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GAUGE THEORIES ON A SIMPLICIAL LATTICE *)

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A B S T R A C T

We present various results obtained for pure gauge theories built on a simplicial lattice, an alternative to the almost always used hypercubical lattice. The main interest is to lead to shorter calculations, with more rapid thermalization of the Monte Carlo simulations. Our results are quite similar to those obtained with hypercubical lattices.

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1. INTRODUCTION

Lattice gauge theories have almost always used a regular hypercubical lattice for regularizing the Yang-Mills gauge action. Only a few attempts towards other prescriptions exist; one of these uses points randomly distributed in space-time⁽¹⁾. Celmaster⁽²⁾ proposes the use of the centered tesseract lattice.

It is known that there exists only three kinds of regular nets in any dimension d . There are, of course, the usual hypercubical lattice, but also its reciprocal and the simplicial lattice. The interest in these other possible discretizations may seem small since they certainly have a more complicated geometry and are nevertheless expected to provide the same results in the physical continuum limit. However, they are interesting, first from a conceptual point of view as they allow a new check of universality (which has recently been a controversial problem⁽³⁾). From the computational point of view, their interest is probably great for the following reasons:

1) There are more plaquettes per site with more orientations. As Celmaster pointed out⁽²⁾, the breaking of rotational invariance is therefore less important. This feature may even be easily improved by using more coupling constants. The singularities associated with the restoration of rotational invariance (maybe the roughening transition⁽⁴⁾) are therefore expected to be weaker, and the bump in the specific heat to be reduced. Hence strong-coupling expansions are more regular, give more accurate results with less terms and are hopefully more easily continued from the strong-coupling phase to the asymptotic freedom regime.

- 2) Similarly, a link interacts with more neighbours and the mean-field analysis is expected to provide better results. For the same reason, Monte Carlo simulations thermalize more rapidly.
- 3) There is an easier check of rotational invariance since different kinds of 2-planes exist and may be used while estimating the string tension. One of these planes contains Wilson loops with the degeneracy of the minimal surface which therefore fluctuates even for strong coupling. This may bring information on the role of the roughening transition⁽⁴⁾.
- 4) The spin analysis of the excitation spectrum will be easier, due to the existence of more plaquette orientations.
- 5) Finally, existing Monte Carlo programs are very easily switched from hypercubical to another regular lattice.

We study here some properties of the simplicial lattice. Its geometry is first summarized in Section 2. The corresponding lattice gauge theories are then shown in Section 3 to reduce to the Yang-Mills theory as the lattice spacing shrinks. This is not a priori evident since extra degrees of freedom are involved for the parametrization of gauge fields and have to disappear in the continuum limit. Finally, we present some Monte Carlo results for pure SU(2) gauge theory and compare them to various theoretical expectations. Some needed technicalities and tables have been relegated to appendices.

2. - GEOMETRY OF THE SIMPLICIAL LATTICE AND THE GAUGE ACTION

Starting from an auxiliary $(d+1)$ - dimensional hypercubical lattice with spacing a , we focus on all points contained in the hyperplane $x_1 + \dots + x_{d+1} = 0$ which will be from now on the d -dimensional physical space. In order to easily compute on this lattice, we start from the orthonormal basis $\{\vec{e}_1, \dots, \vec{e}_{d+1}\}$ in the auxiliary space, which induces a redundant basis

$$\{ \vec{e}_i = e_i - \frac{\vec{e}_1 + \dots + \vec{e}_{d+1}}{d+1}, i = 1, \dots, d+1 \},$$

for the simplicial lattice, obeying the constraint

$$\vec{e}_1 + \dots + \vec{e}_{d+1} = \vec{0}.$$

Sites are characterized by d integers $n_1, \dots, n_d$ and are located at

$$a \sum_{i=1}^{d+1} n_i \vec{e}_i = a \sum_{i=1}^{d+1} n_i \vec{e}_i, \text{ where } n_{d+1} = -n_1 - \dots - n_d.$$

If one uses only the first d vectors of the redundant basis $\{\vec{e}_i\}$, one verifies easily that only the fraction $1/(d+1)$ of the points with integer coordinates are occupied (because the sum of the coordinates in this basis factorizes $(d+1)$). Hence the volume per site is $a^d \sqrt{d+1}$.

Two particular cases are well known: in dimension 2 where the triangular lattice is recovered and dimension 3 where the f.c.c. lattice is recovered.

It is possible to introduce periodic boundary conditions by several different methods. The simplest one is to define the n_i 's modulo L , which yields a number $N = L^d$ sites in the lattice. This prescription, combined with a twist, is certainly the simplest

one to implement for computer simulations (exactly as used in hypercubical lattice calculations). We use it in this paper. But it should be said that this is not adopted when varying the sizes of the lattice independently in various directions, and in particular for finite temperature computations. Other prescriptions have to be used instead.

Nearest neighbours are separated by $a \vec{\mathcal{E}}_{ij} = a(\vec{\mathcal{E}}_i - \vec{\mathcal{E}}_j)$, and the lattice spacing is therefore $b = a\sqrt{2}$. There are $Nd(d+1)/2$ links in the lattice.

The sections of the lattice by 2-planes containing links are displayed in Fig. 1. The triangular pattern of Fig. 1(a) shows that the smallest closed paths are the triangles $(a \vec{\mathcal{E}}_{ij}, a \vec{\mathcal{E}}_{jk}, a \vec{\mathcal{E}}_{ki})$ or $(a \vec{\mathcal{E}}_{ij}, a \vec{\mathcal{E}}_{ki}, a \vec{\mathcal{E}}_{jk})$ (i, j, k all different running from 1 to $d+1$). There are $N d(d^2 - 1)/3$ such triangular plaquettes on the lattice. Other directions also reveal square plaquettes (Fig. 1(b)) ($N d(d^2 - 1)(d - 2)/8$ different squares on the lattice).

Now defining on each link xy a gauge field $U_{xy} = U_{yx}^{-1}$ chosen in a gauge group of $N \times N$ matrices, we will consider the following action

$$S = \beta \sum_{\Delta} \frac{1}{N} \text{Re Tr}(U_{xy} U_{yz} U_{zx}) + \gamma \sum_{\square} \frac{1}{N} \text{Re Tr}(U_{xy} U_{yz} U_{zt} U_{tx}), \quad (1)$$

where the two summations run over all triangular and squared plaquettes, respectively. The second term involving squares can be considered as an extended action term; its role is not essential for our purpose and will no longer be considered in this paper.

3. - RELATION TO THE YANG-MILLS FIELD

It is not clear that the action of Eqn.1 reduces , as the lattice spacing goes to zero, to the continuum Yang-Mills action. There are indeed $d(d+1)/2$ different directions, much more than the d components of the Yang-Mills field. The $d(d-1)/2$ supplementary degrees of freedom will be grouped in an antisymmetric tensor field $T_{\mu\nu}(x)$; we have to show that this field does not propagate in the continuum limit, i.e. it acquires an infinite mass. Its effect only appears in the regularization procedure and, in particular, is expected to affect the value of the Λ parameter.

We parametrize the link variables as

$$U_{x, x+a\vec{e}_{ij}} = \exp\{iag\vec{e}_{ij}^{\mu} A_{\mu}(x + \frac{1}{2}a\vec{e}_{ij}) + \frac{1}{2}iga^2\vec{e}_i^{\mu}\vec{e}_j^{\nu} T_{\mu\nu}(x + \frac{1}{2}a\vec{e}_{ij})\}, \quad (2)$$

where the fields A_{μ} and $T_{\mu\nu}$ belong to the Lie algebra of the gauge group. Now the calculation proceeds as usual. The plaquette term is evaluated using the Baker-Campbell-Hausdorff formula and reads, after some algebra,

$$\text{Tr}(U_{xy}U_{yz}U_{zx}) = \exp\{\frac{1}{2}iga^2\vec{e}_{ij}^{\mu}\vec{e}_{jk}^{\nu} (T_{\mu\nu} \pm F_{\mu\nu}) + O(a^3)\}, \quad (3)$$

where the fields $T_{\mu\nu}$ and $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ig[A_{\mu}, A_{\nu}]$

have been evaluated at the center of the triangle (xyz). The upper and lower signs refer to the two types of triangular paths

$$(a\vec{e}_{ij}, a\vec{e}_{jk}, a\vec{e}_{ki}) \text{ or } (a\vec{e}_{ij}, a\vec{e}_{ki}, a\vec{e}_{jk}).$$

Now expanding the exponential in Eqn.3, summing over all triangles and replacing sums over sites by integrals, we finally

find for the triangular part of the action of Eqn. 1 :

$$S = - \frac{\beta g^2 \sqrt{d+1} a^{4-d}}{6N} \left(\int \text{Tr}(F_{\mu\nu}^2 + T_{\mu\nu}^2) d^d x + O(a) \right). \quad (4)$$

This result shows that the field $T_{\mu\nu}$ does not propagate in this limit; it is, of course, possible to compute its kinetic part, (we do not display this result here), which is of order a^2 and which, therefore, leads to a mass of order $1/a$. This field plays an important role in the regularization procedure; in particular, it must be included in the calculation of the Λ parameter governing the asymptotic freedom region and, therefore, this parameter may be quite different for the hypercubical and the simplicial lattices.

It only remains the usual Yang-Mills field in the continuum limit and, from Eqn. 4, the β parameter is related to the bare coupling constant g_0 by

$$\beta = \frac{6a^{d-4}}{g_0^2 \sqrt{d+1}}, \quad (5)$$

and, in particular, for $d=4$

$$\frac{1}{g_0^2} = \frac{\sqrt{5}}{6} \beta. \quad (6)$$

The role of the second coupling constant γ can be taken into account using similar calculations and leads to the replacement of β in Eqns. 5 and 6 by an effective coupling $(\beta + 3\gamma)$.

In this paper we will present some results for the string tension in 4-dimensional $SU(2)$ gauge theory. The logarithm of the Wilson loops are dominated, as usual, by a term proportional

to the area and the coefficient K of this term, measured in b^2 units, behaves, as predicted by the renormalization group, as

$$K \simeq \frac{\sigma}{\Lambda^2} \left(\frac{4\pi^2\sqrt{5}\beta}{11} \right)^{\frac{102}{121}} \exp \left(-\frac{4\pi^2\sqrt{5}}{11} \beta \right), \quad (7)$$

Λ is the simplicial lattice regularization parameter and σ is the string tension. It can be related to the hypercubical lattice regularization parameter using a background gauge one-loop calculation⁽⁵⁾ (which is not done in this paper).

4. - MONTE CARLO RESULTS

We wish to compare working on a hypercubical and a simplicial lattice. From Table 1 we see that the simplicial lattice iteration will be slower by a factor of $\sqrt{5}$. This is more than compensated for by two factors; first, the thermalization is achieved much more rapidly and second, the density of points needed for accurate results in the continuum limit is less important. Indeed, as we will see, the Λ parameter is about $1/10$ of that on a hypercubical lattice.

We worked on a lattice with 6 lattice sites in each space-time direction and periodic boundary conditions. The method of Metropolis et al.⁽⁶⁾ was used to equilibrate our lattice. To perform our calculations we first carried out 16 iterations through the lattice with 20 Monte Carlo upgrades per link. The space-time lattice equilibrated, over the whole range of inverse coupling constant squared β , in less than 5 iterations. An average was then taken over the next 72 iterations through the lattice taking every second iteration in order to eliminate correlations between iterations. Disordered starting lattices were used for $\beta \leq 1.8$ while ordered starting lattices were used for $\beta \geq 1.9$.

In Fig. 2 we show, as a function of the inverse coupling constant squared β , the average action per plaquette $\langle E_{\Delta} \rangle$. Also shown in Fig. 2 are the 8th and 10th order strong-coupling expansions of Eqn. A-3. The tenth-order expansion agrees well with the data up to the cross-over point at $\beta \approx 1.76$. A Padé continued-fraction approximation hardly improves the result.

Using standard techniques⁽⁷⁾, the partition function is found to be

$$\frac{2 \ln Z}{Nd(d+1)} = \max_K \left\{ \frac{\beta(d-1)}{12} u'^3 + u - K u' \right\},$$

with $u \equiv u(K) = \ln \frac{I_1(2K)}{K}$,

for an average gauge field of the form

$$\langle U \rangle = \frac{1}{2} u' \mathbf{1}.$$

This gives the result

$$\langle E_{\Delta} \rangle = 1 - \frac{1}{2} \langle \text{Re Tr } UUU \rangle = 1 - \frac{1}{8} u'^3,$$

and

$$\langle E_{\square} \rangle = 1 - \frac{1}{2} \langle \text{Re Tr } UUUU \rangle = 1 - \frac{1}{16} u'^4.$$

The mean-field calculation predicts, as usual, an unobserved first-order phase transition at $\beta = 1.74$ (the weak-coupling phase is metastable between $\beta = 1.53$ and $\beta = 1.74$). The mean-field calculation in the weak-coupling region is shown in Fig. 2.

The geometry of the Wilson loops we calculated are shown in Fig. 3. These are averaged over all orientations in our lattice. The numbering scheme for our Wilson loops is also shown in the figure. The Monte Carlo results for the triangular Wilson loops $W_1^{\Delta}, W_2^{\Delta}, W_3^{\Delta}, W_4^{\Delta}, W_6^{\Delta}, W_9^{\Delta}$ and W_{13}^{Δ} are shown in Fig. 4. In the strong-coupling region the loops should obey the leading-order behavior

$$\left(\frac{\beta}{4}\right)^{\text{AREA}}.$$

This behavior is confirmed in Fig. 4.

In order to obtain the area-law behavior of these Wilson loops we form the logarithmic ratios

$$\chi_1 = \frac{2}{\sqrt{3}} \ln \left| \frac{W_4^\Delta}{W_6^\Delta} \right| ,$$

$$\chi_2 = \frac{1}{\sqrt{3}} \ln \left| \frac{W_9^\Delta}{W_{13}^\Delta} \right| ,$$

and

$$\chi_3 = \frac{2}{\sqrt{3}} \ln \left| \frac{W_4^{\Delta^2}}{W_1^\Delta W_9^\Delta} \right| .$$

In these ratios, the perimeter term and the form factor terms individually cancel. The Monte Carlo results for these logarithmic ratios χ_1 , χ_2 and χ_3 are shown in Fig. 5. The strong-coupling expansion of the string tension of Eqn. B-2 of orders 1, 2 and 4 in β are also indicated in Fig. 5. Other quantities such as

$$\frac{W_1^\Delta W_3^\Delta}{W_2^{\Delta^2}} \quad \text{or} \quad \frac{W_2^\Delta W_4^\Delta}{W_3^{\Delta^2}} ,$$

are expected to be close to unity if an exponential form involving only the area and the perimeter of the loop is postulated. This is indeed the case within a range of 1 or 2% and, in any case, within the statistical errors.

Also shown in Fig. 5 is the behavior of Eqn. 7 with

$$\Lambda = (1.2 \times 10^{-3}) \sqrt{K} .$$

This quantity is surprisingly low compared with the corresponding one for the hypercubical lattice

$$\Lambda_H \sim (12 \times 10^{-3}) \sqrt{K} .$$

This confirms the fact that, due to its higher symmetry, the simplicial lattice can accommodate a more coarse grain to obtain

similar results in the continuum limit. Similar curves are obtained for the 2-planes containing square lattices.

It must be pointed out that the Wilson loop in this plane is "rough" for all coupling, even at $\beta = 0$, due to a very large degeneracy for the minimal surface spanned by the loop. Corresponding strong-coupling calculations can be performed by mapping this problem to an F-model, as shown in Appendix 2. From Fig. 5, it also seems that the cross-over is smoother or even perhaps non-existent for such planes.

In Fig. 6 we plot the coefficient of the perimeter term in our Wilson loops. This quantity shows a fast rise as we approach the cross-over region from either strong- or weak-coupling.

Fig. 7 shows the average action per plaquette $\langle E_{\square} \rangle$ for squares as a function of the inverse coupling constant squared β . Also presented in Fig. 7 is the strong-coupling expansion of Eqn. A-5 which agrees with the Monte Carlo data for small β . In Fig. 8 we show the Wilson loops for squares of area up to 9 in lattice units. In the strong-coupling region we show the leading-order behaviour for comparison.

To extract the area-law behaviour of our square Wilson loops we construct the logarithmic ratios

$$x_1 = \ln \left| \frac{W_3^{\square}}{W_4^{\square}} \right|$$

and

$$x_2 = \frac{1}{2} \ln \left| \frac{W_4^{\square 2}}{W_1^{\square} W_9^{\square}} \right| .$$

These ratios are plotted in Fig. 9 along with the strong-coupling expansion of Eqn. B-4. The asymptotic freedom scale parameter Λ obtained in Fig. 5 is also presented in Fig. 9 and it provides a good fit to the Monte Carlo data. Comparing Figs. 5 and 8 shows that rotational invariance is restored, because the string tension is the same at the crossover point at $\beta \sim 1.8$.

5. CONCLUSIONS

The results of the present paper provide no new results; in fact, they merely confirm calculations already carried out on hypercubical lattices. However, the simplicial lattice will have an advantage in computer time needed for Monte Carlo calculations and in the analysis of glueball spectra because we have a finer analyzer of the spin content.

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APPENDIX A - STRONG COUPLING EXPANSIONS ; FREE ENERGY

The framework of strong-coupling expansions^(7,8) can be applied to simplicial lattice gauge theories. However, the diagram tables for this lattice do not exist and must be constructed. Table 2 displays the surfaces homeomorphic to the sphere constructed with at most 10 triangles. Each graph has been mapped on the plane for an easy representation; for each one are indicated the total number of triangles and the number of configurations per site on a d-dimensional simplicial lattice. The $(d+1)_K$ is a shorthand notation for $(d+1)d\cdots(d-K+2)$. Using this basic table and the general techniques⁽⁹⁾ we have computed the logarithm of the partition function per site in the thermodynamic limit. The notation of Ref. 9 has been used here and we get, for any dimension d and any gauge group, the following result:

$$\begin{aligned}
 F = & \frac{d(d^2-1)}{3} \{ \ln A + (d-2) \left[\frac{1}{4} S_4 + \frac{1}{2} (d-3) S_6 \right. \right. \\
 & + \frac{1}{8} (13d^2 - 91d + 157) S_8 + \frac{1}{5} (34d^3 - 408d^2 + 1603d - 2050) S_{10} \\
 & - \frac{1}{8} (4d - 11) S_4^2 - (d-3) \left(3d - \frac{19}{2} \right) S_4 S_6 \\
 & + \frac{1}{2} (d-3) \theta_{331} + \frac{3}{4} (d-3) \theta_{422} + 3(d-3)(d-4) \theta_{531} \\
 & + \frac{1}{6} (d-3)(d-4) \theta_{333} + \frac{9}{2} (d-3)(d-4) \theta_{622} \\
 & + \frac{27}{8} (d-3)(d-4) \theta_{442} + \frac{3}{2} (d-3)(d-4) \zeta_{31213} + \frac{1}{6} (d-3)(d-4) \xi_{3331} \\
 & \left. \left. + \frac{1}{2} (d-3) \phi_{222}^{111} + \frac{1}{20} (d-3) \lambda \right] \right\} , \tag{A-1}
 \end{aligned}$$

where all diagrams up to and including 10 plaquettes have been counted. The group factor λ has not been defined in Ref. 9 and

corresponds to the complete graph with 5 vertices

$$\begin{aligned}
 \lambda = & \sum_{\{n_i\}} \prod_{i=1}^{10} v_i \beta_{n_i} \int \cdots \int \chi_{n_1}(R) \chi_{n_2}(S) \chi_{n_3}(T) \chi_{n_4}(U) \\
 & \times \chi_{n_5}(V) \chi_{n_6}(W) \chi_{n_7}(RST) \chi_{n_8}(RUV) \chi_{n_9}(SWU^{-1}) \\
 & \times \chi_{n_{10}}(T^{-1}WV) DRDSDTDUDVDW.
 \end{aligned} \tag{A-2}$$

For the $SU(2)$ gauge group in four dimensions, this result yields

$$\begin{aligned}
 F = 10 \left(\frac{\beta}{2}\right)^2 + \frac{5}{3} \left(\frac{\beta}{2}\right)^4 - \frac{5}{18} \left(\frac{\beta}{2}\right)^6 - \frac{79}{64} \left(\frac{\beta}{2}\right)^8 \\
 + \frac{1157}{1440} \left(\frac{\beta}{2}\right)^{10} + O(\beta^{12})
 \end{aligned} \tag{A-3}$$

The average action per plaquette for triangles is extracted from (A-3) using

$$\langle E_{\Delta} \rangle = 1 - \frac{1}{20} \frac{\partial F}{\partial \beta} = 1 - \frac{1}{2} \langle \text{Re Tr } \dot{U} U U \rangle. \tag{A-4}$$

The strong-coupling expansion for $\langle E_{\square} \rangle$ is given by

$$\langle E_{\square} \rangle = 1 - 2\left(\frac{\beta}{4}\right)^4 - 16\left(\frac{\beta}{4}\right)^6 + O(\beta^8). \tag{A-5}$$

APPENDIX B - STRONG COUPLING EXPANSIONS ; STRING TENSION

Again the well-known techniques⁽⁷⁻¹⁰⁾ for computing the strong-coupling expansion of the string tension can be used. No difficulty arises in the triangular 2-plane of Fig. 1(a). As we are not interested in this paper in large orders, we just compute the first two corrections (sticking one or two tetrahedrons or one hexahedron on the minimal surface). Using the notation of Ref. 11, we got for any gauge group in any dimension

$$K_{\Delta} = -\frac{4}{\sqrt{3}} \left\{ \ln t + (d-2) \left[t^2 + \frac{B_{31}}{t} + \left(\frac{5}{2}d-8 \right) t^4 - A_4 + \dots \right] \right\}, \quad (B-1)$$

where K_{Δ} is the coefficient of the area, expressed in b^2 units, in the exponential behaviour of large Wilson loops. For $SU(2)$ in 4-dimensions, this yields

$$K_{\Delta} = -\frac{4}{\sqrt{3}} \left\{ \ln \beta - 2 \ln 2 + \frac{1}{12} \beta^2 - \frac{5}{576} \beta^4 + O(\beta^6) \right\}. \quad (B-2)$$

In the squared 2-plane of Fig. 1(b), the problem is more difficult due to entropy factors. An orthogonal projection on the plane $(\vec{\mathcal{E}}_{ij}, \vec{\mathcal{E}}_{kl})$ containing the loop reveals that the triangles with maximal projected area contain either a $\vec{\mathcal{E}}_{ij}$ or a $\vec{\mathcal{E}}_{kl}$ for one side, a $\vec{\mathcal{E}}_{ik}$, a $\vec{\mathcal{E}}_{il}$, a $\vec{\mathcal{E}}_{jk}$ or a $\vec{\mathcal{E}}_{jl}$ for the two other sides. Their projected area is $b^2/4$. Hence the minimal surface will contain four times more triangles than squares and $K_{\square}$ behaves as $-4 \ln \beta$.

However, this minimal surface is not unique. In Fig. 10, the triangles with maximal projected area have been represented; their vertices are heavy dots and they have two dashed lined sides

and one plain lined side. It is clear that any elementary dashed lined square might be covered by two triangles in two ways.

The counting proceeds as follows. Note first that the figure will always be 3-dimensional, using a third axis

$$\frac{1}{2} a(\vec{\mathcal{E}}_{il} + \vec{\mathcal{E}}_{jk}) \equiv \frac{1}{2} (\vec{\mathcal{E}}_{ik} + \vec{\mathcal{E}}_{jl}) .$$

Heavy dots have, in this direction, an integer third coordinate, which will be zero on the bounding Wilson loop. Moreover, two heavy dots connected by a dashed line must differ by exactly 1 in their third coordinate. Therefore, we have to count how many ways integers can be assigned to dots according to this rule. Furthermore, for each of these configurations and for each dashed lined elementary square, there are two ways to cover it by two triangles if both pairs of opposite corners have been assigned to the same integer. If this is not the case, there is however one pair of opposite corners which have been assigned to the same integer and there is just one way to choose the triangles to fit the square.

This counting procedure is then mapped on a known problem. Draw an arrow on each dashed segment to indicate what its highest end point is. Then rotate it around its middle point by an amount $\pi/2$. The problem is reduced to a particular case of the solved 8-vertex model⁽¹²⁾, called the F-model⁽¹³⁾. The six possible cases are displayed on Fig. 11 with the corresponding weights. In these weights, we include the first correction, consisting of replacing a triangle by the three other faces of a tetrahedron constructed on it. Further corrections, or use of the coupling constant γ , lead to a generalization of the obtained F-model, with longer range interactions,

and are not considered here. The free energy of the F-model gives the desired coefficient $K_{\square}$ and hence

$$K_{\square} = -4 \ln t - 4 \ln \frac{\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})} - 4t^2(d - \frac{8}{3}) + O(t^4), \quad (B-3)$$

which yields, in four dimensions, for SU(2) theory

$$K_{\square} = -4 \ln \beta - 4 \ln \frac{\Gamma(\frac{1}{4})}{8\Gamma(\frac{3}{4})} - \frac{1}{3} \beta^2 + O(\beta^4). \quad (B-4)$$

Number of matrix multiplications needed	Hypercubical	Simplicial
per link	12	6
per site	48	60
per volume expressed in lattice spacing units	48	$48\sqrt{5}$

Table 1. Estimation of the time for a Monte Carlo iteration for
a hypercubical and a simplicial lattice.

(4)	$\frac{1}{12} (d+1)_4$
(tetrahedron)
(6)	$\frac{1}{6} (d+1)_5$
(hexahedron)
(8)	$\frac{1}{2} (d+1)_6$
(8)	$\frac{1}{24} (d+1)_4 (d^2 - 7d + 13)$
(octahedron)
(10)	$\frac{1}{3} (d+1)_7$
(10)	$(d+1)_7$
(10)	$\frac{1}{2} (d+1)_7$
(10)	$\frac{1}{3} (d+1)_4 (d^3 - 12d^2 + 48d - 62)$
(10)	$\frac{1}{10} (d+1)_7$
(opposite 5-sided pyramids)		

Table 2. Diagram tables for a simplicial lattice.

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FIGURE CAPTIONS

- Fig. 1. Sections of the lattice in a 2-plane spanned by (a) $(\vec{\mathcal{E}}_{ij}, \vec{\mathcal{E}}_{jk})$ and (b) $(\vec{\mathcal{E}}_{ij}, \vec{\mathcal{E}}_{kl})$.
- Fig. 2. The average action per plaquette $\langle E_{\Delta} \rangle$ for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curves (a) and (b) represent the 8th and 10th order strong-coupling expansion of Eqn. A-3, respectively, curve (c) represents the [2,2] Padé approximation to the expansion and curve (d) represents the weak-coupling mean-field calculation.
- Fig. 3. The geometric form of the Wilson loops for pure SU(2) gauge theory on a simplicial lattice showing our numbering scheme.
- Fig. 4. The Wilson loops for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent W_1^{Δ} , the open circles represent W_2^{Δ} , the full circles represent W_3^{Δ} , the crosses represent W_4^{Δ} , the open upward triangles represent W_6^{Δ} , the full downward triangles represent W_9^{Δ} and the open downward triangles represent W_{13}^{Δ} .
- Fig. 5. The string tension for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent χ_1 , the full circles represent χ_2 and the open circles represent χ_3 . Also shown in the diagram is the behavior of Eqn.7.

Fig. 6. The coefficient of the perimeter term in the Wilson loops for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β .

Fig. 7. The average action per plaquette $\langle E_{\square} \rangle$ for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The curves (a) and (b) represent the 4th and 6th order strong-coupling expansions of Eqn. A-3 while curve (c) represents the weak-coupling mean-field calculation.

Fig. 8. The Wilson loops for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent $W_1^{\square}$, the open circles represent $W_2^{\square}$, the crosses represent $W_3^{\square}$, the open upward triangles represent $W_4^{\square}$, the full downward triangles represent $W_6^{\square}$ and the full circles represent $W_9^{\square}$.

Fig. 9. The string tension for pure SU(2) gauge theory on a simplicial lattice as a function of the inverse coupling constant squared β . The full upward triangles represent χ_1 and the full circles represent χ_2 .

Fig.10. Mapping the problem of counting minimal surfaces onto an F-model.

Fig.11. The 6 configurations of the allowed F-model and the weights.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig.10

$2[t + (d-3)t^3]^2$

$[t + (d-2)t^3]^2$

Fig.11