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ON THE USE OF PRECONDITIONED CONJUGATE GRADIENT METHODS
FOR RED-BLACK ORDERED FIVE-POINT DIFFERENCE SCHEMES

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ABSTRACT

A modified incomplete factorization method is applied to a reduced system of equations, obtained by eliminating one set (the red set) of variables on a red-black ordered grid.

A conjugate gradient method is applied as an iterative refinement procedure.

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1. INTRODUCTION

The red-black (or chequer board) ordering has certain advantages with regard to arithmetic and demand of storage, in particular for large, out of core systems of linear equations corresponding to the five-point difference approximation of the Poisson equation. This is so, because on a $m \times n$ grid, $m \leq n$, one may reduce the problem to an in-core system of half the number of unknowns, without increasing the bandwidth, $2m + 1$. Only this latter system has to be factorized. We refer to, for example, McDonald and Trimble [1] for this and a similar ordering, where also other relevant references may be found.

In particular, when m is large, it may however be of interest to use an incomplete factorization of this reduced system and to solve it by iterative refinement. We show that this is possible and that the number of iterations grow as $O(h^{-1/2})$, as the average mesh size $h \rightarrow 0$, if a conjugate gradient method is used for the iterative refinements. (We assume that we subdivide all mesh sides in the same fraction given a coarse mesh to start with.)

For elongated grids, where m is small relative to n , the incomplete factorization is relatively more accurate than for the case where $m = n$. Hence a fewer number of iterations are needed and, as we will see, the method is still competitive with a full factorization method. For a certain incomplete factorization method used below the demand of storage is about 7.5 mn.

2. INCOMPLETE FACTORIZATION OF RED-BLACK ORDERED DIFFERENCE EQUATIONS

Having a grid Ω_h , consisting of rectangular elements, we number the points as on a chequer board (see Fig. 1). For simplicity we describe the ordering on a rectangular $n \times m$ grid where $m \leq n$. The matrix, corresponding to the difference operator of the Laplacian or of the more general operator

$$\frac{\partial}{\partial x} \left(a \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left(b \frac{\partial}{\partial y} \right), \quad a \geq a_0 > 0, \quad b \geq b_0 > 0,$$

then takes the following form,

$$A = \begin{bmatrix} D_1 & B^T \\ B & D_2 \end{bmatrix},$$

where D_1 and D_2 are positive diagonal matrices. Since we may multiply A by

$$\begin{bmatrix} D_1^{-1/2} & 0 \\ 0 & D_2^{-1/2} \end{bmatrix}$$

from the left and from the right, making corresponding transformations of the unknown and source vectors, we may, without limitation assume that $D_1 = D_2 = I$, the identity matrix. Hence we have

$$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} I & B^T \\ B & I \end{bmatrix}.$$

We now eliminate u_1 from the first set of equations, $u_1 = -B^T u_2 + f_1$. Thus we have to solve the reduced system

$$(I - BB^T)u_2 = f_2 - Bf_1. \quad (2.1)$$

We observe that $\|BB^T\|_\infty \leq \|B\|_\infty \|B^T\|_\infty \leq 1$, and that $I - BB^T$ is an M matrix. (Actually, this matrix corresponds to a nine-point difference operator, as indicated by the circles in Fig. 1.) Hence there exist numerically stable incomplete factorizations of this matrix (see, for example, Axelsson [2], Meijerink and van der Vorst [3] and Gustafsson [4]). Let LL^T be such an incomplete factorization.

3. THE PRECONDITIONED CONJUGATE GRADIENT METHOD

We now solve (2.1) by iteration, using a preconditioned conjugate gradient method. At each step of this we solve the two systems of equations corresponding to the triangular matrices L and L^T .

Since the modified incomplete factorization MIC-methods described in Gustafsson [4] are asymptotically fastest, we will use them in this report.

Then the eigenvalues λ_i of $(LL^T)^{-1}(I - BB^T)$ satisfy $0(1) < \lambda_i < 0(h^{-1})$, $h \rightarrow 0$, $i = 1, 2, \dots, mn/2$. Hence the spectral condition number $\mathcal{K}$ is $0(h^{-1})$, $h \rightarrow 0$, and the conjugate gradient method, preconditioned by LL^T , will converge in $0(h^{-1/2})$ iterations. Different preconditioned conjugate gradient methods are described in, for example, Axelsson [5], Axelsson [6], and Gustafsson [4]. The following one-step version was used (for the solution of a system $Au = f$);

Let u^0 be arbitrary, calculate $r^0 = Au^0 - f$, solve $LL^T\gamma^0 = r^0$ and

calculate $\gamma_0 = (\gamma^0, \gamma^0)$, $d^0 = -\gamma^0$, $dl^0 = Ad^0$, $\sigma_0 = (r^0, \gamma^0)$.

Then for $\ell = 0, 1, \dots$, calculate $\lambda_\ell = \sigma_\ell / (d^\ell, dl^\ell)$,

$u^{\ell+1} = u^\ell + \lambda_\ell d^\ell$, $r^{\ell+1} = r^\ell + \lambda_\ell dl^\ell$,

solve $LL^T\gamma^{\ell+1} = r^{\ell+1}$ and calculate

$\gamma_{\ell+1} = (\gamma^{\ell+1}, \gamma^{\ell+1})$, $\sigma_{\ell+1} = (r^{\ell+1}, \gamma^{\ell+1})$, $\beta_{\ell+1} = \sigma_{\ell+1} / \sigma_\ell$,

$d^{\ell+1} = -\gamma^{\ell+1} + \beta_{\ell+1} d^\ell$, $dl^{\ell+1} = Ad^{\ell+1}$.

Here $(.,.)$ is the usual inner product $(x,y) = x^Ty$. As stopping criterion we have used $\gamma_\ell \leq \varepsilon^2 \gamma_0$ and as incomplete factorization LL^T of $A = I - BB^T$ the MIC(2) algorithm was used.

We observe that the matrix A has only to be formed in order to calculate the incomplete factorization matrix LL^T , but multiplications $Ad^{\ell+1}$ can be performed from $A = I - BB^T$.

We also observe that the vector γ^ℓ may be used for storage of Ad^ℓ . Hence, apart from the matrices B and L , only four vectors of size $mn/2$ have to be stored, and the total requirement of central storage is 7.5 mn.

The number of multiplications (and additions) per iteration is 14 mn. We remark that the MIC(2) method, applied directly on the natural (row-wise or column-wise) ordered mesh, demands 21 mn operations. The factorization work is in both cases about 9 mn operations.

As follows from, for example, Axelsson [5], an upper bound for the number of iterations needed to decrease the relative residual error to $\|r^\ell\|/\|r^0\| \leq \epsilon$ is

$$\ell = \text{int} \left[\frac{1}{2} \sqrt{\kappa} \ln 2/\epsilon + 1 \right],$$

where r^ℓ is the residual vector at the ℓ^{th} iterative step.

The number of iterations ℓ needed for $\epsilon = 10^{-6}$ is given in Table 1 for different values of m and n . The corresponding result for the usual ordering is given in Table 2.

The total solution work per unknown grows as in Fig. 2, where also the corresponding numbers for the natural ordering and for a full factorization of a reduced system, as described in McDonald et al. [1], are given.

3. CONCLUSION

We conclude that the red-black ordering may be advantageous also in connection with incomplete factorization. Furthermore, since for the complete factorization the factorization work is $\sim \frac{1}{2} nm^3$, and the work in our method is $O(nm^{1.5})$, $m \rightarrow \infty$ if the ratio m/n is constant, it is also superior to that method. Only for small systems and for very small bandwidths, is the complete factorization slightly faster. The accuracy depends on the relative accuracy, ϵ in the iterative method. For instance, for $n = 40$, $m = 10$ and $\epsilon = 10^{-3}$, the number of operations per unknown in our method is only 63 multiplications and additions.

We finally remark that our method, incomplete factorization and a conjugate gradient algorithm, is applicable also on diagonally dominant non-symmetric M-matrices, like those arising from (modified) upwind approximations of diffusion-convective equations, for arbitrary large Peclet numbers (see Axelsson and Gustafsson [7]).

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Table 1

The number of iterations for the MICCG(2) method
on the reduced system, $\epsilon = 10^{-6}$

n	8	16		32			40		
m	8	8	16	8	16	32	10	20	40
ℓ	4	5	6	5	7	9	6	8	10

Table 2

The number of iterations for the MICCG(2) method
on the usual ordered system, $\epsilon = 10^{-6}$

n	8	16		32			40		
m	8	8	16	8	16	32	10	20	40
ℓ	5	6	8	7	9	11	8	10	12

Figure captions

Fig. 1 : The red-black ordering for $m = 5$, $n = 6$.

Fig. 2 : The total work per unknown for the MICCG(2) method on the reduced red-black ordered system (RB), the MICCG(2) method on the natural ordered system (NO), and the zebra algorithm described in [1] for different values of n and $E = n/m$, $\epsilon = 10^{-3}$ (in the iterative methods).

Fig. 1

Fig. 2