in the system when their contribution to the free energy F becomes negative. The energy E of a free vortex is recorded in Eq. (2.3) and its entropy S is obtained by counting the number of places it can occur on the lattice,

$$E = \pi J \ln(R/a), \quad S = k \ln(R/a)^2, \quad (2.4a)$$

where k = Boltzmann's constant. So,

$$F = E - TS = (\pi J - 2kT) \ln(R/a) \quad (2.4b)$$

becomes negative for T greater than $T_c = \pi J/2k$. It is easy to argue that for $T < T_c$ the spin-spin correlation function is power-behaved. To see this suppose $T < T_c$ so vortices can be ignored. Replace the action of Eq. (2.2) by its continuum form $J \sum_r [\nabla \theta(r)]^2$ and extend the range of the variable θ to $[-\infty, \infty]$. Then,

$$\langle e^{i(\theta_i - \theta_j)} \rangle = \frac{\int \prod_i d\theta_i \exp[-(J/kT) \sum_r (\nabla \theta)^2] e^{i(\theta_i - \theta_j)}}{\int \prod_i d\theta_i \exp[-(J/kT) \sum_r (\nabla \theta)^2]} \quad (2.5)$$

which is easily evaluated since all the integrals are Gaussian,

$$\langle e^{i(\theta_i - \theta_j)} \rangle = \exp \left[\frac{kT}{2J} \Delta(r_i - r_j) \right] \\ \sim_{|r_i - r_j| \gg a} \exp \left[-\frac{kT}{4\pi J} \ln \left| \frac{r_i - r_j}{r_0} \right| \right], \quad (2.6)$$

where $\Delta(r_i - r_j)$ is a massless propagator (on a lattice) and we have used its asymptotic form in the last step. So, finally,

$$\langle e^{i(\theta_i - \theta_j)} \rangle \sim \left| \frac{r_0}{r_i - r_j} \right|^{kT/4\pi J}, \quad (2.7)$$

for $T < T_c$. We learn from this that spin waves cause the correlation function to be power-behaved so $\langle e^{i\theta_i} \rangle = 0$ and there is no spontaneous magnetization at any T . A renormalization-group analysis¹² shows that the line of fixed points in Eq. (2.7) terminates into a continuous phase transition at T_c above which the correlation function becomes an exponential in $(r_i - r_j)$.

Now return to Eq. (2.1). This theory is trivially soluble. Make a gauge transformation,

$$D_\mu = A_\mu + \frac{1}{g} \partial_\mu \theta, \quad (2.8)$$

where θ is defined via the relation,

$$\phi = \kappa e^{i\theta}, \quad \kappa = \text{positive constant}. \quad (2.9)$$

Substituting into Eq. (2.1) gives

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + g^2 \kappa^2 D_\mu^2, \quad (2.10)$$

which describes a free vector meson of mass

$$m_V = \sqrt{2} g \kappa. \quad (2.11)$$

This theory does not, of course, confine. If we couple charged static sources to the field A_μ , and then imagine slowly separating a pair of such impurities (of charge $+e$ and $-e$, say), the energy required would be finite since the gauge field D_μ is free and massive.

This analysis does not, of course, mean that the continuum limit of a lattice formulation of this model will be Eq. (2.10). For example, the lattice planar model becomes a massless free field in the continuum limit only for sufficiently low temperatures. This fact suggests that the continuum limit of the electrodynamics of the planar model will be Eq. (2.10) only for κ sufficiently large. The analysis presented in this paper will support this view and will make it precise.

Now we shall construct the lattice version of this theory and derive its confining properties. Before beginning a detailed analysis we can anticipate many of the results using the facts concerning the planar model reviewed above. The lattice version of Eq. (2.1) will contain both spin waves and vortices. But, since Eq. (2.1) contains a dimensionful parameter g , or equivalently the mass m_V , the energetics of the theory are different from the planar model. The parameter m_V will act as an infrared cutoff so the action of a vortex will vary as $\ln(m_V a)$. So, for finite $m_V a$, vortices will be important excitations. Their presence will

lead to confinement. For κ^2 sufficiently large confinement will be lost precisely in the continuum limit since the action of a vortex diverges at this point.

The lattice version of Eq. (2.1) is obtained following the original ideas of Wilson² and Polyakov¹ on compact QED. Define the vector variable $B_\mu(r) = agA_\mu(r)$ on the link connecting the sites r and $r + \hat{\mu}$ on a square Euclidean two-dimensional lattice labeled by integers $r = (r_1, r_2)$. Δ_μ will denote a discrete difference operator,

$$\Delta_\mu f(r) = f(r + \hat{\mu}) - f(r).$$

The lattice action replacing Eq. (2.1) is then

$$A = 2\kappa^2 \sum_{r, \mu} \cos[\Delta_\mu \theta(r) + B_\mu(r)] + \frac{1}{2g^2 a^2} \sum_{r, \mu, \nu} \cos[\Delta_\mu B_\nu(r) - \Delta_\nu B_\mu(r)]. \quad (2.12)$$

Some of the important properties of A are the following. It is locally gauge invariant, i.e., the lattice gauge transformation,

$$B_\mu(r) \rightarrow B_\mu(r) + \Delta_\mu \Lambda(r), \quad \theta(r) \rightarrow \theta(r) - \Lambda(r) \quad (2.13)$$

leaves A invariant. $B_\mu(r)$ is effectively an angular variable so in the partition function it will be integrated from 0 to 2π . $\theta(r)$ is also an angular variable.

To discuss the confining properties of this theory we need to construct a suitable correlation function. This is the Wilson loop integral,

$$\left\langle \exp\left(ie \int_C A_\mu dx^\mu\right) \right\rangle = \frac{\int \prod_r d\theta(r) \prod_{r', \mu} dA_\mu(r') \exp(-A + ie \int_C A_\mu dx^\mu)}{\int \prod_r d\theta(r) \prod_{r', \mu} dA_\mu(r') e^{-A}}, \quad (2.14)$$

where C is a closed contour. Equation (2.14) gives the action required to move a static charge e around the contour C . It is convenient to consider a contour C such as the one in Fig. 2 having $T \gg R$. In this case heavy quarks of charge $+e$ and $-e$ are adiabatically separated a distance R , are then held there for $T \gg R$, and are then brought back together. The energy required to separate the quarks a distance R is then obtained from the Wilson line integral through the relation

$$E(R) = -\lim_{T \rightarrow \infty} \frac{1}{T} \ln \left\langle \exp\left(ie \int_C A_\mu dx^\mu\right) \right\rangle \quad (2.15)$$

and $E(R)$ can be used to label the phases of the theory. If $E(R) \rightarrow 0$ as $R \rightarrow \infty$, the theory is in the *unconfined* phase. If $E(R) \sim R$ as $R \rightarrow \infty$ for e different than an integral multiple of g and if $E(R) \rightarrow 0$ as $R \rightarrow \infty$ for e equal to an integral multiple of g , then the theory is in the *confining* phase. These are the standard definitions and the reader should consult the literature for further discussion.¹³

Next we must write $\int A_\mu dx^\mu$ using lattice variables. Introduce the vector field $J_\mu(r)$ to label the contour C :

$$J_\mu(r) = \begin{cases} +1 & \text{if the link } r \rightarrow r + \hat{\mu} \text{ is on the contour } C, \\ -1 & \text{if the link } r + \hat{\mu} \rightarrow r \text{ is on the contour } C, \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition $B_\mu = agA_\mu$ we have the correspondence,

$$e \int_C A_\mu dx^\mu \rightarrow \frac{e}{g} \sum_C B_\mu J_\mu. \quad (2.16)$$

Finally, if we define

$$Z(J) = \int \prod_r d\theta(r) \prod_{r', \mu} dA_\mu(r') \times \exp\left(-A - i \frac{e}{g} \sum_C B_\mu J_\mu\right), \quad (2.17)$$

FIG. 2. The contour C chosen to compute the inter-quark potential.

then Wilson's correlation function is $Z(J)/Z(0)$.

Our next task is the evaluation of Eq. (2.17). This will be done using the Villain approximation.¹⁴ To motivate the following we concede that we can only evaluate Gaussian functional integrals in closed form. However, the action A depends on angular variables θ and B_μ and we must preserve this property of the model in any approximation procedure. Villain invented the periodic Gaussian approximation for just this situation. He noted that for large β ,

$$e^{-\beta(1-\cos\theta)} \approx \frac{1}{\sqrt{2\pi\beta}} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-n^2/2\beta}, \quad (2.18a)$$

which is just the Fourier series analysis of the periodic function $\exp[-\beta(1-\cos\theta)]$. We will make the following replacements in the action Eq. (2.12),

$$e^{-2\kappa^2[1-\cos(\Delta_\mu\theta-B_\mu)]} \rightarrow \sum_{n_\mu=-\infty}^{\infty} e^{in_\mu(\Delta_\mu\theta-B_\mu)} e^{-n_\mu^2/4\kappa^2}, \quad (2.18b)$$

$$\exp\left(-\frac{1}{2g^2a^2}[1-\cos(\Delta_\mu B_\nu-\Delta_\nu B_\mu)]\right)$$

$$\rightarrow \sum_{l_{\mu\nu}=-\infty}^{\infty} e^{il_{\mu\nu}(\Delta_\mu B_\nu-\Delta_\nu B_\mu)} e^{-g^2a^2l_{\mu\nu}^2/4}.$$

These replacements preserve the periodicity of the variables θ and B_μ , the local gauge invariance and the continuum limit. Therefore, they serve as another *definition* of the lattice theory for all values κ^2 and g^2a^2 of the model. Since the Villain form of the theory is tractable we concentrate on it from here on. In addition, to preserve periodicity in the presence of an external current loop, we should give the matter field a charge p and consider external charges $q=0,1,\dots,p$. This point will be left implicit below.

Using the replacements Eq. (2.18) in the partition function $Z(J)$ we can then do the functional integrations over $\theta(r)$ and $B_\mu(r)$,

$$Z(J) = \prod_{r,\mu,\nu} \sum_{l_{\mu\nu}(r)} \prod_{r',\mu'} \delta_{\Delta_\nu l_{\mu\nu}(r') + (e/g)J_\mu(r') + n_\mu(r'),0}$$

$$\times \prod_{r''} \sum_{n_\mu(r'')} \prod_{r'''} \delta_{\Delta_\mu n_\mu(r'''),0}$$

$$\times e^{-n_\mu^2(r'')/4\kappa^2} e^{-g^2a^2l_{\mu\nu}^2(r)/4}. \quad (2.19)$$

These integrals have generated constrained sums over integer-valued fields. This phenomenon has been noted and studied recently by several authors.^{15,16} It is best to solve the constraints explicitly. Since the fields n_μ and J_μ are divergence free, they can be written as curls of other integer-

valued fields,

$$n_\mu = \epsilon_{\mu\nu} \Delta_\nu n, \quad J_\mu = \epsilon_{\mu\nu} \Delta_\nu J. \quad (2.20)$$

Finally, the constraint

$$\Delta_\nu l_{\mu\nu} + \frac{e}{g} J_\mu + n_\mu = 0$$

can be solved,

$$l_{\mu\nu} = m^\mu (m \cdot \Delta)^{-1} \left(\frac{e}{g} J^\nu + n^\nu \right) - m^\nu (m \cdot \Delta)^{-1} \left(\frac{e}{g} J^\mu + n^\mu \right), \quad (2.21)$$

where the expression $(m \cdot \Delta)^{-1}$ means a discrete line integral along the direction m ,

$$(m \cdot \Delta)^{-1} f(r_1, r_2) = \sum_{m'=1}^{r_2} f(r_1, m'). \quad (2.22)$$

We choose m in the two direction for convenience. This choice will soon disappear from the mathematics. Next we can write the Gaussian functions appearing in Eq. (2.19) in terms of n , and J ,

$$n_\mu n_\mu = (\Delta_\mu n)^2, \quad (2.23)$$

$$l_{\mu\nu} l_{\mu\nu} = 2 \left(\frac{e}{g} J + n \right)^2.$$

So, now $Z(J)$ has the simpler form,

$$Z(J) = \sum_{n(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r [\Delta_\mu n(r)]^2 - \frac{g^2a^2}{2} \sum_r \left(n(r) + \frac{e}{g} J(r) \right)^2 \right]. \quad (2.24)$$

If g^2 were zero, we could recognize this expression as the interface-roughening model.¹⁷ Equation (2.24) is a convenient expression in our case only when g^2a^2 is large so that the sum over $n(r)$ is quickly convergent. We will return to Eq. (2.24) below, but now we will write $Z(J)$ in a convenient form for small g^2a^2 . To do this we borrow another trick from the recent studies of the planar model¹⁶—the Poisson summation formula which allows us to rewrite the sum over integers in a different way,

$$\sum_{l=-\infty}^{\infty} h(l) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi h(\phi) e^{2\pi i m \phi}, \quad (2.25)$$

where h is arbitrary. Applying this trick to Eq. (2.24),

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\phi(r) \sum_{m(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_\mu \phi)^2 - \frac{g^2 a^2}{2} \sum_r \left(\phi + \frac{e}{g} J \right)^2 + 2\pi i \sum_r m(r) \phi(r) \right]. \quad (2.26)$$

Doing the integrations over $\phi(r)$ and dropping an overall constant we have

$$Z(J) = \sum_{m(r)=-\infty}^{\infty} \exp \left(-4\pi^2 \kappa^2 \sum_{r,r'} m(r) G(r-r'; m_v a) m(r') - 4\kappa^2 \pi i \frac{e}{g} (g^2 a^2) \sum_{r,r'} m(r) G(r-r'; m_v a) J(r') \right. \\ \left. + 4\kappa^2 (g^2 a^2)^2 \frac{e^2}{g^2} \sum_{r,r'} J(r) G(r-r'; m_v a) J(r') - \frac{1}{2} g^2 a^2 \frac{e^2}{g^2} \sum_r J(r)^2 \right), \quad (2.27)$$

where G is the lattice propagator coming from inverting the lattice form of the operator $(\vec{\nabla}^2 + m_v^2 a^2)$,

$$G(r; m_v a) = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{4 - 2\cos k_x - 2\cos k_y + m_v^2 a^2}. \quad (2.28)$$

To expose the physics in Eq. (2.27) consider the case $J=0$,

$$Z(J=0) = \sum_{m(r)=-\infty}^{\infty} \exp \left(-4\pi^2 \kappa^2 \sum_{r,r'} m(r) \times G(r-r'; m_v a) m(r') \right), \quad (2.29)$$

which expresses the partition function as integer-valued fields $m(r)$ interacting through short-range potentials $G(r, m_v a)$. As will become clearer below $m(r)$ represent the vortices which are interacting through the field identified in Eq. (2.10)—the massive, free vector field. Note that if $g=0$ here $Z(0)$ reduces to the two-dimensional Coulomb gas.¹⁶

Equation (2.27) is a convenient form of the partition function for small a . We will now evaluate it for a suitable contour ($T \gg R$) and verify that for κ sufficiently large the theory confines except at $a=0$. This analysis will be easy for two reasons. First, the self-energy of a vortex diverges as $a \rightarrow 0$, so their density on the lattice will be small. In addition, they interact through short-range forces, so for “ a ” small they can be approximated by a free, noninteracting gas. The term $r=r'$ in the sum of Eq. (2.29) gives the self-energy

$$G(0, m_v a) \sim -\frac{1}{2\pi} \ln(m_v a), \quad (2.30)$$

so neglecting interactions between vortices,

$$Z(0) \approx \sum_{m(r)} \exp \left(2\pi \kappa^2 \ln(\sqrt{2} \kappa g a) \sum_r m(r)^2 \right). \quad (2.31)$$

The interactions between vortices can be ignored if they usually lie outside each other's range of influence which is $\sim (m_v a)^{-1}$. But Eq. (2.31) implies that the probability to find a vortex on a given lattice site is $(m_v a)^{2\pi \kappa^2}$. So, the probability that a second vortex lies close enough to interact with a given one is roughly

$$(m_v a)^{2\pi \kappa^2} (m_v a)^{-2} = (m_v a)^{2\pi \kappa^2 - 2}. \quad (2.32)$$

So, if $\kappa^2 > 1/\pi$ this quantity vanishes as $a \rightarrow 0$ and Eq. (2.31) is a sensible approximation to $Z(J=0)$. If κ violates this inequality the dilute-gas approximation fails and other methods of analysis (an attractive possibility is a theory of screening implemented via a renormalization group) are necessary. Strong-coupling expansions will be used in the next section of this paper to investigate the theory for arbitrary κ^2 .

The analysis of Eq. (2.27) is now straightforward and bears a strong resemblance to the instanton calculations of Ref. 18. First note that for the contour of Fig. 2, $J(r) = +1$ if r is inside the contour and zero otherwise. Then $J_\mu = \epsilon_{\mu\nu} \Delta_\nu J$ is $+1$ on the left vertical piece of C and -1 on the right vertical piece. Aside from an overall factor of $g^2 a^2 (e^2/g^2)$ the last two terms in Eq. (2.27) become

$$(m_v a)^2 \sum_{\substack{r, r' \\ \text{inside}}} G(r-r'; m_v a) - RT. \quad (2.33)$$

But for R and $T \gg (m_v a)^{-1}$, the dominant piece of the double sum in Eq. (2.33) is easily obtained. Fix r' inside and not near the contour. Then the sum over r can be done because then the extremities in the sum can be ignored. But

$$\sum_r G(r; m_v a) = \frac{1}{(m_v a)^2}, \quad (2.34)$$

so Eq. (2.33) becomes

$$(m_v a)^2 \left(\sum_{\substack{r' \\ \text{inside}}} \right) \left(\sum_r G(r; m_v a) \right) - RT \\ = (m_v a)^2 \frac{1}{(m_v a)^2} RT - RT = 0 \quad (2.35)$$

to leading order in RT . Similar approximations give

$$\sum_{r, r'} m(r) G(r - r'; m_v a) J(r') \approx \frac{1}{m_v^2} \sum_r^{\text{inside}} m(r). \quad (2.36)$$

So, finally we have a simple expression for $Z(J)$,

$$Z(J) = \sum_{m(r)} \exp \left(2\kappa^2 \pi \ln(\sqrt{2} \kappa g a) \sum_r m^2(r) - 2\pi i \frac{e}{g} \sum_r^{\text{inside}} m(r) \right). \quad (2.37)$$

Since we are taking a to be small the sum over vortices can be truncated after $m(r) = 0, \pm 1$. Then if r is inside the contour,

$$\sum_{m(r)} \exp \left(2\pi \kappa^2 \ln(m_v a) m(r)^2 - 2\pi i \frac{e}{g} m(r) \right) \\ \approx 1 + 2 \exp \left[2\pi \kappa^2 \ln(m_v a) \right] \cos \left(2\pi \frac{e}{g} \right) \\ \approx \exp \left[2e^{2\pi \kappa^2 \ln(m_v a)} \cos \left(2\pi \frac{e}{g} \right) \right]. \quad (2.38)$$

So, we obtain for the Wilson loop integral,

$$\frac{Z(J)}{Z(0)} \approx \exp \left\{ 2(m_v a)^{2\pi \kappa^2} \sum_r^{\text{inside}} \left[\cos \left(2\pi \frac{e}{g} \right) - 1 \right] \right\}. \quad (2.39)$$

So,

$$\frac{Z(J)}{Z(0)} \cong \exp \left[-2(m_v a)^{2\pi \kappa^2} \left(1 - \cos 2\pi \frac{e}{g} \right) RT \right], \quad (2.40)$$

from which we read off $E(R)$ from the prescription in Eq. (2.15)

$$E(R) = 2m_v^2 (m_v a)^{2\pi \kappa^2 - 2} \left(1 - \cos 2\pi \frac{e}{g} \right) R, \quad (2.41)$$

where we have restored physical units (i.e., R could be in GeV^{-1} units).

We read off several facts from Eq. (2.41). The coefficient of the linear force law is periodic in e/g . If e/g is an integer it vanishes. Thus, we have the standard picture of confinement. Note that the coefficient of $E(R)$ vanishes in the continuum limit for $\kappa^2 > 1/\pi$ and confinement is lost precisely at $a = 0$.

Finally, let us briefly consider confinement for large a . Then it is best to return to Eq. (2.24) for $Z(J)$,

$$Z(J) = \sum_{n(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_\mu n)^2 - \frac{g^2 a^2}{2} \sum_r \left(n(r) + \frac{e}{g} J(r) \right)^2 \right]. \quad (2.24)$$

The absence of a long-range force in the case $e/g = \text{integer}$ is obvious from this expression—simply translate the sum over $n(r)$ and the factor $(e/g)J(r)$ disappears except for some short-range effects. Now suppose $e/g \neq \text{integer}$. Then the most important configuration $\{n(r)\}$ will be

inside C : $n(r) = \text{nearest integer to } -\frac{e}{g}$,

outside C : $n(r) = 0$.

Suppose $|e/g|$ is much smaller than unity, so the dominant configuration $\{n(r)\}$ is zero everywhere. Aside from some short-range terms, the Wilson loop integral becomes

$$Z(J)/Z(0) \approx e^{-e^2 a^2 RT/2}, \quad (2.42a)$$

so that

$$E(R) = \frac{1}{2} e^2 R \quad (2.42b)$$

in physical units. Recognize this result as characteristic of an electric flux tube, completely unaffected by the vacuum, extending between the impurities $\pm e$. The behavior of $E(R)$ for larger $|e|$ is now clear: $E(R)$ rises as e^2 until $|e| = \frac{1}{2}|g|$ where its slope changes sign. It vanishes at $|e| = |g|$. In general, $E(R)$ is a periodic function of $|e|$ with period $|g|$.

In summary now, we see that this theory displays confinement both for strong coupling (large $g^2 a^2$) and for weak coupling (small $g^2 a^2$) and loses this property precisely at $a^2 = 0$. Although we have not analyzed the intermediate coupling regime in detail, there is no reason to expect a transition separating weak and strong coupling. We could, of course, use the interface roughening form of the theory, Eq. (2.24), and develop a strong-coupling expansion for the string tension. Instead, we turn to the Hamiltonian form of the theory and use strong coupling and Padé approximants to analyze the same physics for all values of κ^2 and κ .

III. HAMILTONIAN FORMULATION

In order to study the theory in more detail we return to real space-time and consider the Hamiltonian formulation of the theory on a one-dimensional spatial lattice, with time kept continuous.³ We choose that class of gauges for which $A_0 = 0$. For the Lagrangian of Eq. (2.1) where $\phi^\dagger \phi = \kappa^2$, our momentum conjugate to $\theta(\phi \equiv \kappa e^{i\theta})$ is

$$\pi = 2\kappa^2 \partial_0 \theta \quad (3.1)$$

and the Hamiltonian density

$$\begin{aligned} \mathcal{H} &= \frac{1}{4\kappa^2} \pi^2 + \kappa^2 (\partial_1 \theta)^2 + 2\kappa^2 g (\partial_1 \theta) A_1 + \kappa^2 g^2 A_1^2 + \frac{1}{2} E^2 \\ &= \frac{1}{4\kappa^2} \pi^2 + \partial_1 \phi^\dagger \partial_1 \phi - ig \phi^\dagger \bar{\partial}_1 \phi A_1 + \kappa^2 g^2 A_1^2 + \frac{1}{2} E^2. \end{aligned} \quad (3.2)$$

Now consider the transition to the lattice of spacing a . The equal-time commutation relation

$$[\pi(x), \theta(y)] = -i\delta(x-y) \quad (3.3a)$$

becomes

$$[\pi(n), \theta(m)] = -i\delta_{mn}, \quad (3.3b)$$

where

$$\begin{aligned} \pi(n) &\equiv a\pi(na), \\ \theta(n) &\equiv \theta(na), \end{aligned} \quad (3.4)$$

with n an integer. In terms of $\phi(n) = \kappa e^{i\theta(n)}$ Eq. (3.3) becomes

$$\begin{aligned} [\pi(n), \phi(m)] &= -\delta_{mn} \phi(n), \\ [\pi(n), \phi^\dagger(m)] &= \delta_{mn} \phi^\dagger(m). \end{aligned}$$

Following the technique used for (1+1)-dimensional QED¹⁹ and both (1+1)- and (3+1)-dimensional QCD,³ we assign our electromagnetic field to the links of the lattice and define

$$U(n, n+1) = \exp[igaA_1(n)]. \quad (3.5)$$

Our gauge-invariant Hamiltonian $H = \int dx \mathcal{H}(x)$ then becomes

$$\begin{aligned} H &= \frac{1}{2} \sum_n ag^2 L^2(n) + \sum_n \frac{1}{2a\kappa^2} \pi^2(n) \\ &\quad - \frac{1}{a} \sum_n [\phi^\dagger(n) U(n, n+1) \phi(n+1) \\ &\quad + \phi^\dagger(n+1) U^\dagger(n, n+1) \phi(n) - 2\kappa^2], \end{aligned} \quad (3.6)$$

on the lattice where

$$L(n) = \frac{1}{g} E(na). \quad (3.7)$$

Since E and A_1 are canonically conjugate, the lattice variables $L(n)$ and $agA_1(n)$ are canonically conjugate,

$$[L(n), agA_1(m)] = \delta_{n,m}. \quad (3.8)$$

For the purpose of setting up the lattice perturbation theory, we define

$$\begin{aligned} W &= \frac{2}{ag^2} H = \sum_n \left\{ L^2(n) + \frac{x}{\kappa^2} \pi^2(n) \right. \\ &\quad \left. - 4x [\phi^\dagger(n) U(n, n+1) \phi(n+1) \right. \\ &\quad \left. + \text{H.c.} - 2\kappa^2] \right\}, \end{aligned} \quad (3.9)$$

where $x = 1/(2a^2g^2)$. We can now formulate perturbation theory about $a = \infty$, i.e., $x = 0$ and attempt to continue series expansions to $a = 0$, i.e., $x = \infty$, by use of Padé approximants.

It will often be found useful to consider the simpler Hamiltonian obtained from the above when $\pi^2(n) = 0$, i.e., ϕ is treated classically (unquantized). In this approximation the modes for each link decouple and the theory is exactly soluble. The Hamiltonian is (up to an overall factor of $2/ag^2$)

$$\begin{aligned} W' &= \sum_n \{ L^2(n) - 4x [\phi^\dagger(n) U(n, n+1) \phi(n+1) \\ &\quad + \text{H.c.} - 2\kappa^2] \}. \end{aligned} \quad (3.10)$$

A simple gauge transformation reduces all the $\phi(n)$'s to κ giving

$$\begin{aligned} W' &= \sum_n \{ L^2(n) - 4x\kappa^2 [U(n, n+1) \\ &\quad + U^\dagger(n, n+1) - 2] \}. \end{aligned} \quad (3.11)$$

As promised, the Hamiltonians for each link are decoupled. For any link the Hamiltonian is of the form

$$W_{\text{link}} = L^2 - 4x\kappa^2 (U + U^\dagger - 2). \quad (3.12)$$

If we set $U = e^{i\chi}$ then on the space of functions of χ ,

$$W_{\text{link}} = -\frac{d^2}{d\chi^2} - 8x\kappa^2 (\cos\chi - 1). \quad (3.13)$$

Making the substitution $\eta = \frac{1}{2}\chi$, the eigenvalue equation

$$W_{\text{link}} \psi(\eta) = \omega_i \psi(\eta) \quad (3.14)$$

becomes

$$\left[\frac{d^2}{d\eta^2} + 32x\kappa^2 (\cos\eta - 1) + 4\omega_i \right] \psi(\eta) = 0, \quad (3.15)$$

which is Mathieu's equation. In later calculations we will take advantage of the fact that the eigenvalues ω_i have been computed and tabulated.²⁰

IV. PERTURBATION THEORY AND THE VACUUM

In order to perturb around $x=0$, we use our unperturbed Hamiltonian

$$W_0 = \sum_n \left[L^2(n) + \frac{x}{\kappa^2} \pi^2(n) + 8x\kappa^2 \right], \quad (4.1a)$$

our perturbation then being

$$4xV = -4x \sum_n [\phi^\dagger(n)U(n, n+1)\phi(n+1) + \text{H.c.} - 2\kappa^2]. \quad (4.1b)$$

The zeroth-order ground state is then clearly a state for which

$$L^2(n)|0\rangle = 0. \quad (4.2a)$$

Considering the first-order contribution from W_0 , we see that the ϕ dependence of the ground state is

$$\pi^2(n)|0\rangle = 0. \quad (4.2b)$$

We then note that

$$\begin{aligned} L^2(n)[U(n, n+1)]^m|0\rangle &= m^2[U(n, n+1)]^m|0\rangle, \\ \pi^2(n)[\phi(n)]^m|0\rangle &= m^2[\phi(n)]^m|0\rangle, \end{aligned} \quad (4.3)$$

enabling us to calculate the intermediate state energies required in the perturbation expansion. Since we include the diagonal $(x/\kappa^2)\pi^2$ in the unperturbed Hamiltonian, we only have even orders in the perturbation theory for the ground state, odd orders being obtained by expanding the x -dependent energy denominators.

Thus the zeroth-order energy of the vacuum state is 0, the first order is $8x\kappa^2N$. The second order is given by the graph of Fig. 3. The perturbation creates a single ϕ excitation on one site and an anti- ϕ excitation on an adjacent site, joined by an electric flux string U . Clearly on an N site (link) lattice this can be done in $2N$ ways. The amplitude for this "creation" is $-4x\kappa^2$. The second perturbation annihilates this excitation to return us to the initial state. The energy denominator is clearly

FIG. 3. Second-order graph contributing to the vacuum energy.

$$\omega_0^{(0)} - \omega_0' = -\left(1 + \frac{2x}{\kappa^2}\right), \quad (4.4)$$

using Eq. (4.1a). Thus the total second-order contribution is

$$-\frac{2(4x\kappa^2)^2N}{(1+2x/\kappa^2)}. \quad (4.5)$$

The fourth-order contribution is given by the graphs in Fig. 4 and their various time orderings. Fig. 4(a) represents the contribution where excitations are produced on different, nonadjacent links and later annihilated. This yields a total contribution

$$-\frac{(4x\kappa^2)^4N(N-3)\times 2\times 4}{2(1+2x/\kappa^2)^3}. \quad (4.6a)$$

Figures 4(b) and 4(c) give the contributions where the perturbation is applied on adjacent links. In Fig. 4(b) both excitations are created with a $\phi(\phi^\dagger)$ on the common site, while in Fig. 4(c) one excitation is created with a ϕ and the other a $\phi^\dagger$ on the common link. The contributions are

$$-\frac{8N}{[1+2(x/\kappa^2)]^2[2+6(x/\kappa^2)]}(4x\kappa^2)^4 \quad (4.6b)$$

and

$$-\frac{8N}{[1+2(x/\kappa^2)]^2[2+2(x/\kappa^2)]}(4x\kappa^2)^4, \quad (4.6c)$$

respectively. Finally, in Fig. 4(d) we have the case where the two excitations are "created" (then annihilated) on the same link, giving

$$-\frac{2N}{[1+2(x/\kappa^2)]^2[4+8(x/\kappa^2)]}(4x\kappa^2)^4. \quad (4.6d)$$

FIG. 4. Fourth-order graphs contributing to the vacuum energy.

In addition, at fourth order we have the term of the Rayleigh-Schrödinger theory,¹⁹

$$-(4x\kappa^2)^4 \left\langle 0 \left| V \frac{1}{\omega_0^{(0)} - W_0} V \right| 0 \right\rangle \left\langle 0 \left| V \frac{1}{(\omega_0^{(0)} - W_0)^2} V \right| 0 \right\rangle \\ = \frac{(4x\kappa^2)^4 N^2 4}{(1 + 2x/\kappa^2)^3}, \quad (4.7)$$

which just cancels the N^2 contribution leaving the required extensive result.

This calculation can easily be extended to sixth order. The eighth order can then be trivially obtained using the following observation.

If we are terminating our series at eighth order, we can set x to zero in the energy denominators of our eighth-order calculation. This is equivalent to calculating our eighth-order graphs using a Hamiltonian with $\pi^2(n) \equiv 0$. This problem has already been solved in the preceding section, since it reduces to solving Mathieu's equation on each link. We thus need the eighth-order term of $N\omega_0$, where ω_0 is the appropriate ground-state energy (a_0) of the Mathieu equation. From Ref. 20, p. 724, we find

$$\omega_0 = \frac{1}{4}a_0(16x\kappa^2). \quad (4.8)$$

Thus the contribution of the eighth-order graphs is

$$\frac{68687.4^7}{18874368} N(4x\kappa^2)^8. \quad (4.9)$$

Such methods were used for the highest-order graphs for all the states we have calculated.

After expanding in powers of x we end up with the following power series expansion for ω_0 , the ω energy of the vacuum,

$$\omega_0 = 2(4x) - 2(4x)^2 + (4x)^3 + 3(4x)^4 - 5(4x)^5 \\ - \frac{595}{72}(4x)^6 + \frac{4199}{144}(4x)^7 + \frac{107911}{5760}(4x)^8 + \dots, \quad (4.10)$$

for $\kappa = 1$, for example.

We can learn a great deal about this model if we generalize this calculation to the energy density of the ground state of the theory formulated in a background electric field α . As discussed in Ref. 13 we can implement this idea by placing widely separated equal and opposite (fractional) charges into the system. Then a calculation of the ground-state energy density gives us the interquark potential which has already been discussed for $\kappa^2 > 1/\pi$ in the Euclidean formulation of the theory. Note that the versions of the theory with different α ($|\alpha| < \frac{1}{2}$) cannot mix with each other through the perturbation V . At zeroth order in x these states

have the properties

$$L(n)|\alpha\rangle = \alpha|\alpha\rangle, \\ \pi(n)|\alpha\rangle = 0, \quad (4.11)$$

for all links and sites. The first statement of Eq. (4.11) states that the fractional charge α is accompanied by a flux tube of the correct strength to satisfy Gauss's law.

Before turning to the actual calculation it is interesting to interpret the parameter α in the decoupled $[\pi^2(n) = 0]$ Mathieu theory. There α labels a state in the lowest *band* for which $\alpha = 0$ is the lowest-lying state.²⁰ If we replace the Mathieu equation by the harmonic oscillator and call the lattice spacing 1, α corresponds to choosing a lattice with origin α rather than zero. In this case it is clear that α cannot effect the continuum limit of the theory since it just corresponds to a slightly different lattice formulation of a potential model. In other words the entire $\alpha < \frac{1}{2}$ band must become degenerate with the $\alpha = 0$ state in the continuum limit (a well-known property of the Mathieu equation), i.e., the interquark potential vanishes in the decoupled theory in the limit $x \rightarrow \infty$. Of course, the full lattice field theory will not be so simple.

A perturbative calculation of the ω energy of an " α vacuum" involves the same graphs as that for the "true" vacuum. The zeroth-order ω energy is

$$\omega_0^{(0)} = \alpha^2 N, \quad (4.12)$$

while the first order is simply

$$\omega_\alpha^{(1)} = 8x\kappa^2 N. \quad (4.13)$$

The second-order contribution is given by the graph of Fig. 3 and is,

$$- \left[\frac{1}{(1 + 2\alpha + 2x/\kappa^2)} + \frac{1}{(1 - 2\alpha + 2x/\kappa^2)} \right] (4x^2\kappa^4) N. \quad (4.14)$$

A graphical calculation including sixth order can then be supplemented by a single link (Mathieu) calculation to obtain the series for ω_α through eighth order.

Now we can extract the physics from these calculations. The computation of ω_α produces a series,

$$\omega_\alpha = NS_\alpha,$$

where S_α is a dimensionless Taylor series in x which is independent of N , the number of links of the lattice. The real energy of the α vacuum above the $\alpha = 0$ vacuum is

$$E_\alpha - E_0 = \frac{ag^2}{2} (\omega_\alpha - \omega_0) = g \frac{ag}{2} N(S_\alpha - S_0), \quad (4.15)$$

where we have used Eq. (3.8) relating W to H . The number of links making up the lattice is L/a where L is the length of the system in physical units, say. Therefore,

$$\frac{1}{L}(E_\alpha - E_0) = \frac{g^2}{2}(S_\alpha - S_0). \quad (4.16)$$

We shall call $(E_\alpha - E_0)/L$ the "string tension" τ_α and compute

$$\tau_\alpha / (\frac{1}{2}g^2) = S_\alpha - S_0. \quad (4.17)$$

Now it is clear how to compute $\tau_\alpha / (\frac{1}{2}g^2)$ from the series for S_α —form diagonal Padé approximants for $S_\alpha - S_0$ and take the continuum limit $\chi \rightarrow \infty$ directly. This is the same method used in other lattice theories^{3,19} to take the continuum limit. In particular, *dimensionless* quantities were identified (such as ratio of masses or dimensionless matrix elements) and the strong-coupling expansions were extrapolated to the continuum limit using diagonal Padé approximants as in analogous statistical-mechanics calculations.²¹

Using the expansions for $S_\alpha - S_0$ which we have calculated to eighth order in χ for arbitrary α and κ^2 , we form the $[4, 4]$ Padé approximants and take the continuum limits. In Table I we show the results for $(1/\alpha^2)(S_\alpha - S_0)$ as a function of κ^2 for $\alpha = 0.3$. The division by α^2 is done to measure the string tension relative to the strong-coupling ($\chi = 0$) value of $(1/\alpha^2)(S_\alpha - S_0) = 1$. (See Tables II and III.) The results in Table III are also plotted in Fig. 5. Similar results are obtained for all values of α , $|\alpha| < \frac{1}{2}$. We learn from this curve that the string tension *persists* in the continuum limit for $\kappa^2 < \kappa_c^2 \approx 0.7$ but vanishes for $\kappa^2 > \kappa_c^2$. So, our suspicion that this theory confines in the continuum limit for κ^2 small enough appears very possible. The fact that the string tension is slightly negative for $\kappa^2 > \kappa_c^2$ should not (we think) be taken seriously. The tension is within 7% of zero for $\kappa^2 > \kappa_c^2$ so within the context of an approximate eighth-order calculation this result is compatible with zero, the expected answer for large κ^2 .

Since the Hamiltonian and Euclidean formulations of the model use different cutoff procedures, the parameter κ maps in a nontrivial fashion from one formulation to the other. Our estimate $\kappa_c^2 \approx 0.6-0.8$ applies to the Hamiltonian formulation and $\kappa_c^2 \approx 1/\pi$ applies to the Euclidean formulation. (The reader should bare this in mind later in the text where we do not make this distinction explicitly.)

V. SIMPLE MESONIC STATES

To understand this lattice theory and its continuum limit in more detail, it is necessary to study its low-lying mass spectrum. In the strong-cou-

TABLE I. The series of $S_\alpha - S_0$ for various values of κ^2 . The coefficients are defined by the relation $S_\alpha - S_0 = A_0 + A_1y + A_2y^2 + A_3y^3 + \dots$ where $y = \chi/\kappa^2$. The chosen value of α is 0.3.

κ^2	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
0.2	0.09	0.0	-0.028 800	0.237 60	-1.410 41	7.421 62	-36.834 53	1.762 22 $\times 10^2$	-8.180 18 $\times 10^2$
0.4	0.09	0.0	-0.460 80	3.801 60	-20.215 87	70.738 60	-31.622 96	-1.961 01 $\times 10^3$	1.931 19 $\times 10^4$
0.6	0.09	0.0	-2.332 80	19.245 60	-50.776 77	-6.950 46 $\times 10^2$	1.045 70 $\times 10^4$	-5.504 93 $\times 10^4$	-2.857 12 $\times 10^5$
0.8	0.09	0.0	-7.372 80	60.825 60	2.782 98 $\times 10^2$	-1.115 81 $\times 10^4$	8.208 64 $\times 10^4$	9.673 66 $\times 10^5$	-2.744 79 $\times 10^7$
1.0	0.09	0.0	-18.000 00	1.485 0 $\times 10^3$	2.938 22 $\times 10^3$	-7.337 34 $\times 10^4$	-2.911 88 $\times 10^4$	2.603 51 $\times 10^7$	-3.457 10 $\times 10^8$
1.2	0.09	0.0	-37.324 80	3.079 30 $\times 10^2$	1.460 98 $\times 10^4$	-3.260 97 $\times 10^5$	-4.400 15 $\times 10^6$	2.715 51 $\times 10^8$	-9.275 88 $\times 10^8$
1.4	0.09	0.0	-69.148 80	5.704 78 $\times 10^2$	5.305 14 $\times 10^4$	-1.134 84 $\times 10^6$	-3.949 31 $\times 10^7$	1.835 28 $\times 10^9$	1.916 20 $\times 10^{10}$
1.6	0.09	0.0	-1.179 65 $\times 10^2$	9.732 10 $\times 10^2$	1.585 01 $\times 10^5$	-3.324 74 $\times 10^6$	-2.240 66 $\times 10^8$	9.374 78 $\times 10^9$	2.905 95 $\times 10^{11}$
1.8	0.09	0.0	-1.889 57 $\times 10^2$	1.558 89 $\times 10^3$	4.122 88 $\times 10^5$	-8.560 65 $\times 10^6$	-9.823 80 $\times 10^8$	3.910 58 $\times 10^{10}$	2.370 67 $\times 10^{12}$
2.0	0.09	0.0	-2.880 00 $\times 10^2$	2.376 00 $\times 10^3$	9.652 13 $\times 10^5$	-1.992 68 $\times 10^7$	-3.603 03 $\times 10^9$	1.396 30 $\times 10^{11}$	1.421 85 $\times 10^{13}$

TABLE II. The $[4, 4]$ Padé approximant for $S_\alpha - S_0$. The coefficients are defined by $(N_0 + N_1y + N_2y^2 + N_3y^3 + N_4y^4)/(D_0 + D_1y + D_2y^2 + D_3y^3 + D_4y^4)$, where $y = x/\kappa^2$. Again, $\alpha = 0.3$.

κ^2	N_0	N_1	N_2	N_3	N_4	D_0	D_1	D_2	D_3	D_4
0.2	0.09	1.188 21	5.400 17	9.649 43	5.991 79	1.00	13.202 33	60.321 88	$1.088 01 \times 10^2$	66.695 45
0.4	0.09	1.441 63	10.515 69	37.546 37	57.347 14	1.00	16.018 13	$1.219 6 \times 10^2$	$4.569 55 \times 10^2$	$8.096 46 \times 10^2$
0.6	0.09	1.803 00	16.463 84	61.523 96	39.022 54	1.00	20.033 37	$2.088 52 \times 10^2$	$9.890 25 \times 10^2$	$2.127 27 \times 10^3$
0.8	0.09	1.763 65	23.997 36	$1.055 16 \times 10^2$	-21.929 13	1.00	19.596 11	$3.485 57 \times 10^2$	$2.101 88 \times 10^3$	$1.197 41 \times 10^4$
1.0	0.09	1.795 93	41.195 59	$2.039 68 \times 10^2$	-3.203 20	1.00	19.954 80	$6.577 29 \times 10^2$	$4.607 27 \times 10^3$	$6.241 44 \times 10^4$
1.2	0.09	1.859 86	71.688 99	$3.776 61 \times 10^2$	-1.486 66	1.00	20.665 12	$1.211 26 \times 10^3$	$9.345 03 \times 10^3$	$2.527 81 \times 10^5$
1.4	0.09	1.936 25	$1.200 33 \times 10^2$	$6.520 33 \times 10^2$	-5.118 57	1.00	21.513 88	$2.102 02 \times 10^3$	$1.743 57 \times 10^4$	$8.323 19 \times 10^5$
1.6	0.09	2.009 76	$1.917 17 \times 10^2$	$1.057 66 \times 10^3$	-1.473 67	1.00	22.330 64	$3.440 91 \times 10^3$	$3.020 76 \times 10^4$	$2.343 73 \times 10^6$
1.8	0.09	2.071 54	$2.934 83 \times 10^2$	$1.632 28 \times 10^3$	-3.734 92	1.00	23.017 07	$5.360 45 \times 10^3$	$4.914 02 \times 10^4$	$5.859 71 \times 10^6$
2.0	0.09	2.119 34	$4.333 66 \times 10^2$	$2.421 07 \times 10^3$	-8.585 84	1.00	23.548 22	$8.015 17 \times 10^3$	$7.585 51 \times 10^4$	$1.334 83 \times 10^7$

TABLE III. Values of the string tension, $\tau_\alpha/(\frac{1}{2}g^2)$, divided by the strong-coupling limit for various values of κ^2 . The right-hand column is obtained from the Padé approximants of Table II in the continuum limit, $x \rightarrow \infty$.

κ^2	$\frac{1}{\alpha^2}(S_\alpha - S_0)$
0.2	0.998
0.4	0.787
0.6	0.204
0.8	-0.020
1.0	-0.057
1.2	-0.065
1.4	-0.068
1.6	-0.070
1.8	-0.071
2.0	-0.071

pling limit (x small) simple gauge-invariant states are made by applying $\phi(n)$ to one site, $\phi^\dagger(n+1)$ to a nearest neighbor and connecting them by a flux line U . There are two such states whose zero-momentum unperturbed forms are

$$\frac{1}{\sqrt{2N}} \sum_n [\phi^\dagger(n)U(n, n+1)\phi(n+1) + \text{H.c.}] |0\rangle \quad (5.1a)$$

and

$$\frac{1}{\sqrt{2Ni}} \sum_n [\phi^\dagger(n)U(n, n+1)\phi(n+1) - \text{H.c.}] |0\rangle. \quad (5.1b)$$

It is easy to see that in the *naive* continuum limit the latter state becomes a single massive vector meson $B_1[B_\mu \equiv A_\mu + (1/g)\partial_\mu \theta]$ for which $m_{B_1}/g = \sqrt{2}\kappa$. The former (symmetric) state is identified with the unbound scalar state of mass $2m_\nu$. In this respect these states are analogous to the low-lying vector and (unbound) scalar states of the massless Schwinger model.¹⁹

FIG. 5. String tension, $\tau_\alpha/(\frac{1}{2}g^2)$, for background field $\alpha=0.3$ plotted against κ^2 .

To calculate the masses of the states in Eq. (5.1) perturbatively, it is convenient to deal with the states

$$|+\rangle = \frac{1}{\sqrt{N}} \sum_n \phi^\dagger(n) U(n, n+1) \phi(n+1) |0\rangle \quad (5.2a)$$

and

$$|-\rangle = \frac{1}{\sqrt{N}} \sum_n \phi^\dagger(n+1) U^\dagger(n, n+1) \phi(n) |0\rangle \quad (5.2b)$$

and consider a 2×2 mass matrix. We will label the diagonal $(++)$ matrix element of the Raleigh-Schrödinger perturbation theory A and the mixing $(+-)$ between the states B . The energy of the vector (scalar) state is then $A - B(A+B)$.

To zeroth order such states have ω energies $1 + 2x/\kappa^2$. The graphs contributing second-order terms to A are listed in Fig. 6. The ω energy of each intermediate state is just the sum of the squares of the fluxes on each link plus x/κ^2 times the sum of the squares of the number of ϕ excitations on sites. The energy denominator is then $1 + 2x/\kappa^2$ minus this ω energy for each graph of Fig. 6. We have for each graph

$$\begin{aligned} (a) &= -\frac{2(N-3)}{(1+2x/\kappa^2)} (4\kappa^2 x)^2, \\ (b) &= -\frac{2}{(1+4x/\kappa^2)} (4\kappa^2 x)^2, \\ (c) &= -2(4\kappa^2 x)^2, \\ (d) &= -\frac{1}{(3+6x/\kappa^2)} (4\kappa^2 x)^2, \\ (e) &= \frac{N}{(1+2x/\kappa^2)} (4\kappa^2 x)^2, \\ (f) &= -\frac{(N-3)}{(1+2x/\kappa^2)} (4\kappa^2 x)^2, \\ (g) &= -2(4\kappa^2 x)^2. \end{aligned} \quad (5.3)$$

FIG. 6. Graphs contributing to the amplitude A in second order.

The graphs contributing to B at second order are shown in Fig. 7. Their values are

$$\begin{aligned} (a) &= \frac{N}{(1+2x/\kappa^2)} (4\kappa^2 x)^2, \\ (b) &= -\frac{(N-3)}{(1+2x/\kappa^2)} (4\kappa^2 x)^2, \\ (c) &= -\frac{2}{(1+4x/\kappa^2)} (4\kappa^2 x)^2. \end{aligned} \quad (5.4)$$

The calculation of the amplitudes A and B have been done by computer to sixth order. The eighth-order coefficient is obtained by using Mathieu's equation [one needs the characteristic roots labeled b_2 (vector) and a_2 (scalar)]. The series for the ω energies are given in Tables IV and V, while the $[4, 4]$ Padé approximants are given in Tables VI and VII. We note that all the Padé approximants are free of zeros or poles for positive x . This is an important result because it indicates that none of the series have a tendency to vanish for finite x . In other words, the theory's mass gap (in units of the lattice spacing) does not appear to vanish for finite x . Thus, we find no evidence for any continuous phase transitions for finite x . However, we do find qualitatively different numerical values for the Padé approximants above and below $\kappa^2 = \kappa_c^2 \approx 0.6-0.8$. In particular consider the ratio of the vector-state mass at various κ^2 , $m_V(\kappa^2)$, to the vector-state mass at $\kappa^2 = 2$, well above the suspected critical point. If we compute this ratio by taking the ratio of the $[4, 4]$ Padé approximants for the corresponding ω energies and then let $x \rightarrow \infty$, Table VIII results. Table VIII also contains a column showing

$$\frac{m_V(\kappa^2)}{m_V(\kappa^2=2)} \left(\frac{2}{\kappa^2} \right)^{1/2} \text{ vs } \kappa.$$

This particular quantity is chosen because if the lattice theory had precisely the naive continuum limit for all κ^2 , then $m_V(\kappa^2) = \sqrt{2}g\kappa$ and the quantity $[m_V(\kappa^2)/m_V(2)](2/\kappa^2)^{1/2}$ would be unity. This table and Fig. 8 indicate that the relationship $m_V(\kappa^2) = \sqrt{2}g\kappa$ is fairly well obeyed above $\kappa_c^2 \approx 0.6-0.8$, but the vector-state mass becomes huge below the critical point.

FIG. 7. Graphs contributing to the amplitude B in second order.

TABLE IV. The series for the ω energy of the vector state. The coefficients are defined by $\omega_V = V_0 + V_1 v + V_2 v^2 + \dots$ where $v = x/\kappa^2$.

κ^2	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
0.2	1.0	2.0	0.04267	-0.29013	0.57799	-1.11919	2.15730	-4.14320	8.03219
0.4	1.0	2.0	0.68267	-4.64213	8.70095	-7.98491	-3.95779	28.42896	-24.70768
0.6	1.0	2.0	3.45600	-23.50080	32.05177	1.77244 $\times 10^2$	-7.73018 $\times 10^2$	-5.23024 $\times 10^2$	1.17943 $\times 10^4$
0.8	1.0	2.0	10.92267	-74.27413	-0.781383	2.41231 $\times 10^3$	-6.52592 $\times 10^3$	-7.73965 $\times 10^4$	4.58557 $\times 10^5$
1.0	1.0	2.0	26.66667	-1.81333 $\times 10^2$	-5.27407 $\times 10^2$	1.54240 $\times 10^4$	-9.82856 $\times 10^3$	-1.39647 $\times 10^6$	5.06393 $\times 10^6$
2	1.0	2.0	4.32667 $\times 10^2$	-2.90133 $\times 10^3$	-2.22056 $\times 10^5$	4.12262 $\times 10^6$	1.99802 $\times 10^8$	-6.46628 $\times 10^9$	-2.17421 $\times 10^{11}$
4	1.0	2.0	6.82667 $\times 10^3$	-4.64213 $\times 10^4$	-5.82391 $\times 10^7$	1.058177 $\times 10^9$	8.81265 $\times 10^{11}$	-2.66798 $\times 10^{13}$	1.66236 $\times 10^{16}$
6	1.0	2.0	3.45600 $\times 10^4$	-2.35008 $\times 10^5$	-1.49451 $\times 10^9$	2.71237 $\times 10^{10}$	1.14778 $\times 10^{14}$	-3.46295 $\times 10^{15}$	-1.10028 $\times 10^{19}$
8	1.0	2.0	1.09227 $\times 10^5$	-7.42741 $\times 10^5$	-1.49315 $\times 10^{10}$	2.70938 $\times 10^{11}$	3.62578 $\times 10^{15}$	-1.09330 $\times 10^{17}$	-1.09921 $\times 10^{21}$
10	1.0	2.0	2.66667 $\times 10^5$	-1.81333 $\times 10^6$	-8.90038 $\times 10^{10}$	1.61493 $\times 10^{12}$	5.27713 $\times 10^{16}$	-1.59099 $\times 10^{18}$	-3.90653 $\times 10^{22}$

TABLE V. The same as Table IV except for the scalar particle.

κ^2	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
0.2	1.0	2.0	0.093867	-0.18773	-0.44834	4.19192	-21.39511	94.15659	-3.91273 $\times 10^2$
0.4	1.0	2.0	1.50187	-3.00373	-8.27958	73.70732	-3.30910 $\times 10^2$	1.21541 $\times 10^3$	-4.10908 $\times 10^3$
0.6	1.0	2.0	7.60320	-15.20640	-66.18046	5.18734 $\times 10^2$	-1.24272 $\times 10^3$	-2.05455 $\times 10^3$	2.25535 $\times 10^4$
0.8	1.0	2.0	24.02987	-48.05973	-4.15635 $\times 10^2$	2.87829 $\times 10^3$	4.403081 $\times 10^3$	-1.22836 $\times 10^5$	2.01797 $\times 10^5$
1.0	1.0	2.0	58.66667	-1.17333 $\times 10^2$	-2.07763 $\times 10^3$	1.34044 $\times 10^4$	1.07898 $\times 10^5$	-1.44144 $\times 10^6$	-5.53039 $\times 10^6$
2	1.0	2.0	9.38667 $\times 10^2$	-1.87733 $\times 10^3$	-4.65313 $\times 10^5$	2.80690 $\times 10^8$	4.36319 $\times 10^8$	-4.43812 $\times 10^9$	-5.15356 $\times 10^{11}$
4	1.0	2.0	1.50187 $\times 10^4$	-3.00373 $\times 10^4$	-1.18055 $\times 10^8$	7.08572 $\times 10^8$	1.77260 $\times 10^{12}$	-1.77449 $\times 10^{13}$	-3.35817 $\times 10^{16}$
6	1.0	2.0	7.60320 $\times 10^4$	-1.52064 $\times 10^5$	-3.02417 $\times 10^9$	1.81462 $\times 10^{10}$	2.29882 $\times 10^{14}$	-2.29930 $\times 10^{15}$	-2.20491 $\times 10^{19}$
8	1.0	2.0	2.40299 $\times 10^5$	-4.80597 $\times 10^5$	-3.02051 $\times 10^{10}$	1.81234 $\times 10^{11}$	7.25665 $\times 10^{15}$	-7.25713 $\times 10^{16}$	-2.19979 $\times 10^{21}$
10	1.0	2.0	5.86667 $\times 10^5$	-1.17333 $\times 10^6$	-1.80032 $\times 10^{11}$	1.08020 $\times 10^{12}$	1.05596 $\times 10^{17}$	-1.05599 $\times 10^{18}$	-7.81509 $\times 10^{22}$

TABLE VI. The $[4, 4]$ Padé approximant coefficients for the vector particle ω_V . The coefficients are defined by $(A_0 + A_1y + A_2y^2 + A_3y^3 + A_4y^4)/(B_0 + B_1y + B_2y^2 + B_3y^3 + B_4y^4)$.

κ^2	A_0	A_1	A_2	A_3	A_4	B_0	B_1	B_2	B_3	B_4
0.2	1.0	-5.700 42	-42.300 02	-71.147 52	-32.395 96	1.0	-7.700 42	-26.941 85	-16.645 14	-0.768 297
0.4	1.0	35.914 52	1.453 21 $\times 10^2$	2.577 20 $\times 10^2$	80.394 59	1.0	33.914 52	76.809 76	85.589 93	5.514 07
0.6	1.0	6.901 10	32.844 96	64.543 5	56.235 30	1.0	4.901 103	19.586 76	31.932 56	7.806 44
0.8	1.0	6.196 72	69.683 56	1.531 87 $\times 10^2$	6.236 67 $\times 10^2$	1.0	4.196 72	50.367 46	80.886 43	2.242 36 $\times 10^2$
1.0	1.0	6.416 67	1.529 18 $\times 10^2$	3.680 86 $\times 10^2$	3.670 98 $\times 10^3$	1.0	4.416 67	1.174 18 $\times 10^2$	1.968 05 $\times 10^2$	1.474 52 $\times 10^3$
1.2	1.0	6.957 40	3.015 20 $\times 10^2$	7.949 82 $\times 10^2$	1.532 90 $\times 10^4$	1.0	4.957 40	2.363 09 $\times 10^2$	4.242 52 $\times 10^2$	6.352 68 $\times 10^3$
1.4	1.0	7.694 74	5.391 13 $\times 10^2$	1.567 58 $\times 10^3$	5.078 27 $\times 10^4$	1.0	5.694 74	4.252 81 $\times 10^2$	8.302 40 $\times 10^2$	2.126 47 $\times 10^4$
1.6	1.0	8.514 86	8.922 37 $\times 10^2$	2.860 58 $\times 10^3$	1.422 48 $\times 10^5$	1.0	6.514 86	7.044 44 $\times 10^2$	1.501 52 $\times 10^3$	5.972 78 $\times 10^4$
1.8	1.0	9.307 63	1.391 74 $\times 10^3$	4.874 02 $\times 10^3$	3.516 06 $\times 10^5$	1.0	7.307 63	1.097 19 $\times 10^2$	2.537 53 $\times 10^3$	1.475 76 $\times 10^5$
2.0	1.0	9.999 96	2.074 25 $\times 10^3$	7.819 19 $\times 10^3$	7.900 27 $\times 10^5$	1.0	7.999 96	1.631 58 $\times 10^3$	4.044 04 $\times 10^3$	3.310 63 $\times 10^5$

TABLE VII. The same as Table VI except for the scalar particle.

κ^2	A_0	A_1	A_2	A_3	A_4	B_0	B_1	B_2	B_3	B_4
0.2	1.0	30.299 09	1.986 76 $\times 10^2$	4.697 66 $\times 10^2$	3.831 98 $\times 10^2$	1.0	28.299 09	1.419 84 $\times 10^2$	1.833 30 $\times 10^2$	8.971 11
0.4	1.0	10.839 31	47.081 68	1.028 99 $\times 10^2$	94.718 41	1.0	8.839 31	27.901 19	36.825 08	13.989 90
0.6	1.0	15.080 63	88.743 30	3.393 26 $\times 10^2$	4.829 67 $\times 10^2$	1.0	13.080 63	54.978 84	1.451 197 $\times 10^2$	39.802 18
0.8	1.0	2.973 66	64.799 56	4.780 30	6.492 98 $\times 10^2$	1.0	0.973 66	38.822 37	-48.201 61	2.752 34 $\times 10^2$
1.0	1.0	3.949 00	1.669 69 $\times 10^2$	2.069 20 $\times 10^2$	5.236 01 $\times 10^3$	1.0	1.949 01	1.044 04 $\times 10^2$	1.103 76	1.415 09 $\times 10^3$
1.2	1.0	4.130 28	3.348 95 $\times 10^2$	4.823 18 $\times 10^2$	2.209 51 $\times 10^4$	1.0	2.130 28	2.089 84 $\times 10^2$	48.502 72	5.409 34 $\times 10^3$
1.4	1.0	4.243 53	6.040 19 $\times 10^2$	9.234 55 $\times 10^2$	7.298 93 $\times 10^4$	1.0	2.243 53	3.741 58 $\times 10^2$	1.202 52 $\times 10^2$	1.706 86 $\times 10^4$
1.6	1.0	4.324 05	1.012 48 $\times 10^3$	1.600 72 $\times 10^3$	2.062 29 $\times 10^5$	1.0	2.324 05	6.233 56 $\times 10^2$	2.294 22 $\times 10^2$	4.703 02 $\times 10^4$
1.8	1.0	4.378 10	1.604 02 $\times 10^3$	2.587 97 $\times 10^3$	5.187 65 $\times 10^5$	1.0	2.378 10	9.834 00 $\times 10^2$	3.883 17 $\times 10^2$	1.165 85 $\times 10^5$
2.0	1.0	4.413 51	2.427 88 $\times 10^3$	3.967 16 $\times 10^3$	1.189 74 $\times 10^6$	1.0	2.413 51	1.484 39 $\times 10^2$	6.102 376 $\times 10^2$	2.650 12 $\times 10^5$

TABLE VIII. The continuum limit of the mass of the vector state at various κ^2 compared to its value at $\kappa^2 = 2$ vs the parameter κ^2 .

κ^2	$m_V(\kappa^2)/m_V(2)$	$\left(\frac{2}{\kappa^2}\right)^{1/2} \frac{m_V(\kappa^2)}{m_V(2)}$
0.2	17.66	55.84
0.4	6.10	13.64
0.6	3.01	5.50
0.8	1.16	1.83
1.0	1.04	1.47
1.2	1.00	1.29
1.4	1.00	1.20
1.6	1.00	1.19
1.8	1.00	1.05
2.0	1.00	1.00

It is also informative to study the mass of the scalar state as a function of κ^2 in precisely the same way as we studied the vector. The corresponding table for the scalar mass $m_S(\kappa^2)$ and a plot of $m_S(\kappa^2)/m_S(2)$ vs κ^2 are given in Table IX and Fig. 9, respectively. These results indicate that the scalar particle becomes *massless* at the critical point. This means that the phase transition at κ_c^2 is *continuous*. In addition, the scalar particle, like the vector particle, becomes extremely massive in the small κ^2 phase.

One can also study the mass ratio m_S/m_V as a function of κ^2 . The most reliable procedure to employ in calculating this ratio in the continuum limit is to note that

$$\frac{m_S}{m_V} = \frac{\omega_S}{\omega_V}$$

is a dimensionless quantity which should be a finite real number. So, using the strong-coupling expansions for ω_S and ω_V , we obtain the Taylor series to eighth order for ω_S/ω_V by division and continue it to $x = \infty$ using a diagonal [4, 4] Padé

FIG. 8. Plot of the ratio of the [4, 4] Padé approximants for $m_V(\kappa^2)$ and $m_V(\kappa^2=2)$ vs κ^2 .

TABLE IX. The same as Table VIII except for the scalar particle.

κ^2	$\frac{m_S(\kappa^2)}{m_S(2)}$	$\frac{m_S(\kappa^2)}{m_S(2)} \left(\frac{2}{\kappa^2}\right)^{1/2}$
0.2	9.52	30.10
0.4	1.51	3.38
0.6	2.70	4.93
0.75	0.26	0.42
0.8	0.53	0.84
1.0	0.82	1.16
1.2	0.91	1.17
1.4	0.95	1.14
1.6	0.98	1.10
1.8	0.99	1.04
2.0	1.00	1.00

approximant. The results of this computation are given in Table X and plotted in Fig. 10. They indicate, in agreement with the analysis above, that the scalar state becomes massless at κ_c^2 . However, one must be wary of this calculation since we have seen in Figs. 8 and 9 that the series for both ω_S and ω_V change their character as κ^2 passes through κ_c^2 . Therefore, taking the ratio of the individual series and forming a Padé approximant might be quite sensitive to the "noise" in each series. In fact, the series for ω_S/ω_V and the Padé approximant itself are quite irregular just below κ_c^2 .

Figure 10 also indicates that in the large- κ^2 phase,

$$\frac{m_S}{m_V} \cong 1.84 \quad (\kappa^2 > 5). \quad (5.5)$$

This ratio lies 8% below the value 2.00 expected in a naive continuum limit. This is satisfactory agreement, but it is not a particularly profound test of our calculational method. The reason is that for large κ^2 the momentum conjugate to $\theta(n)$ enters the Hamiltonian in the expression $\pi^2(n)/\kappa^2$.

FIG. 9. Same as Fig. 8 except for the scalar particle.

TABLE X. Continuum limit of the [4,4] Padé approximant for $m_s/m_V = \omega_s/\omega_V$ as a function of κ^2 . The ratios are quite irregular (and presumably unreliable) in the immediate vicinity of $\kappa_c^2 \approx 0.75-0.85$.

κ^2	m_s/m_V	κ^2	m_s/m_V
0.05	1.0001	0.60	2.037
0.10	1.002	0.65	2.316
0.15	1.008	0.70	2.802
0.20	1.025	0.75	4.429
0.25	1.060	0.80	...
0.30	1.120	0.85	0.272
0.35	1.209	0.90	0.961
0.40	1.329	0.95	1.238
0.45	1.474	1.00	1.390
0.50	1.640	2.00	1.807
0.55	1.825	4.00	1.834
		6.00	1.836
		...	...
		20.00	1.836

Hence, the quantum character of $\theta(n)$ becomes numerically insignificant for large κ^2 and the calculation of masses becomes essentially identical to the calculation of the level structure of the harmonic oscillator. (This observation was also made in Sec. III.) Nonetheless, Eq. (5.5) is quite satisfactory, although the possibility exists that the Padé results for m_s/m_V will not converge monotonically to 2.00 since convergence is known to be quite irregular for potential models.²²

Now it is interesting to abstract the phase diagram of the electrodynamics of the planar model from these calculations. For $\kappa^2 > \kappa_c^2 \approx 0.6-0.8$ the continuum limit of the theory exhibits the Higgs mechanism and for κ^2 well above κ_c^2 it is well approximated by a free massive vector meson. However, for $\kappa^2 < \kappa_c^2$, the continuum limit of the theory appears to be the free, massless electromagnetic field. This last fact should be so because for $\kappa^2 < \kappa_c^2$ the theory confines in the continuum limit and

FIG. 10. Plot of the [4,4] Padé approximant for the ratio m_s/m_V vs κ^2 .

has no finite mass spectrum. In other words, we are interpreting the graphs of Figs. 8 and 9 to mean that the vector and scalar mesons move to *infinite* mass, leaving behind the free electromagnetic field. Presumably if higher orders were calculated $m_V(\kappa^2)/m_V(2)$ would steadily increase for $\kappa^2 < \kappa_c^2$. We feel that this is a reasonable interpretation of Figs. 8 and 9, since the plots change so dramatically as κ^2 passes through κ_c^2 .

There are other intuitive reasons for suspecting this result. From the perspective of Sec. II, below κ_c^2 the vortices become dense in the continuum limit and can be expected to remain relevant. They then preserve the confining property of the theory as $a \rightarrow 0$. In addition, they disorder the theory on the length scale of the lattice spacing itself. The size of the vortices, g , becomes unimportant since they overlap. Therefore, it is reasonable to expect the masses of the theory to be $O(1/a)$ or precisely zero. Hence, the vector and scalar states disappear as $a \rightarrow 0$, and a massless confining potential remains behind.

VI. THE EUCLIDEAN FORMULATION REVISITED

It is interesting to return to the Euclidean formulation and search for the phases of the model in light of the results found in Secs. IV and V. Consider Eq. (2.26)

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\phi(r) \sum_{m(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_r \phi)^2 - \frac{g^2 a^2}{2} \sum_r \left(\phi + \frac{e}{g} J \right)^2 + 2\pi i \sum_r m(r) \phi(r) \right]. \quad (2.26)$$

Following the strategy used to understand the planar model,¹⁶ we generalize this expression to one which contains an adjustable activation energy of an isolated bare vortex,

$$Z(J, A) = \int_{-\infty}^{\infty} \prod_r d\phi(r) \sum_{m(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_r \phi)^2 - \frac{g^2 a^2}{2} \sum_r \left(\phi + \frac{e}{g} J \right)^2 - A \sum_r m(r)^2 + 2\pi i \sum_r m(r) \phi(r) \right]. \quad (6.1)$$

It is interesting to suppose that $A \gg 1$, so the sum over $m(r)$ can be well approximated by keeping just the

$m(r) = 0, \pm 1$ terms. Then,

$$\sum_{m(r)=-\infty}^{\infty} e^{-Am(r)^2 + 2\pi i m(r)\phi(r)} \approx 1 + 2e^{-A} \cos[2\pi\phi(r)] \approx e^{2e^{-A} \cos[2\pi\phi(r)]}. \quad (6.2)$$

So, in this approximation,

$$Z(J, A) \approx \int_{-\infty}^{\infty} \prod_r d\phi(r) \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_\nu \phi)^2 - \frac{g^2 a^2}{2} \sum_r \left(\phi + \frac{e}{g} J \right)^2 + 2e^{-A} \sum_r \cos[2\pi\phi(r)] \right]. \quad (6.3)$$

It is best to rescale $\phi(r) \rightarrow \phi(r)/\sqrt{2}\kappa$ and replace sums over lattice sites by integrals over space-time,

$$Z(J, \mu) \approx \int_{-\infty}^{\infty} \prod_r d\phi(r) \exp \left\{ - \int d^2x \left[\frac{1}{2} (\partial_\nu \phi)^2 + \frac{2\kappa^2 g^2}{2} \left(\phi + \frac{e}{\sqrt{2}\kappa g} J \right)^2 - \mu \cos(2\pi\sqrt{2}\kappa\phi) \right] \right\} \quad (6.4)$$

where $\mu = 2e^{-A}/a^2$. Setting $J=0$, we see that our generalized model is equivalent to the lattice sine-Gordon theory with a mass term $\frac{1}{2}m_v^2\phi^2$, where $m_v = \sqrt{2}\kappa g$. In addition, for $\kappa > 1/\sqrt{\pi}$ this model is known to be equivalent to the electrodynamics of the Thirring model.²³ Observe also that at $\kappa = 1/\sqrt{\pi}$ the argument of the cosine term becomes $\sqrt{8\pi}\phi$ —this is the special point where the ground state of the continuum model falls to minus infinity.²⁴ For $\kappa = 1/\sqrt{2\pi}$ the argument of the cos term becomes $\sqrt{4\pi}$ and the theory reduces to the massive Schwinger model [the fermion mass ~ the parameter μ in Eq. (6.4)].

Of course, what we really need to know is the quantity μ itself. In recent studies of the pure planar model the renormalization-group trajectories in the (μ, κ) plane were determined.^{12,16} In this way one related the theory at one point (μ_1, κ_1) to the theory at another (μ_2, κ_2) . Then the introduction of the parameter μ could be justified as just a helpful device. These studies found the fixed line of spin wave theory and the Kosterlitz-Thouless phase transition was studied in quantitative detail.^{12,16}

A detailed renormalization-group analysis of this model can be done and will be reported elsewhere. Here we will only present a crude “two-step” renormalization which is adequate for $\kappa^2 > 1/\pi$ and a sketch of the general approach. Return to Eq. (2.29),

$$\mathcal{Z}(J=0) = \sum_{m(r)} \exp \left(-4\pi^2 \kappa^2 \sum_{r, r'} m(r) \times G(r-r'; m_v a) m(r') \right). \quad (2.29)$$

Recall that if interactions between vortices are neglected, one obtains Eq. (2.31),

$$Z(J=0) \approx \sum_{m(r)} \exp \left(2\pi \kappa^2 \ln(m_v a) \sum_r m(r)^2 \right). \quad (2.31)$$

So, the activation energy for an isolated vortex is

$$\bar{A} = -2\pi \kappa^2 \ln(m_v a). \quad (6.5)$$

We have seen in Sec. II that for $m_v a$ sufficiently small and κ^2 sufficiently large, the gas of vortices is dilute and Eq. (2.31) is a good approximation to the real system. These observations suggest the following two-step renormalization procedure appropriate for $\kappa^2 > 1/\pi$.

Consider Eq. (6.1) again and organize the integration over $\phi(r)$ into a “high” momentum slice and a “low” momentum slice. Include in the “high” momentum slice all the frequency components of ϕ greater than $(m_v a)^{-1}$ and include all other momenta in the “low” momentum slice. Let $\bar{\phi}$ denote the field with the high-momentum components removed. Then explicitly integrating the high-momentum components of $\phi(r)$ out of Eq. (2.26) gives

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\bar{\phi}(r) \sum_{m(r)=-\infty}^{\infty} \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_\nu \bar{\phi})^2 - \frac{g^2 a^2}{2} \sum_r \left(\bar{\phi} + \frac{e}{g} J \right)^2 - 4\pi^2 \kappa^2 \sum_{r, r'} m(r) \bar{G}(r-r'; m_v a) m(r') + 2\pi i \sum_r m(r) \bar{\phi}(r) \right], \quad (6.6)$$

where $\tilde{G}$ is the Green's function for $\phi - \bar{\phi}$, the high-momentum components of $\phi(r)$. Let us replace $\tilde{G}(r-r'; m_V a)$ by $-\tilde{A}\delta_{r,r'}$. As reviewed above, this approximation is good for $\kappa^2 > 1/\pi$. Then the manipulations in Eq. (6.2) can be applied again giving

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\bar{\phi}(r) \exp \left[-\frac{1}{4\kappa^2} \sum_r (\Delta_\nu \bar{\phi})^2 - \frac{g^2 a^2}{2} \sum_r \left(\bar{\phi} + \frac{e}{g} J \right)^2 + 2e^{-\tilde{A}} \sum_r \cos(2\pi \bar{\phi}(r)) \right],$$

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\bar{\phi}(r) \exp \left\{ - \int d^2x \left[\frac{1}{2} (\partial_\nu \bar{\phi})^2 + \frac{2\kappa^2 g^2}{2} \left(\bar{\phi} + \frac{e}{\sqrt{2}\kappa g} J \right)^2 - 2m_V^2 (m_V a)^{2(\pi\kappa^2-1)} \cos(2\pi\sqrt{2}\kappa\bar{\phi}) \right] \right\}. \quad (6.7)$$

Now we can let $a \rightarrow 0$. For $\kappa^2 > 1/\pi$, the coefficient of the cosine term vanishes in the continuum limit and we have

$$Z(J) = \int_{-\infty}^{\infty} \prod_r d\bar{\phi}(r) \exp \left\{ - \int d^2x \left[\frac{1}{2} (\partial_\nu \bar{\phi})^2 + \frac{2\kappa^2 g^2}{2} \left(\bar{\phi} + \frac{e}{2\sqrt{\kappa}g} J \right)^2 \right] \right\}, \quad (6.8)$$

which the reader can identify as the massless Schwinger model. This theory exhibits the Higgs mechanism and does not confine—by translating $\bar{\phi}$, the external source J is eliminated from the expression for the free energy. In addition, the mass of the boson ϕ is $m_V = \sqrt{2}\kappa g$ in agreement with our

earlier analyses.

For general κ^2 such a naive two-step renormalization must be replaced by an infinite sequence of integrations. Consider Eq. (6.3) after restoring continuum space-time notation,

$$Z(J, \mu) = \int_{-\infty}^{\infty} \prod_r d\phi(r) \exp \left[- \int d^2x \frac{1}{4\kappa^2} (\partial_\nu \phi)^2 + \frac{g^2}{2} \left(\phi + \frac{e}{g} J \right)^2 - \mu \cos[2\pi\phi(r)] \right], \quad (6.9)$$

where κ^2 and μ are considered to be bare parameters of the theory. The lattice provides an ultraviolet cutoff on the theory, the largest permissible Euclidean momentum is $p < 1/a$. Now consider integrating in a sequence of infinitesimal steps all the momenta in the theory between $1/a$ and a finite value on the order of, but considerably larger than $1/g$. Then in each of these steps the superrenormalizable mass term $\frac{1}{2}g^2[\phi + (e/g)J]^2$ in Eq. (6.9) is irrelevant and can be ignored. Therefore, the infinite renormalizations in the model of Eq. (6.9) are the same as those of the ordinary sine-Gordon theory. But this is (essentially) the model studied by Kosterlitz,¹² so we can apply his analysis directly. In particular, for $\kappa^2 < 1/\pi$, his work suggested that the (κ^2, μ) trajectories flow to a fixed point where both κ^2 and μ are large (presumably infinite). In other words, the equivalent temperature of the theory formulated on a coarse lattice is infinite and the ground state consists of a condensate of vortices. If the coefficient of the cosine term in Eq. (6.9) becomes large, then ϕ will freeze at values where $\cos[2\pi\phi(r)] = 1$. In addition, if the coefficient of the gradient term in Eq. (6.9) becomes large, $\phi(r)$ will become a con-

stant. Requiring that the partition function be nontrivial, then picks out the value zero for $\phi(r)$. The Wilson loop integral becomes

$$Z(J)/Z(0) \rightarrow \exp \left(-\frac{1}{2}e^2 \int J^2(r) d^2x \right), \quad (6.10)$$

which implies an interquark potential of

$$E(R) = \frac{1}{2}e^2 R. \quad (6.11)$$

So, an unscreened electric flux tube extends between the quarks in agreement with the strong-coupling expansions for $\kappa^2 < \kappa_c^2$.

In summary, this renormalization-group-analysis and the results of strong-coupling expansions suggest that there is a fixed line of theories for $\kappa^2 > \kappa_c^2$ which exhibit the Higgs mechanism, that the fixed line terminates and that for $\kappa^2 < \kappa_c^2$ the renormalization-group trajectories flow to a $\kappa^2 = 0$ fixed point which is the free electromagnetic field.

VII. EXCITED STATES (LOCAL EXCITATIONS)

Naively, following our experience with the Schwinger model¹⁹ we would expect that local ex-

cited states could only be created from the vacuum by successive applications of the gauge-invariant operators $\phi^\dagger(n)U(n, n+1)\phi(n+1)$ and its Hermitian conjugate. Actually there are a far larger number of gauge-invariant operators allowed in this theory since there clearly exists a sensible definition of the operator $[\phi^\dagger(n)U(n, n+1)\phi(n+1)]^\alpha$ and its Hermitian conjugate for any real α .

Hence we will consider those states obtained by applying $[\phi^\dagger(n)U(n, n+1)\phi(n+1)]^\alpha$ to the vacuum, or their zero-momentum analogs obtained by summing over n . We will first consider the case in which $\alpha \neq \frac{1}{2}n$ (n an integer) and also not equal to certain other values, determined later, for which a degenerate perturbation theory calculation is necessary.

The zeroth- and first-order contributions are clearly

$$\omega_\alpha^{(0)} = \alpha^2 \left(1 + \frac{2x}{\kappa^2}\right) \quad (7.1)$$

after vacuum subtraction. The second-order contribution to the ω energy is clearly given by the graphs of Fig. 11 where the dashed lines represent the fractionally charged ϕ^α 's and their associated fluxes U^α . In each of these graphs, the perturbation creates a scalar-antiscalar particle pair with its associated flux $U(U^\dagger)$ and then annihilates it on the same link. Figure 11(a) is the contribution where the vacuum excitation is not adjacent to the initial excited link, and contributes

$$\frac{-(N-3)2}{(1+2x/\kappa^2)} (4x\kappa^2)^2. \quad (7.2a)$$

In Fig. 11(b), the perturbation excites a link adjacent to the initial excited link yielding

$$-2 \left\{ \frac{1}{[1+(2+2\alpha)x/\kappa^2]} + \frac{1}{[1+(2-2\alpha)x/\kappa^2]} \right\} (4\kappa^2 x)^2. \quad (7.2b)$$

Finally we have the case shown in Fig. 11(c) where the perturbation acts on the excited link changing its flux to $(\alpha \pm 1)$ giving

$$- \left[\frac{1}{(1+2\alpha)} + \frac{1}{(1-2\alpha)} \right] \frac{1}{(1+2x/\kappa^2)} (4\kappa^2 x)^2. \quad (7.2c)$$

This calculation has been extended to sixth order in x , (see Fig. 12) and by use of the single-link

FIG. 11. Second-order graphs contributing to the ω energy of an α excitation.

FIG. 12. Sixth-order graphs contributing to the ω energy of an α excitation.

(Mathieu) calculation, to eighth order.

The quantity $m_\alpha/g = \omega_\alpha/\sqrt{8x}$ has been calculated for a range of values of α and κ^2 , and it has been plotted as a function of x in Figs. 13(a) and 13(b) for $\kappa^2=1$. In order to dispose of the factor $\sqrt{8x}$ which prevents us from taking the limit $x \rightarrow \infty$ directly, we have considered the ratio $m_\alpha/m_{0.75}$, expanded it as a power series in x to eighth order and expressed it as a $[4, 4]$ Padé approximant for a range of values of κ^2 . The results appear in Table XI and are plotted in Figs. 14(a) and 14(b). Before discussing these results we must mention a peculiar property of these states: An α excita-

FIG. 13. The masses of α excitations m'_α/g plotted vs κ^2 for α between 0.05 and 0.45 in (a); and between 0.45 and 1.55 in (b).

TABLE XI. Continuum limit of the $[4, 4]$ Padé approximant calculation of $m_\alpha/m_{0.75}$ as a function of κ^2 and α .

$\alpha \backslash \kappa^2$	0.2	0.4	0.6	0.8	1.0	1.4	1.8
0.05	0.004 15	0.001 40	-0.000 19	-0.000 64	-0.000 83	-0.000 95	-0.000 98
0.15	0.037 26	0.012 03	-0.001 88	-0.006 61	-0.009 34	-0.011 59	-0.012 23
0.25	0.102 73	0.029 99	-0.005 29	-0.018 70	-0.027 97	-0.036 62	-0.039 27
0.35	0.351 69	0.050 10	-0.007 79	-0.020 64	-0.024 36	-0.026 30	-0.026 74
0.45	0.287 75	0.129 61	0.097 73	0.091 46	0.089 70	0.088 78	0.088 57
0.55	0.611 29	0.759 27	0.789 15	0.795 05	0.796 71	0.797 57	0.797 76
0.65	0.751 96	0.907 50	0.948 22	0.958 86	0.962 05	0.963 74	0.964 12
0.75	1.000 00	1.000 00	1.000 00	1.000 00	1.000 00	1.000 00	1.000 00
0.85	1.276 55	1.239 92	-2.741 60	0.948 46	0.980 00	0.990 49	0.992 45
0.95	1.587 40	1.437 86	1.354 42	1.718 33	-0.742 46	0.621 47	0.695 70
1-	2.060 04	0.529 88	0.012 21	1.578 5	1.080 27	0.964 76	0.944 46
1+	1.857 80	-0.329 29	1.276 93	1.385 73	1.401 69	1.403 12	1.402 40
1.05	1.934 26	1.578 33	0.953 84	1.241 05	1.277 22	1.287 99	1.289 76
1.15	2.314 96	1.692 25	1.544 87	1.454 85	1.423 84	1.406 23	1.402 11
1.25	2.696 86	2.129 89	1.741 43	1.625 13	1.588 67	1.569 04	1.564 58

tion on one link cannot be annihilated by the perturbation and recreated on another link. Therefore, these states have no momentum dependence. We call such states which cannot move "arthritic."

Let us first consider those states for which $|\alpha| < \frac{1}{2}$. They have the property that the perturbation does not mix them with states of lower (zeroth order) energy. The corresponding band of states of the decoupled mode Hamiltonian (Mathieu equa-

tion)²⁵ attain zero energy as $x \rightarrow \infty$. From Fig. 14(b) we see that they do indeed form a separate band of zero mass for $\kappa^2 \geq \kappa_c^2 \approx 0.6$. We believe that the small (less than 10%) negative values of $m_\alpha/m_{0.75}$ just represent limitations of $[4, 4]$ Padé approximants in approximating zero. Below κ_c^2 we expect the confining property of the continuum limit to cause $m_\alpha/m_{0.75}$ to be a monotonic function of α for $|\alpha| < \frac{1}{2}$. This is, in fact, what we learn

FIG. 14. Plot of the $[4, 4]$ Padé approximant to $m_\alpha/m_{0.75}$ vs α for κ^2 between 0.2 and 0.6 in (a); and between 0.8 and 1.8 in (b).

from Table XI and Fig. 14(a).

In summary, the $|\alpha| < \frac{1}{2}$ band of states is degenerate with the vacuum in the continuum limit for $\kappa^2 > \kappa_c^2$. As discussed in Sec. IV the α vacuums become degenerate in the continuum limit for $\kappa^2 > \kappa_c^2$ because of screening (the Higgs mechanism). Clearly the local, arthritic states (for $|\alpha| < \frac{1}{2}$) become degenerate with the vacuum because of the same mechanism.

It is clear that the arthritic states considered here are only a small subset of all such vacuum excitations. All gauge-invariant states for which the flux on each link is less than $\frac{1}{2}$ fall into this category. That they are stable, despite the fact that their energies above the vacuum can be arbitrarily large (for a sufficiently coarse lattice) is a consequence of the inability of the confining theory to screen nonintegral charge.

Let us now move to the consideration of the next band, $\frac{1}{2} < |\alpha| < 1$, its boundaries defined by a zeroth-order degeneracy and its associated level repulsion. The decoupled Hamiltonian²⁵ leads us to associate this band with the first excited state, the vector particle, of the continuum limit for $\kappa^2 > \kappa_c^2$. We see from Table X that the [4, 4] Padé approximants for the mass ratios $m_\alpha/m_{0.75}$ lie within 20% of the expected value 1 independent of α for $0.6 < \alpha < 1$. Although we could have hoped for better numerical values, the fact that the $\frac{1}{2} < |\alpha| < 1$ is qualitatively distinct from the vacuumlike $0 < |\alpha| < \frac{1}{2}$ band is certainly clear from Fig. 14(b) where a plateau exists for $0.6 < \alpha < 1$. In addition we have calculated the antisymmetric $\alpha = 1$ state's mass to eighth order and find $m_1^{(-)}/m_{0.75} = 1.080271$ which falls within 9% of unity. For $\kappa^2 < \kappa_c^2$ the $\frac{1}{2} < |\alpha| < 1$ band does not become degenerate in the continuum limit.

Thus, considering mass alone, our interpretation of the band $\frac{1}{2} < |\alpha| < 1$ as being a set of states which become degenerate representations of the vector particle of mass $\sqrt{2}\kappa g$ seems good (for $\kappa^2 > \kappa_c^2 \approx 0.6$, of course). However, this interpretation poses a conceptual problem. For the uppermost state of the band ($\alpha = 1$) the single-particle interpretation is clear since the excitation can be moved by the perturbation from one site to another. The other states of the band are, however, arthritic, so how can one interpret them as single-particle states?

To shed light on this, consider a link having flux β and suppose that $-\frac{1}{2} < \beta < 0$. Consider now the action of $\phi^\dagger(n)U(n, n+1)\phi(n+1)$ or its Hermitian conjugate on each link of the lattice. If it were applied uniformly to the $\alpha = 0$ vacuum state, we would obtain the vector or scalar state with $\alpha = \pm 1$. It has a zeroth-order ω energy of 1. In the case at hand, if the operator is applied on any link

other than the excited one we get an excitation with flux ± 1 on this link, while leaving flux β on the excited link. An ω energy $1 + \beta^2 > 1$ ensues. If, however, the operator acts on the excited link it produces a flux $1 + \beta = \alpha$ and an ω energy $\alpha^2 < 1$. Its Hermitian conjugate produces a flux $-1 + \beta$ and an ω energy $(1 - \beta)^2 > \beta^2$. The interpretation is now clear. The vacuum excitation β acts as a scattering center (defect) situated on one link of the lattice. For small x there exists one bound state with zeroth-order ω energy α^2 , and scattering states with zeroth-order ω energy $1 + \beta^2$ and $(1 - \beta)^2$ indicating a potential which is attractive or repulsive, depending on the parity of the incoming particle. The spectrum of scattering states only appears strange because momentum dependence of the scattering state energies only appears at order x^2 when the higher momentum scalar states can feel the repulsion and connect with the state of ω energy $(1 - \beta)^2$. Now we see why our states of flux $\frac{1}{2} < |\alpha| < 1$ are arthritic: It is simply because the vector particle is bound to the vacuum excitation β which is itself arthritic. As $x \rightarrow \infty$, the vacuum excitation becomes a genuine vacuum and the particle unbinds, which is the reason that this state becomes a representative of the vector particle state in the continuum limit.

It is more difficult to identify the third and higher-lying bands in Fig. 14(b). There is an indication of a band at $m_\alpha/m_{0.75} \approx 1.6-1.7$ which we would like to identify with the scalar particle (the precise ratio should be 2). Apparently our eighth-order calculation is not powerful enough to adequately describe the continuum limit of higher-lying bands.

Finally we note that new degeneracies not present in the decoupled theory occur causing higher bands to break up into sub-bands, presumably a prelude to their becoming continuum rather than discrete states when $a = 0$. The first such new boundary is at $\alpha = \frac{5}{4}$. In general they occur at $\alpha = \frac{1}{2}(n + n/m)$, where n and m are integers.

VIII. THE VACUUM IN BACKGROUND FIELD $\frac{1}{2}$

The problem of the α vacuums becomes more difficult when $\alpha = \frac{1}{2}$, since the perturbation connects the state with all fluxes $\frac{1}{2}$ to states where some fluxes are $-\frac{1}{2}$. To zeroth order these states are degenerate with the initial state. Moreover, a state with a single charged ϕ (or $\phi^\dagger$) excitation on 1 site can have fluxes $\frac{1}{2}$ on one side, $-\frac{1}{2}$ on the other, a state which is again degenerate with the vacuum at zeroth order. Thus we see a phenomenon first noticed in the massive Schwinger model and true for the lattice Schwinger model even for zero mass⁷: Charged particles are unconfined for a vacuum field $\frac{1}{2}$.

In order to explore this problem more fully it is expedient to write down an effective Hamiltonian whose eigenstates solve the zeroth-order degeneracy problem. The Hamiltonian is

$$W = \sum_n \{ L^2(n) + x\kappa^{-2}\pi^2(n) - 4x[\phi^\dagger(n)U(n, n+1)\phi(n+1) + \text{H.c.} - 2\kappa^2] \}. \quad (8.1)$$

For the states of interest $|\alpha| = \frac{1}{2}$ on each link. Thus

$$\sum_n L^2(n) = \frac{1}{4}N$$

and can be dropped. There are only two allowed states for each link $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. The operators $\phi^\dagger(n)U(n, n+1)\phi(n+1)$ and its Hermitian conjugate simply take us from one of these states to the other. We therefore define a spin variable which is $+\frac{1}{2}$ for $\alpha = \frac{1}{2}$ and $-\frac{1}{2}$ for $\alpha = -\frac{1}{2}$. The $\phi^\dagger(n)U(n, n+1)\phi(n+1)$ and its Hermitian conjugate are just κ^2 times the spin raising and lowering operators $\sigma_+(n)$, $\sigma_-(n)$. As for $\pi^2(n)$, it yields 0 if the two links adjacent the n th site have the same flux, and 1 if one has $\alpha = \frac{1}{2}$, the other $\alpha = -\frac{1}{2}$. Thus we can write it in terms of spin variables as

$$\pi^2(n) = \frac{1}{2}[1 - \sigma_+(n)\sigma_-(n+1)]. \quad (8.2)$$

Thus, removing an irrelevant constant

$$W = -\frac{1}{2}x\kappa^{-2} \sum_n [\sigma_+(n)\sigma_-(n+1) + 8\kappa^4\sigma_x(n)], \quad (8.3)$$

which is the Hamiltonian for the one-dimensional Ising model in a transverse magnetic field. It is a soluble model²⁶ and has the following two phases:

A. $8\kappa^4 < 1$. The system is magnetized, i.e., the ground-state expectation value of σ_x is nonvanishing.

B. $8\kappa^4 > 1$. The system is disordered and the ground state consists of a kink condensate.²⁷ In terms of the disorder parameter,

$$\mu_x(m) = \prod_{n>m} \sigma_x(n), \quad (8.4)$$

the ground state satisfies $\langle 0 | \mu_x(m) | 0 \rangle \neq 0$. These two phases are separated by a second-order phase transition.

The physical meaning of the phases A and B considered in terms of the original variables ϕ and U is as follows. For $8\kappa^4 > 1$, the ground state is a kink condensate. But the kink operator $\mu_x(m)$ is just a phase operator $\exp[i\phi(m)]$ with a string of flux extending to the right. This is so because

FIG. 15. The phase diagram of the electrostatics of the planar model found in this approximate study.

$$\sigma_x(n) \leftrightarrow \phi^\dagger(n)U(n, n+1)\phi(n+1) + \text{H.c.}$$

and from Eq. (8.4),

$$\mu_x(m) \leftrightarrow \phi^\dagger(m) \prod_{n>m} U(m, m+n). \quad (8.5)$$

So, the disordered phase of the Ising model corresponds to a condensation of gauge-invariant matter fields in the original model, i.e., the Higgs mechanism. So, in this phase confinement is surely lost even in the lattice theory ($a \neq 0$). For $8\kappa^4 < 1$ the expectation value of the operator $\mu_x(m)$ vanishes in the ground state and the theory confines for strong coupling.

IX. CLOSING REMARKS

Rather than summarize all the calculations and detailed conclusions occurring in this study, we will just remind the reader of the points we feel are most interesting.

According to our analysis the phase diagram of this model lattice gauge theory is that shown in Fig. 15. For fixed finite x and any κ^2 , the theory confines. In the continuum limit and $\kappa^2 > \kappa_c^2 \approx 0.6-0.8$ the theory displays the Higgs mechanism. The simple possibility that the theory is then just a free massive vector meson with $m_V = \sqrt{2}\kappa g$ is supported by several calculations. For $\kappa^2 < \kappa_c^2$ the

FIG. 16. A speculative phase diagram. "Pure electrostatics" is a linear potential while "confinement" has a nontrivial mass spectrum.

continuum limit appears to be just the free electromagnetic field. It confines and is simply a linear, static potential. We have argued that the existence of κ_c^2 is linked to the presence of vortices in the lattice gauge theory and that for $\kappa^2 > \kappa_c^2$ they become infinitely dilute and irrelevant in the continuum limit while for $\kappa^2 < \kappa_c^2$ they do not.

Might the phase diagram really be more complicated than this? Since our calculations were only approximate, they could have missed some interesting structure. Since the lattice theory contains two-dimensional parameters g and a , the critical region $\kappa^2 \approx \kappa_c^2$ could be more complicated than the simple ending of a fixed line. Another interesting possibility is that the pure electrodynamics phase really extends a finite distance into the phase diagram as shown in Fig. 16. We have searched for this possibility using our series expansions and found no evidence for it. We certainly, however, cannot rule it out.

The original motivation for studying this model was to gain some insight into a field theory which could lose confinement precisely at $a=0$. We have

seen that Hamiltonian strong-coupling expansions were quite efficient and unambiguous in uncovering this fact. This gives us some confidence that if a similar "catastrophe" occurs in non-Abelian gauge theories in 3+1 dimensions, it should show up plainly in our work.

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