Appendix A
Vector Calculus

A.1 Differential Operators

Rectangular Coordinates

\[ \text{grad } \Phi = \hat{e}_x \frac{\partial \Phi}{\partial x} + \hat{e}_y \frac{\partial \Phi}{\partial y} + \hat{e}_z \frac{\partial \Phi}{\partial z} \]  
(A.1)

\[ \text{div } \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \]  
(A.2)

\[ \text{curl } \mathbf{F} = \hat{e}_x \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{e}_y \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{e}_z \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \]  
(A.3)

\[ \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}. \]  
(A.4)

Cylindrical Coordinates

\[ \text{grad } \Phi = \hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \hat{e}_z \frac{\partial \Phi}{\partial z} \]  
(A.5)

\[ \text{div } \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \]  
(A.6)

\[ \text{curl } \mathbf{F} = \hat{e}_r \left[ \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right] + \hat{e}_\theta \left[ \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right] + \hat{e}_z \frac{1}{r} \left[ \frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \theta} \right] \]  
(A.7)
\[ \nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}. \]  

(A.8)

**Spherical Coordinates**

\[ \text{grad} \; \Phi = \hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\phi \frac{1}{r \sin \phi} \frac{\partial \Phi}{\partial \phi} + \hat{e}_\phi \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \]  

(A.9)

\[ \text{div} \; \mathbf{F} = \frac{1}{r^2 \frac{\partial}{\partial r} \left( r^2 F_r \right)} + \frac{1}{r \sin \phi} \frac{\partial F_\phi}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} \left( F_\phi \sin \phi \right) \]  

(A.10)

\[ \text{curl} \; \mathbf{F} = \hat{e}_r \frac{1}{r \sin \phi} \left[ \frac{\partial}{\partial \phi} \left( F_\phi \sin \phi \right) - \frac{\partial F_r}{\partial \phi} \right] + \hat{e}_\phi \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r F_\phi \right) - \frac{\partial F_r}{\partial \phi} \right] \]  

\[ + \hat{e}_\phi \frac{1}{r \sin \phi} \left[ \frac{1}{\sin \phi} \frac{\partial F_r}{\partial \phi} - \frac{\partial F_\phi}{\partial r} \left( r F_\phi \right) \right] \]  

(A.11)

\[ \nabla^2 \Phi = \frac{1}{r^2 \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right)} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 \Phi}{\partial \phi^2} \]  

\[ + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial \Phi}{\partial \phi} \right). \]  

(A.12)

**A.2 Differential Operator Identities**

\[ \text{div} \; \text{grad} \; \Phi = \nabla^2 \Phi \]  

(A.13)

\[ \text{div} \; \text{curl} \; \mathbf{F} = 0 \]  

(A.14)

\[ \text{curl} \; \text{grad} \; \Phi = 0 \]  

(A.15)

\[ \text{curl} \; \text{curl} \; \mathbf{F} = \text{grad} \; \text{div} \; \mathbf{F} - \nabla^2 \mathbf{F} \]  

(A.16)

\[ \text{grad} \; (\Phi \Psi) = \Psi \; \text{grad} \; \Phi + \Phi \; \text{grad} \; \Psi \]  

(A.17)

\[ \text{grad} \; (\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \text{grad} \; \mathbf{G}) + \mathbf{F} \times (\text{curl} \; \mathbf{G}) \]  

\[ + (\mathbf{G} \cdot \text{grad} \; \mathbf{F}) + \mathbf{G} \times (\text{curl} \; \mathbf{F}) \]  

(A.18)

\[ \text{div} \; (\Phi \mathbf{F}) = (\text{grad} \; \Phi) \cdot \mathbf{F} + \Phi \; \text{div} \; \mathbf{F} \]  

(A.19)

\[ \text{div} \; (\mathbf{F} \times \mathbf{G}) = (\text{curl} \; \mathbf{F}) \cdot \mathbf{G} - (\text{curl} \; \mathbf{G}) \cdot \mathbf{F} \]  

(A.20)

\[ \text{curl} \; (\Phi \mathbf{F}) = (\text{grad} \; \Phi) \times \mathbf{F} + \Phi \; \text{curl} \; \mathbf{F} \]  

(A.21)

\[ \text{curl} \; (\mathbf{F} \times \mathbf{G}) = (\text{div} \; \mathbf{G}) \mathbf{F} - (\text{div} \; \mathbf{F}) \mathbf{G} \]  

\[ + (\mathbf{G} \cdot \text{grad} \; \mathbf{F}) - (\mathbf{F} \cdot \text{grad} \; \mathbf{G}) \]  

(A.22)
Appendix B
Dirac Delta Sequences

Here are some examples of $\delta$– sequences. The most useful in physics are the first three.

(a.)
\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \exp\left[ ik (x' - x) \right] = \delta(x - x') \quad \text{(B.1)}
\]

(b.)
\[
\left( \frac{1}{2\pi} \right)^3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d^3k \exp\left[ ik \cdot (\mathbf{r} - \mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}') \quad \text{(B.2)}
\]

(c.)
\[
\text{div grad} \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r}') \quad \text{(B.3)}
\]

(d.)
\[
\lim_{a \to 0} \left( \frac{1}{\pi^{1/2} a} \right) e^{-\frac{x^2}{a^2}} = \delta(x) \quad \text{(B.4)}
\]

(e.)
\[
\lim_{a \to 0} \left( \frac{\alpha}{\pi} \right) \frac{\sin^2 \left( \frac{x}{a} \right)}{x^2} = \delta(x) \quad \text{(B.5)}
\]

(f.)
\[
\lim_{a \to 0} \left( \frac{\alpha}{\pi} \right) \frac{1}{(x^2 + \alpha^2)} = \delta(x) \quad \text{(B.6)}
\]
Appendix C
Divergence and Curl of $B$

We begin with the integral form of the magnetic field (5.47)

$$B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times \mathbf{J}' dV'$$

(C.1)

Using (A.20) the divergence of (C.1) is

$$\text{div } B(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V'} \mathbf{J}' \cdot \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dV'$$

$$= 0,$$

(C.2)

since curl grad $\equiv 0$. Therefore

$$\text{div } B = 0$$

(C.3)

Taking the curl of $B$ with respect to the field coordinates, we have

$$\text{curl } B(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int_{V'} dV' \text{ curl } \left[ \mathbf{J}' \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]$$

(C.4)

Since $\mathbf{J}'$ is independent of $\mathbf{r}$, using (A.22) we have

$$-\text{curl } \left[ \mathbf{J}' \times \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]$$

$$= -\mathbf{J}' \text{ div } \nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) + \left( \mathbf{J}' \cdot \nabla' \right) \text{ grad } \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$= 4\pi \mathbf{J}' \delta(\mathbf{r} - \mathbf{r}') - \left( \mathbf{J}' \cdot \nabla' \right) \text{ grad } \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right).$$

(C.5)

Where we have used (2.110) and the fact that grad $f(\mathbf{r} - \mathbf{r}') = -\text{grad}' f(\mathbf{r} - \mathbf{r}')$. 


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Now, we note that
\[
\text{div}' \left[ J' \ \text{grad} \left( \frac{1}{|r - r'|} \right) \right] = (\text{div}' J') \ \text{grad} \left( \frac{1}{|r - r'|} \right) \\
+ (J' \cdot \text{grad}') \ \text{grad} \left( \frac{1}{|r_2 - r_1|} \right)
\]
\[
= (J' \cdot \text{grad}') \ \text{grad} \left( \frac{1}{|r_2 - r_1|} \right),
\]
(C.6)
since, in the static situation
\[
\text{div}' J' = 0.
\]
(C.7)
Then (C.5) becomes
\[
-\text{curl} \left[ J' \times \text{grad} \left( \frac{1}{|r - r'|} \right) \right] = 4\pi J' \delta (r - r') - \text{div}' \left[ J' \ \text{grad} \left( \frac{1}{|r - r'|} \right) \right]
\]
(C.8)
and
\[
\text{curl} \ \mathbf{B} (\mathbf{r}) = \mu_0 \int_V J' \delta (r' - r) \ dV' - \frac{\mu_0}{4\pi} \int_V \text{div}' \left[ J' \ \text{grad} \left( \frac{1}{|r - r'|} \right) \right] \ dV'
\]
\[
= \mu_0 J (\mathbf{r}) - \frac{\mu_0}{4\pi} \sum_{\alpha=1}^3 \hat{e}_\alpha \int_V \text{div}' \left[ J' G_\alpha (\mathbf{r}, \mathbf{r}') \right] \ dV'.
\]
(C.9)
where
\[
G_\alpha (\mathbf{r}, \mathbf{r}') = \frac{\partial}{\partial x_\alpha} \left( \frac{1}{|r_2 - r_1|} \right)
\]
(C.10)
is a scalar. Applying Gauss’ Theorem,
\[
\int_V \text{div}' \left[ J' G_\alpha (\mathbf{r}, \mathbf{r}') \right] \ dV' = \oint_S J' G_\alpha (\mathbf{r}, \mathbf{r}') \cdot d\mathbf{S}'.
\]
(C.11)
But here \( S \) is the surface of the conductor carrying the current density \( J' \). The current density vector is parallel to this surface everywhere. Therefore \( J' G_\alpha (\mathbf{r}, \mathbf{r}') \cdot d\mathbf{S}' = 0 \) everywhere and
\[
\text{curl} \ \mathbf{B} (\mathbf{r}) = \mu_0 J (\mathbf{r}).
\]
(C.12)
Appendix D

Green’s Theorem

Green’s Theorem is a valuable integral theorem involving analytic functions.

**Theorem D.1. Green’s Theorem.** If $\Phi$ and $\Psi$ are analytic everywhere within $V$ then

$$\int_V \left[ \Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi \right] dV = \oint_S \left[ \Phi \text{ grad } \Psi - \Psi \text{ grad } \Phi \right] \cdot dS$$

**Proof.** The divergence of the product $\Phi F$ is

$$\text{div } \Phi F = \Phi \text{ div } F + \text{grad } \Phi \cdot F.$$  \hfill (D.1)

With $F = \text{grad } \Psi$ we then have

$$\Phi \nabla^2 \Psi = \text{div } (\Phi \text{ grad } \Psi) - \text{grad } \Phi \cdot \text{grad } \Psi,$$

and

$$\Psi \nabla^2 \Phi = \text{div } (\Psi \text{ grad } \Phi) - \text{grad } \Psi \cdot \text{grad } \Phi.$$  

Then, using Gauss’ Theorem

$$\int_V \left[ \Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi \right] dV = \int_V \text{div } \left[ \Phi \text{ grad } \Psi - \Psi \text{ grad } \Phi \right] dV$$

$$= \oint_S \left[ \Phi \text{ grad } \Psi - \Psi \text{ grad } \Phi \right] \cdot dS.$$
Appendix E
Laplace’s Equation

**Theorem E.1.** If \( \Phi_1 \) and \( \Phi_2 \) are solutions of Laplace’s equation, then \( (a\Phi_1 + b\Phi_2) \) is as well.

*Proof.* Since the Laplacian is a linear operator

\[
\nabla^2 (a\Phi_1 + b\Phi_2) = a \nabla^2 \Phi_1 + b \nabla^2 \Phi_2
\]

\[= 0.\]

**Theorem E.2.** If \( \nabla^2 \Phi = 0 \) in a region \( V \) and \( \Phi = 0 \) on the surface \( S \) of \( V \), then \( \Phi = 0 \) everywhere in \( V \).

*Proof.* Since \( \nabla^2 \Phi = 0 \),

\[
\Phi \nabla^2 \Phi = 0,
\]

and

\[
\int_V \Phi \nabla^2 \Phi dV = 0.
\]

From (D.1)

\[
\Phi \nabla^2 \Phi = \text{div} (\Phi \text{ grad } \Phi) - \text{ grad } \phi \cdot \text{grad } \Phi = 0.
\]

by hypothesis. Then

\[
0 = \int_V \left[\text{div} (\Phi \text{ grad } \Phi) - \text{ grad } \phi \cdot \text{grad } \Phi\right] dV
\]

\[= \oint_S (\Phi \text{ grad } \Phi) \cdot dS - \int_V |\text{grad } \Phi|^2 dV \quad \text{(E.1)}
\]

using Gauss’ Theorem.
Since \( \Phi = 0 \) on \( S \),
\[
\oint_S (\Phi \text{ grad } \Phi) \cdot dS = 0,
\]
and
\[
\int_V |\text{grad } \Phi|^2 dV = 0 \tag{E.2}
\]
Now \( |\text{grad } \Phi|^2 \geq 0 \). Therefore, in order for the volume integral (E.2) to vanish,
\[
\text{grad } \Phi = 0
\]
in \( V \). That is \( \Phi = \text{constant in } V \). But \( \Phi = 0 \) on the boundary. Therefore \( \Phi = 0 \) in \( V \).

**Corollary E.1.** If \( \nabla^2 \Phi = 0 \) in a region \( V \) and \( \partial \Phi / \partial n = 0 \) on the surface \( S \) of \( V \), then \( \Phi = \text{constant in } V \).

**Proof.** As in the proof of Theorem E.2,
\[
0 = \oint_S (\Phi \text{ grad } \Phi) \cdot dS - \int_V |\text{grad } \Phi|^2 dV.
\]
Since \( \partial \Phi / \partial n = \hat{n} \cdot \text{grad } \Phi = 0 \), \( \text{grad } \Phi = 0 \) on \( S \). Therefore
\[
\oint_S (\Phi \text{ grad } \Phi) \cdot dS = 0,
\]
and
\[
\int_V |\text{grad } \Phi|^2 dV = 0.
\]
So \( \Phi = \text{constant in } V \).

**Corollary E.2.** If \( \nabla^2 \Phi = 0 \) in all space and \( r \Phi (r) \rightarrow \text{function of } (\partial, \phi) \) alone as \( r \rightarrow \infty \) then \( \Phi = 0 \) everywhere.

**Proof.** By hypothesis
\[
\Phi = \frac{f(\partial, \phi)}{r}
\]
on \( S \). Then
\[
\frac{\partial \Phi}{\partial r} = -\frac{f(\partial, \phi)}{r^2}
\]
on \( S \). On the surface at infinity
\[
\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial n} = \hat{n} \cdot \text{grad } \Phi
\]
and
\[ \hat{n} \cdot \text{grad } \Phi \, dS = \text{grad } \Phi \cdot dS. \]

Therefore
\[ \oint_S \Phi \, \text{grad } \Phi \cdot dS \propto - \int_\Omega \frac{f(\partial, \phi)}{r} \frac{f(\partial, \phi)}{r^2} r^2 d\Omega. \]

Here we have written the differential surface area as \( r^2 d\Omega \) where \( \Omega \) is the solid angle with \( d\Omega = \sin \phi d\theta d\phi \). Then
\[ \oint_S \Phi \, \text{grad } \Phi \cdot dS \propto - \frac{1}{r} \int_\Omega \frac{|f(\partial, \phi)|^2}{r^2} d\Omega, \]
and
\[ \lim_{r \to \infty} \frac{1}{r} \int_\Omega \frac{|f(\partial, \phi)|^2}{r} d\Omega = 0. \]

Therefore
\[ \oint_S \Phi \, \text{grad } \Phi \cdot dS = 0. \]

Then, using (E.1), which is valid if \( \Phi \) satisfies Laplace’s Equation, we have
\[ \int_V |\text{grad } \Phi|^2 \, dV = 0 \]
and, as a consequence,
\[ \text{grad } \Phi = 0 \]
and
\[ \Phi = \text{constant} \]
in \( V \). Then, since we require also that \( \lim_{r \to \infty} r \Phi \) is independent of \( r \), the only constant value of \( \Phi \) that is possible is \( \Phi = 0 \).

**Theorem E.3.** If \( \nabla^2 \Phi = 0 \) in \( V \) and \( \Phi \) takes on specified values on the surface \( S \) bounding \( V \), then if a solution exists for \( \Phi \) it is unique.

**Proof.** Assume \( \Phi_1 \) and \( \Phi_2 \) are two distinct solutions. Define
\[ \Phi = \Phi_1 - \Phi_2. \]

Then
\[ \nabla^2 \Phi = \nabla^2 \Phi_1 - \nabla^2 \Phi_2 = 0. \]

Since \( \Phi_1 \) and \( \Phi_2 \) are solutions to the same problem they have the same values on the boundary. Therefore \( \Phi = 0 \) on \( S \). Then from **Theorem E.2** \( \Phi = 0 \) everywhere in \( V \). This is true if and only if \( \Phi_1 = \Phi_2 \).
**Theorem E.4.** If \( \mathbf{r} \) and \( \mathbf{r}' \) are position vectors from the origin in \( V \), then

\[
\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = 0
\]

if \( \mathbf{r} \neq \mathbf{r}' \).

*Proof.* See exercises in Chap. 2.

**Theorem E.5.** If \( \Phi \) is continuous and has continuous first derivatives at \( \mathbf{r} = \mathbf{r}' \) and \( S_R \) is the surface of a sphere of radius \( R \) centered at \( \mathbf{r}' \) then

\[
\lim_{R \to 0} \oint_{S_R} \left[ \Phi \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \Phi \right] \cdot dS = -4\pi \Phi(\mathbf{r}')
\]

*Proof.* Since \( \Phi \) is continuous and has continuous first derivatives, \( R \) can be chosen so small that \( \Phi \) is a constant (or deviates from a constant value by an amount \( < \varepsilon \)) over \( S \). Then the integral above is

\[
\Phi(\mathbf{r}') \oint_{S_R} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS - \Phi(\mathbf{r}') \oint_{S_R} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS.
\]

Here \( \Phi(\mathbf{r}') \) is the value of \( \Phi \) at \( \mathbf{r} \) and, therefore, the value of \( \Phi \) on \( S_R \). Now (see exercises in Chap. 2)

\[
\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r} - \mathbf{r}'|^3}.
\]

And, since \( dS \) points outward from the volume centered on \( \mathbf{r}' \), and the point \( \mathbf{r} \) is the location of \( dS \),

\[
dS = \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r} - \mathbf{r}'|} dS.
\]

Then

\[
\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS = -\frac{1}{R^2} dS = -d\Omega,
\]

the differential solid angle (see proof of corollary E.2), and the first integral becomes

\[
\Phi(\mathbf{r}') \oint_{S_R} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS = -\Phi(\mathbf{r}') \oint_{S_R} d\Omega
\]

\[
= -4\pi \Phi(\mathbf{r}').
\]

The second integral is
\[
\n\text{grad} \Phi (\mathbf{r'}) \cdot \oint_{S_R} \frac{1}{|\mathbf{r} - \mathbf{r'}|} \, dS = \text{grad} \Phi (\mathbf{r'}) \cdot \oint_{S_R} \frac{1}{R} \frac{\mathbf{r'} - \mathbf{r}}{R} \, dS \\
= \text{grad} \Phi (\mathbf{r'}) \cdot \oint_{S_R} (\mathbf{r'} - \mathbf{r}) \, d\Omega.
\]

This integral vanishes by symmetry. For every \( \mathbf{r} \) on the surface there is another point diametrically opposed, which produces a contribution which is the negative of that from \( \mathbf{r} \).

Therefore

\[
\lim_{R \to 0} \oint_{S_R} \left[ \Phi \text{grad} \frac{1}{|\mathbf{r} - \mathbf{r'}|} - \frac{1}{|\mathbf{r} - \mathbf{r'}|} \text{grad} \Phi \right] \cdot dS = -4\pi \Phi (\mathbf{r'})
\]

**Theorem E.6.** If \( \nabla^2 \Phi = 0 \) in \( V \) and \( \mathbf{r'} \) is a point in \( V \), then

\[
\Phi (\mathbf{r'}) = -\frac{1}{4\pi} \oint_{S} \left[ \Phi \text{grad} \frac{1}{|\mathbf{r} - \mathbf{r'}|} - \frac{1}{|\mathbf{r} - \mathbf{r'}|} \text{grad} \Phi \right] \cdot dS,
\]

where \( S \) bounds \( V \).

**Proof.** We divide \( V \) into the regions \( V - O \) and \( O \), where the region \( O \) is a very small spherical region centered on \( \mathbf{r'} \) that we will shrink to zero. Green’s Theorem, written for the region \( V - O \) is

\[
\int_{V - O} \left[ \Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right] \, dV = \oint_{S + O} [\Phi \text{grad} \psi - \psi \text{grad} \Phi] \cdot dS.
\]

The region \( V - O \) is bounded by the original boundary \( S \) and the surface surrounding \( \mathbf{r'} \).

We now choose

\[
\psi = \frac{1}{|\mathbf{r} - \mathbf{r'}|}.
\]

Since the small sphere surrounding the point \( \mathbf{r'} \) has been eliminated, from *Theorem E.4* we have then

\[
\nabla^2 \psi = 0.
\]

And, because \( \nabla^2 \Phi = 0 \),

\[
\int_{V - O} \left[ \Phi \nabla^2 \psi - \psi \nabla^2 \Phi \right] \, dV = 0.
\]
Then

\[ 0 = \oint_S \left[ \Phi \frac{\nabla}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \Phi \right] \cdot d\mathbf{S} \]

\[ -\oint_O \left[ \Phi \frac{\nabla}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \Phi \right] \cdot d\mathbf{S}, \]

the negative sign coming from the convention on \( d\mathbf{S} \) as pointing out of the original volume, i.e. into \( O \). From Theorem E.5 the integral over \( O \) is \(-4\pi\Phi(\mathbf{r}')\). Then

\[ \Phi(\mathbf{r}') = -\frac{1}{4\pi} \oint_S \left[ \Phi \frac{\nabla}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla \Phi \right] \cdot d\mathbf{S} \]
Appendix F
Poisson’s Equation

Theorem F.1. If \( \nabla^2 \Phi_1 = -g_1 \) and \( \nabla^2 \Phi_2 = -g_2 \) then
\[
\nabla^2 (\Phi_1 + \Phi_2) = -(g_1 + g_2).
\]

Proof. The proof is obvious.

Theorem F.2. If \( \nabla^2 \Phi = -g \) and \( \Phi \) takes on specified values on the surface \( S \) of \( V \), then \( \Phi \) is uniquely determined in \( V \).

Proof. Assume that there are two separate solutions \( \Phi_1 \) and \( \Phi_2 \) satisfying
\[
\nabla^2 \Phi_1 = -g
\]
and
\[
\nabla^2 \Phi_2 = -g
\]
in \( V \). If we define
\[
\Phi = \Phi_1 - \Phi_2,
\]
then
\[
\nabla^2 \Phi = 0
\]
from Theorem F.1. Because \( \Phi_1 \) and \( \Phi_2 \) have the same values on the surface \( S \), \( \Phi = 0 \) on the surface \( S \). Therefore, from Theorem E.2 \( \Phi = 0 \) throughout \( V \).

Theorem F.3. If \( \nabla^2 \Phi = -g \) in \( V \), then
\[
\Phi (\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{g(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'
\]
is a particular solution.

Proof. We again exclude the point \( \mathbf{r}' \) from the volume \( V \) by enclosing \( \mathbf{r}' \) within a small volume \( O \). Then, if the theorem is valid we have
\[ \nabla^2 \Phi (\mathbf{r}) = \frac{1}{4\pi} \int_{V-O} g (\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' + \frac{1}{4\pi} \int_{O} g (\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV'. \]

Since the integral is over the primed coordinates, \( \nabla^2 \) can be brought inside the integral where it does not operate on \( g (\mathbf{r}') \). The first integral on the right hand side is zero from Theorem E.4, since throughout the volume \( V - O \) we have \( \mathbf{r} \neq \mathbf{r}' \). Since
\[ \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} \]
(see exercises Chap. 2), we can write the second integral on the right hand side as
\[ \frac{1}{4\pi} \int_{O} g (\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi} \int_{O} g (\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV'. \]

Since this integral vanishes everywhere except when \( \mathbf{r} = \mathbf{r}' \), we can exchange the primed and unprimed variables in the integration and
\[ \frac{1}{4\pi} \int_{O} g (\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi} \int_{O} g (\mathbf{r}) \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV. \]

The volume \( O \) is a very small (infinitesimal) volume, and, since \( \rho \) is a continuous function of spatial coordinates, we may consider it to be constant throughout the volume \( O \). Gauss’ Theorem then results in
\[ \frac{1}{4\pi} \int_{O} g (\mathbf{r}) \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV = \frac{1}{4\pi} g (\mathbf{r}) \oint_{S_O} \text{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS \]

where \( S_O \) is the surface around the volume \( O \). Since
\[ \text{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS = -d\Omega \]
(see Theorem E.5), the integral becomes
\[ \frac{1}{4\pi} g (\mathbf{r}) \oint_{S_O} \text{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS = -g (\mathbf{r}). \]

And the result is equal to \( \nabla^2 \Phi (\mathbf{r}) \). This establishes the theorem.
Appendix G
Helmholtz’ Equation

Theorem G.1. If \((\nabla^2 + K^2) \Phi_1 = -h_1\) and \((\nabla^2 + K^2) \Phi_2 = -h_2\) then
\[
(\nabla^2 + K^2) (\Phi_1 + \Phi_2) = -(h_1 + h_2).
\]

Proof. The proof is obvious.

Theorem G.2. If \((\nabla^2 + K^2) \Phi = -h\) and \(\Phi\) takes on specified values on the surface \(S\) of \(V\), then \(\Phi\) is uniquely determined in \(V\).

Proof. Assume that there are two separate solutions \(\Phi_1\) and \(\Phi_2\) satisfying
\[
(\nabla^2 + K^2) \Phi_1 = -h
\]
and
\[
(\nabla^2 + K^2) \Phi_2 = -h
\]
in \(V\). If we define
\[
\Phi = \Phi_1 - \Phi_2,
\]
then
\[
(\nabla^2 + K^2) \Phi = 0
\]
from Theorem G.1. Because \(\Phi_1\) and \(\Phi_2\) have the same values on the surface \(S\), \(\Phi = 0\) on the surface \(S\). Therefore, from Theorem E.2 \(\Phi = 0\) throughout \(V\).

Theorem G.3. If \((\nabla^2 + K^2) \Phi = -h\) in \(V\), then
\[
\Phi (\mathbf{r}) = \frac{1}{4\pi} \int_V \frac{h (\mathbf{r'}) \exp(\pm iK |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV'
\]
is a particular solution.

Proof. We again exclude the point \(\mathbf{r}'\) from the volume \(V\) by enclosing \(\mathbf{r}'\) within a small volume \(O\). Then, if the theorem is valid we have
\[(\nabla^2 + K^2) \Phi (\mathbf{r}) = \frac{1}{4\pi} \int_{V-O} h (\mathbf{r}') (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV' + \frac{1}{4\pi} \int_{O} h (\mathbf{r}') (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV'.\]

Since the integral is over the primed coordinates, \(\nabla^2\) can be brought inside the integral where it does not operate on \(\mathbf{h}\). The first integral on the right hand side is zero from Theorem E.4, since throughout the volume \(V - O\) we have \(\mathbf{r} \neq \mathbf{r}'\). Since

\[\left(\nabla^2 + K^2\right) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} = \left(\nabla^2 + K^2\right) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|}\]

(see exercises Chap. 2), we can write the second integral on the right hand side as

\[\frac{1}{4\pi} \int_{O} h (\mathbf{r}') (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi} \int_{O} h (\mathbf{r}') (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV'.\]

Since this integral vanishes everywhere except when \(\mathbf{r} = \mathbf{r}'\), we can exchange the primed and unprimed variables in the integration and

\[\frac{1}{4\pi} \int_{O} h (\mathbf{r}) (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV' = \frac{1}{4\pi} \int_{O} h (\mathbf{r}) (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV.\]

The volume \(O\) is a very small (infinitesimal) volume, and, since \(\rho\) is a continuous function of spatial coordinates, we may consider it to be constant throughout the volume \(O\). Gauss’ Theorem then results in

\[\frac{1}{4\pi} \int_{O} h (\mathbf{r}) (\nabla^2 + K^2) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV = \frac{1}{4\pi} h (\mathbf{r}) \oint_{S_0} \mathbf{\nabla} \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{S} + \frac{K^2}{4\pi} \int_{O} h (\mathbf{r}) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV \quad \text{(G.1)}\]

where \(S_0\) is the surface around the volume \(O\). As the volume \(O\) shrinks to zero the distance between the points \(|\mathbf{r} - \mathbf{r}'|\) also shrinks to zero. Then, as the volume \(O\) shrinks to zero
and the first integral on the right hand side of (G.1) becomes

\[
\frac{1}{4\pi} h (\mathbf{r}) \oint_{S_0} \text{grad} \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \cdot dS = \frac{1}{4\pi} h (\mathbf{r}) \oint_{S_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS. 
\]

(G.3)

Since

\[
\text{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot dS = -d\Omega
\]

(see Theorem E.5), the integral on the right hand side of (G.3) is

\[
\frac{1}{4\pi} h (\mathbf{r}) \oint_{S_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} dS = -h (\mathbf{r}).
\]

(G.4)

With (G.2) the second integral on the right hand side of (G.1) becomes

\[
\frac{K^2}{4\pi} \int_{\Omega} h (\mathbf{r}) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV = \frac{K^2}{4\pi} \int_{\Omega} h (\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV.
\]

(G.5)

For infinitesimal \( \Omega \) and analytic \( h (\mathbf{r}) \) the integral on the right hand side of (G.5) is

\[
\frac{K^2}{4\pi} \int_{\Omega} h (\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{r}'|} dV = K^2 h (\mathbf{r}) \lim_{R \to 0} \int_{\Omega} RdR = 0.
\]

Therefore using (G.4), (G.1) is

\[
(\nabla^2 + K^2) \left\{ \frac{1}{4\pi} \int_{\Omega} h (\mathbf{r}) \frac{\exp (\pm i K |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} dV \right\} = -h (\mathbf{r}),
\]

which establishes the theorem.
Appendix H
Legendre’s Equation

The differential equation
\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{d}{dx} P_n \right] - n (n + 1) P_n = 0 \quad \text{(H.1)}
\]
or
\[
(x^2 - 1) \frac{d^2}{dx^2} P_n + 2x \frac{d}{dx} P_n - n (n + 1) P_n = 0 \quad \text{(H.2)}
\]
is Legendre’s Equation. It is solved by the polynomials
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad \text{(H.3)}
\]

Which are the Legendre Polynomials. Our proof of this will follow the suggested approach in ([16], p. 86).

We define the function
\[
u(x) = (x^2 - 1)^n, \quad \text{(H.4)}
\]
for which we have the identity
\[
(x^2 - 1) \frac{du}{dx} = 2nx (x^2 - 1)^n = 2nxu. \quad \text{(H.5)}
\]

We shall now show that the \((n+1)\)th order derivative of \((H.5)\) produces Legendre’s equation \((H.2)\) for the polynomial
\[
y_n(x) = \frac{d^n u}{dx^n} = \frac{d^n}{dx^n} (x^2 - 1)^n, \quad \text{(H.6)}
\]

which differs from the \(n\)th order Legendre polynomial \((H.3)\) by the constant factor \(1/(2^n n!)\), which has no effect on the solution.
By carrying out the derivatives we see that

\[
\frac{d}{dx} \left[ (x^2 - 1) \frac{du}{dx} \right] = 2x \frac{du}{dx} + (x^2 - 1) \frac{d^2u}{dx^2},
\]

\[
\frac{d^2}{dx^2} \left[ (x^2 - 1) \frac{du}{dx} \right] = 2 \frac{du}{dx} + 4x \frac{d^2u}{dx^2} + (x^2 - 1) \frac{d^3u}{dx^3},
\]

\[
\frac{d^3}{dx^3} \left[ (x^2 - 1) \frac{du}{dx} \right] = 6 \frac{d^2u}{dx^2} + 6x \frac{d^3u}{dx^3} + (x^2 - 1) \frac{d^4u}{dx^4},
\]

\[
\frac{d^4}{dx^4} \left[ (x^2 - 1) \frac{du}{dx} \right] = 12 \frac{d^3u}{dx^3} + 8x \frac{d^4u}{dx^4} + (x^2 - 1) \frac{d^5u}{dx^5},
\]

\[\vdots\]

\[
\frac{d^{n+1}}{dx^{n+1}} \left[ (x^2 - 1) \frac{du}{dx} \right] = n (n + 1) \frac{d^n u}{dx^n} + 2 (n + 1) x \frac{d^{n+1} u}{dx^{n+1}} + (x^2 - 1) \frac{d^{n+2} u}{dx^{n+2}}. \tag{H.7}
\]

and

\[
\frac{d}{dx} (2nxu) = 2nu + 2nx \frac{du}{dx},
\]

\[
\frac{d^2}{dx^2} (2nxu) = 4n \frac{du}{dx} + 2nx \frac{d^2u}{dx^2},
\]

\[
\frac{d^3}{dx^3} (2nxu) = 6n \frac{d^2u}{dx^2} + 2nx \frac{d^3u}{dx^3},
\]

\[\vdots\]

\[
\frac{d^{n+1}}{dx^{n+1}} (2nxu) = 2n (n + 1) \frac{d^n u}{dx^n} + 2nx \frac{d^{n+1} u}{dx^{n+1}}. \tag{H.8}
\]

Then, equating the identities (H.7) and (H.8), we have

\[
(x^2 - 1) \frac{d^{n+2} u}{dx^{n+2}} + 2x \frac{d^{n+1} u}{dx^{n+1}} - n (n + 1) \frac{d^n u}{dx^n} = 0. \tag{H.9}
\]

with (H.6) equation (H.9) becomes

\[
(x^2 - 1) \frac{d^2 y_n}{dx^2} + 2x \frac{dy_n}{dx} - n (n + 1) y_n = 0. \tag{H.10}
\]

Multiplying (H.10) by 1/ \((2^n n!)\), we have
\[ (x^2 - 1) \frac{d^2 P_n}{dx^2} + 2x \frac{dP_n}{dx} - n(n + 1) P_n = 0, \]  
(H.11)

which is Legendre’s Equation (H.2).
Appendix I
Jacobians

The Jacobian or Jacobian determinant is used to transform the integral

\[ \int \int_R f(x, y) \, dx \, dy \]

performed over \( x \) and \( y \) in a region \( R \) in the \((x, y)\) plane into

\[ \int \int_{R'} f(u, v) \, du \, dv \]

over \( u \) an \( v \) in a region \( R' \) defined by those variables.

Here we will perform the transformation in two steps for the sake of clarity. The first step will transform only the \( y \) and the second only the \( x \). We designate the first transformation as from \( R \) to \( B \) and the second as from \( B \) to \( R' \).

We write the first transformation step as

\[ x = x \\
y = \Phi(v, x) . \tag{I.1} \]

We have illustrated this in the Fig. I.1, where we have drawn cells \( \Delta R_{ij} \) and \( \Delta B_{ij} \) in the two regions \( R \) and \( B \) indicated in panels (a) and (b) of Fig. I.1.

In panel (a) we have the rectangular Cartesian grid of the \((x, y)\) plane. In panel (b) we have the transformed grid. In this we assume that the partial derivative \( \partial \Phi/\partial y = \Phi_v \neq 0 \) everywhere.

The Riemann integral is

\[ \int \int_R f(x, y) \, dx \, dy = \lim_{N \to \infty, \Delta R_{ij} \to 0} \sum_{i,j}^N f_{ij} \Delta R_{ij} \tag{1.2} \]
\[ \lim_{N \to \infty, \Delta B_{ij} \to 0} \sum_{i,j} f_{ij} \Delta B_{ij} \]  
(I.3)

where
\[ f_{ij} = f(x_i, y_j) \]  
(I.4)

and
\[ \Delta B_{ij} = \int_{x_i}^{x_i+w} \left[ \Phi(v_j, x) - \Phi(v_j + h, x) \right] dx. \]  
(I.5)

Now
\[ \lim_{h \to 0} \frac{\Phi(v, x) - \Phi(v + h, x)}{h} = \frac{\partial \Phi}{\partial v} = \Phi_v(v, x), \]  
(I.6)

which, from (I.1) is
\[ \Phi_v(v, x) = \frac{\partial y}{\partial v}. \]  
(I.7)

Using (I.6) (I.5) becomes
\[ \Delta B_{ij} = h \int_{x_i}^{x_i+w} \Phi_v(v_j, x) dx, \]  
(I.8)

in which \( v_j \) is a point within the range of \( v \) chosen to best approximate the area. Integrating over \( x \) we have
\[ \Delta B_{ij} = hw \Phi_v(v_j, \bar{x}_i), \]  
(I.9)

where \( \bar{x}_i \) provides the best approximation to the area. We may recognize that (I.9) is the central limit theorem. With (I.9) equation (I.3) becomes
\[ \int \int_R f(x, y) \, dx \, dy = \lim_{N \to \infty, \Delta B_{ij} \to 0} \sum_{i,j} f(\bar{x}_i, \Phi_v(v_j, \bar{x}_i)) h w \Phi_v(v_j, \bar{x}_i), \]  
(I.10)

since \( \bar{y}_j = \Phi_v(v_j, \bar{x}_i) \). Taking the limits specified in (I.10) we have
\[
\int \int_{\mathbb{R}} f(x, y) \, dx \, dy = \int \int_{\mathbb{B}} f(x, \psi(x, y)) \, dx \, dy.
\] (I.11)

We can now transform \(x\) in the same fashion. We define this step by

\[
v = \psi(u, v).
\] (I.12)

The result is

\[
\int \int_{\mathbb{R}} f(x, y) \, dx \, dy = \int \int_{\mathbb{R}'} f(u, \psi(u, v)) \, du \, dv.
\] (I.13)

where

\[
\psi_u = \frac{\partial x}{\partial u}.
\] (I.14)

The Jacobian determinant is defined as

\[
\frac{\partial (x, y)}{\partial (u, v)} = \det \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{bmatrix}
\]

\[
= \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}.
\] (I.15)

Then

\[
\frac{\partial (x, y)}{\partial (x, y)} = \begin{vmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
1 & 0 \\
0 & 1
\end{vmatrix}
\]

\[
= \frac{\partial y}{\partial v}.
\] (I.16)

since \(x\) and \(y\) are independent variables. Likewise
\[
\frac{\partial (x, v)}{\partial (u, v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial v}{\partial u} & \frac{\partial v}{\partial v}
\end{vmatrix}
\]
\[
= \frac{\partial x}{\partial u}
\]
(1.17)

since \( u \) and \( v \) are independent variables.

Jacobians obey a chain rule of the form

\[
\frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial (x, y)}{\partial (\xi, \eta)} \frac{\partial (\xi, \eta)}{\partial (u, v)}.
\]
(1.18)

To establish the validity of (1.18) we consider the transformations

\[
\xi = \phi (x, y) ; \eta = \psi (x, y)
\]
and

\[
u = \Phi (\xi, \eta) ; v = \Psi (\xi, \eta) .
\]

The partial derivatives in the Jacobian are then

\[
\frac{\partial u}{\partial x} = \frac{\partial \Phi}{\partial \xi} \frac{\partial \xi}{\partial x} - \frac{\partial \Phi}{\partial \eta} \frac{\partial \eta}{\partial x}
= \Phi_\xi \phi_x + \Phi_\eta \psi_x,
\]

\[
\frac{\partial u}{\partial y} = \Phi_\xi \phi_y + \Phi_\eta \psi_y,
\]

\[
\frac{\partial v}{\partial x} = \Psi_\xi \phi_x + \Psi_\eta \psi_x,
\]

and

\[
\frac{\partial v}{\partial y} = \Psi_\xi \phi_y + \Psi_\eta \psi_y.
\]

The Jacobian is then

\[
\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{vmatrix}
\]
which establishes the chain rule for Jacobians.

Combining (I.7), (I.14), (I.16), (I.14), with (I.18) we then have

\[
\frac{\partial (x, y)}{\partial (u, v)} = \frac{\partial (x, y) \partial (x, v)}{\partial (x, v) \partial (u, v)} = \Phi_x, \Phi_y.
\]

With (I.19) (I.13) becomes

\[
\int \int_R f(x, y) \, dx \, dy = \int \int_{R'} f(u, v) \frac{\partial (x, y)}{\partial (u, v)} \, du \, dv.
\]

Equation (I.20) provides a transformation from one integral into another.

This result can be extended in the same fashion to any number of variables. That is

\[
dx_1 \, dx_2 \cdots \, dx_n = \frac{\partial (x_1, x_2, \cdots, x_n)}{\partial (\xi_1, \xi_2, \cdots, \xi_n)} \, d\xi_1 \, d\xi_2 \cdots \, d\xi_n.
\]
Appendix J
Dispersion

This treatment of the energy in a damped and dispersed wave in a nonmagnetic medium will be more general than the treatment in Chap. 16. Here we will consider the tensor character of the conductivity and conduct an expansion of the wave equation in Fourier space. Our discussion parallels that of [4] and [90]. Abraham Bers has also provided a short treatment of this general situation in [7].

Our treatment here may be considered to be more elegant than that in Chap. 16. But the results are fundamentally unchanged.

As in our treatment in Chap. 16, we will again consider waves

\[ \mathbf{E}(\mathbf{r}, t) = \frac{1}{2} [\mathbf{E} \exp (i \omega t - i \mathbf{k} \cdot \mathbf{r}) \]
\[ + \mathbf{E}^* \exp (-i \omega^* t + i \mathbf{k}^* \cdot \mathbf{r})] \]  \hspace{1cm} (J.1)

and

\[ \mathbf{B}(\mathbf{r}, t) = \frac{1}{2} [\mathbf{B} \exp (i \omega t - i \mathbf{k} \cdot \mathbf{r}) \]
\[ + \mathbf{B}^* \exp (-i \omega^* t + i \mathbf{k}^* \cdot \mathbf{r})], \]  \hspace{1cm} (J.2)

that are very nearly monochromatic. As we pointed out in Chap. 16 the field vectors \( \mathbf{E} \) and \( \mathbf{B} \) in (J.1) and (J.2), which are not functions of \( (\mathbf{k}, \omega) \).

We accept Ohm’s Law

\[ \mathbf{J}(\mathbf{k}, \omega) = \sigma(\mathbf{k}, \omega) \cdot \mathbf{E} \]  \hspace{1cm} (J.3)

as valid and base our treatment on the Fourier transformed form of Maxwell’s Equations in (16.8). The wave vectors \( \mathbf{E} \) or \( \mathbf{B} \) must then satisfy

\[ \mathbf{D}(\mathbf{k}, \omega) \cdot (\mathbf{E} \text{ or } \mathbf{B}) = 0, \]  \hspace{1cm} (J.4)
where

\[ D(\mathbf{k}, \omega) = \left( k^2 - \frac{\omega^2}{c^2} K \right) \mathbf{1} - \omega k + i \omega \mu_0 \sigma, \]  

(J.5)

where \( K \) is the dielectric constant for the matter.

Equation (J.4) is the wave equation in Fourier space. Written for the electric field in Einstein subscript notation the wave (J.4) is

\[
\left[ \left( k^2 - \frac{\omega^2}{c^2} K \right) \delta_{\alpha\beta} - k_{\alpha} k_{\beta} + i \omega \mu_0 \sigma_{\alpha\beta} \right] E_{\beta} = 0
\]  

(J.6)

For the undamped and undispersed case, \( k \) and \( \omega \) are real. Taking the Hermitian conjugate (adjoint) of (J.6) for real \( k \) and \( \omega \) we have

\[
E_{\beta} \left[ \left( k^2 - \frac{\omega^2}{c^2} K \right) \delta_{\beta\alpha} - k_{\beta} k_{\alpha} - i \omega \mu_0 \sigma^*_{\beta\alpha} \right] = 0
\]  

(J.7)

The wave (J.6) and (J.7) are identical if

\[
\sigma^*_{\beta\alpha} = -\sigma_{\alpha\beta}
\]  

(J.8)

which is the requirement that the conductivity tensor is antihermitian at the undamped condition.

The dispersion relation for undamped and undispersed plane waves is

\[
\det D = 0.
\]  

(J.9)

We are interested in wavelike solutions that differ only slightly from the undamped and undispersed solutions. For these waves there will be imaginary contributions to \( k \) and \( \omega \), which we shall simply identify as \( \Delta k \) and \( \Delta \omega \). There will also be a change in the conductivity tensor \( \sigma \) resulting from the imaginary additions to \( k \) and \( \omega \) as well as the addition of a small structural change to the form of \( \sigma \). The wave (J.6), for the slightly damped wave is

\[
0 = \left\{ \left[ (k + \Delta k)^2 - \frac{(\omega + \Delta \omega)^2}{c^2} K \right] \mathbf{1} - (k + \Delta k)(k + \Delta k) \right. 
+ i (\omega + \Delta \omega) \mu_0 (\sigma + \Delta \sigma) \right\} \cdot \mathbf{E}.
\]  

(J.10)

If we multiply (J.10) on the left by \( \mathbf{E}^* \) and hold terms to first order in \( \Delta \) we have

\[
0 = \mathbf{E}^* \cdot \left[ \left( 2k \cdot \Delta k - \frac{2\omega \Delta \omega}{c^2} K \right) \mathbf{1} - k \Delta k - \Delta kk \right.
+ i \Delta \omega \mu_0 \sigma + \omega \mu_0 \Delta \sigma] \cdot \mathbf{E}.
\]  

(J.11)
From the Fourier Transformed form of Maxwell’s Equations (16.8) we find that
\[
\omega A k \cdot (E^* \times B + E \times B^*)
= 2A k \cdot k E^2 - E^* \cdot k A k \cdot E - E \cdot k A k \cdot E^*, \quad (J.12)
\]
and
\[
\omega \left( \varepsilon E^2 - \frac{1}{\mu_0} B^2 \right) = i E^* \cdot \sigma \cdot E \quad (J.13)
\]
for real \( k \) and \( \omega \).

With (J.12) and (J.13) equation (J.11) becomes
\[
-\Delta \omega \left( \varepsilon E^2 + \frac{1}{\mu_0} B^2 \right) + A k \cdot \frac{1}{\mu_0} (E^* \times B + E \times B^*)
= -i E^* \cdot A \sigma \cdot E \quad (J.14)
\]

We now identify
\[\Delta \omega = i \omega_i \text{ and } A k = i k_i. \quad (J.15)\]

Then equation (J.14) is
\[
-\omega_i \left( \varepsilon E^2 + \frac{1}{\mu_0} B^2 \right) + k_i \cdot \frac{1}{\mu_0} (E^* \times B + E \times B^*)
= -E^* \cdot A \sigma \cdot E. \quad (J.16)
\]
If we add (J.16) to its complex conjugate we obtain
\[
-2\omega_i \left( \varepsilon E^2 + \frac{1}{\mu_0} B^2 \right) + 2k_i \cdot \frac{1}{\mu_0} (E^* \times B + E \times B^*)
= -E \cdot (A \sigma + A \sigma^*) \cdot E^*, \quad (J.17)
\]
where \( A \sigma^* \) is the Hermitian conjugate of \( A \sigma \).

Upon comparing (J.17) with (16.38) from Chap. 16 we see that (J.17) is the field energy equation for waves in dispersive media.

The perturbation of the conductivity tensor \( \sigma \) is
\[
\Delta \sigma = \Delta \omega \frac{\partial}{\partial \omega} \sigma + A k \cdot \frac{\partial}{\partial k} \sigma + A^i \sigma
= i \omega_i \frac{\partial}{\partial \omega} \sigma + i k_i \cdot \frac{\partial}{\partial k} \sigma + A^i \sigma, \quad (J.18)
\]
where we have used (J.15). In (J.18) we have designated the slight structural change in \( \sigma \) that results at the damped condition as \( A^i \sigma \). In (J.18) we have evaluated \( A^i \sigma \).
using the undamped, real values of $k$ and $\omega$. With (J.18) we find that

$$
\Delta \sigma + \Delta \sigma^+ = i \omega \frac{\partial}{\partial \omega} (\sigma - \sigma^+) + i k \cdot \frac{\partial}{\partial k} (\sigma - \sigma^+) + (\Delta' \sigma + \Delta' \sigma^+)
$$

$$
= -2\omega \frac{\partial}{\partial \omega} \sigma - 2k \cdot \frac{\partial}{\partial k} \sigma^{(A)} + 2 \Delta' \sigma^{(H)}. \quad (J.19)
$$

The antihermitian and Hermitian parts of the tensors $\sigma$ and $\Delta' \sigma$ are defined by

$$
\sigma^{(A)} = \frac{1}{2i} (\sigma - \sigma^+) \quad (J.20)
$$

and

$$
\Delta' \sigma^{(H)} = \frac{1}{2} (\Delta' \sigma + \Delta' \sigma^+) \quad (J.21)
$$

We recall that the conductivity tensor is antihermitian at the undamped condition. And we see that the Hermitian part of the structural change in $\sigma$ enters the energy equation as a loss term. With (J.19) the energy (J.17) becomes

$$
-2\omega \left( \varepsilon E^2 + \frac{1}{\mu_0} B^2 + E \cdot \frac{\partial}{\partial \omega} \sigma^{(A)} \cdot E^* \right)
$$

$$
+ 2k \cdot \left[ \frac{1}{\mu_0} \left( E^* \times B + E \times B^* \right) - E \cdot \frac{\partial \sigma^{(A)}}{\partial k} \cdot E^* \right]
$$

$$
= -2E^* \cdot \Delta' \sigma^{(H)} \cdot E. \quad (J.22)
$$

Using Faraday’s Law we can show that

$$
\frac{1}{\mu_0} \left( E^* \times B + E \times B^* \right) = \frac{1}{\mu_0 \omega} \frac{\partial}{\partial k} \left[ E^2 k^2 - (k \cdot E) (E^* \cdot k) \right]
$$

$$
= \frac{1}{\mu_0 \omega} \frac{\partial}{\partial k} \left[ E \cdot (1k^2 - k \cdot k) \cdot E^* \right] \quad (J.23)
$$

(see equation (16.52)). Then using (J.5) at the propagation condition, we find that (J.23) becomes

$$
\frac{1}{\mu_0} \left( E^* \times B + E \times B^* \right)
$$

$$
= \frac{1}{\omega} \frac{\partial}{\partial k} \left[ E \cdot (\omega \sigma^{(A)} + \omega^2 \varepsilon I) \cdot E^* \right]
$$

$$
= \left[ \left( E \cdot \frac{\sigma^{(A)}}{\omega} \cdot E^* + E \cdot \frac{\partial \sigma^{(A)}}{\partial \omega} \cdot E^* + 2\varepsilon I : E^* E \right) \frac{\partial \omega}{\partial k} \right]
$$

$$
+ E \cdot \frac{\partial \sigma^{(A)}}{\partial k} \cdot E^* \quad (J.24)
$$
Then, using Faraday’s and Ampère’s Laws, we find that we can obtain $B^2$ in terms of the electric field as

$$\omega B^2 = -E^* \cdot (i\mu_0\sigma - \omega\mu_0\varepsilon_0) \cdot E,$$

which, after some lines of algebra, becomes

$$\frac{1}{\mu_0} B^2 = E \cdot \frac{1}{\omega} \sigma^{(A)} \cdot E^* + \varepsilon E^2. \quad (J.25)$$

With (J.24) and (J.25) the energy (J.22) becomes

$$-2\omega \left( 2\varepsilon E^2 + E \cdot \frac{1}{\omega} \sigma^{(A)} \cdot E^* + E \cdot \frac{\partial}{\partial \omega} \sigma^{(A)} \cdot E^* \right)$$

$$+ 2k_1 \cdot \frac{\partial \omega}{\partial k} \left( 2\varepsilon E^2 + E \cdot \frac{\sigma^{(A)}}{\omega} \cdot E^* + E \cdot \frac{\partial \sigma^{(A)}}{\partial \omega} \cdot E^* \right)$$

$$= -2E^* \cdot \Delta' \sigma^{(H)} \cdot E. \quad (J.26)$$

Upon comparison with the treatment in Chap. 16 we see that (J.26) is the general form of (16.57) for slightly damped waves in nonmagnetic matter.

We can then identify the total energy in the damped and dispersed wave as

$$\langle E_{\text{wave}} \rangle_{T,L} = 2\varepsilon E^2 + E \cdot \frac{1}{\omega} \sigma^{(A)} \cdot E^* + E \cdot \frac{\partial}{\partial \omega} \sigma^{(A)} \cdot E^* \quad (J.27)$$

and the total Poynting Vector as

$$\langle S_{\text{wave}} \rangle_{T,L} = \frac{\partial \omega}{\partial k} \left( 2\varepsilon E^2 + E \cdot \frac{\sigma^{(A)}}{\omega} \cdot E^* + E \cdot \frac{\partial \sigma^{(A)}}{\partial \omega} \cdot E^* \right). \quad (J.28)$$

By comparing (J.27) and (J.28) that the total Poynting Vector is equal to the total energy multiplied by the group velocity, i.e.

$$\langle S_{\text{wave}} \rangle_{T,L} = \langle E_{\text{wave}} \rangle_{T,L} \cdot \frac{\partial \omega}{\partial k}. \quad (J.29)$$

Although we have not used a time and space average in the derivation of (J.26), it is identical to the equation we obtain from the time and space average. And, as we point out in Chap. 16, the time and space average has an experimental meaning. We, therefore, include the subscript T,L notation here.

The understanding of

$$E \cdot \frac{\partial \sigma^{(A)}}{\partial \omega} \cdot E^* = \sum_{\alpha} \left( T^{(\alpha)}_{\text{hydro}} + T^{(\alpha)}_{\text{thermal}} \right). \quad (J.30)$$
the coherent particle energy is unchanged. And the loss term \(-2E^* \cdot \Delta \sigma^{(H)} \cdot E\) in (J.26) is the transport of the coherent particle energy to the heating of the background matter results as the coherence is lost remains unchanged.
Appendix K
Answers to Selected Exercises

2.11 \( \cos \alpha = 1/\sqrt{3} \)

2.14 (b) These vectors are not linearly independent.

2.15 \( \mathbf{F} \) is generally not perpendicular to \( \text{curl} \mathbf{F} \).

2.19 Note that the order of partial differentiation makes no difference.

2.20 Note that the order of partial differentiation makes no difference.

2.24 (a) Perform a contour integration between two distinct points. The result is contour independent. So choose the contour to make the integration as simple as possible.

\[ \frac{d}{dx} \phi(x, y) = -3x^2 - xy \]

2.25 (a), (c), and (e) are conservative.

2.26 \( \text{div} \mathbf{E} = \rho/\varepsilon_0, \text{curl} \mathbf{B} = \mu_0 \mathbf{J} \)

2.27 \( \int_{-\infty}^{\infty} dx \delta(x - 1) \exp(-\alpha x^2 + \beta x) = \exp(\beta - \alpha) \),

\( \int_{0}^{\infty} dx \delta(x + 1) \exp(-\alpha x^2 + \beta x) = 0 \),

\( \int_{-\infty}^{\infty} dx \delta(x + \lambda) \cos(2\pi x/\lambda) \exp(-x^2/\lambda^2) = \exp(-1) \),

\( \int_{0}^{10} dx \delta(x + 5) (6x^2 + 2x - 3) = 0 \),

\( \int_{-\infty}^{\infty} dx \delta(x + 5) (6x^2 + 2x - 3) = 137 \),

\( \int_{-\infty}^{\infty} dx \delta(x + \lambda) \sin(2\pi x/\lambda) \exp(-x^2/\lambda^2) = 0 \),

\( \int_{-\infty}^{\infty} dx \delta(x - 1) J_n(x) = J_n(1) \),

\( \int_{-\infty}^{\infty} dx \delta(x) \text{erf}(x) = 0 \) since \( \text{erf}(0) = 0 \)

3.2 (b) \( \rho = 0 \) (c) two charged, flat conducting plates arranged parallel to one another, with a positive charge on one and a negative charge on the other will produce this electrostatic field.

3.5 (a) zero (b) \( \sigma_a = q/(4\pi a^2) = (\varepsilon_0 V/a)(b/(b-a)) \), \( \sigma_b = Q_b/(4\pi b^2) = -((\varepsilon_0 V/b)(a/(b-a))) \) electric field is equal to zero for \( r > b \).

3.6 \( \int_V \text{div} \mathbf{E} dV = \frac{\mathbf{q}}{\varepsilon_0} R^5 \) Gauss’ Law is then no longer valid.

3.7 \( E(r) = \rho_0 R^2/(4\varepsilon_0 r) \)
3.8 The charge induced on the surface of the small hollow in the conductor must be $-q$ to counterbalance the $q$ inserted.

3.9 (a) $4.3976 \times 10^{-14}$ C m$^3$ b) $-1.3281 \times 10^{-9}$ C m$^2$ c) $-6.7742 \times 10^5$ C.

d) This charge comes from lightning striking the earth. In a plasma we can consider the massive ions to be immovable and the electrons to be the charge carriers. So in a lightning bolt electrons carry the negative charge to the earth. Positive and negative lightning can exist, but positive is rarer. In the negative case the upper part of a cloud becomes positively charged and the lower negative. When the negative charge on the cloud is high enough lightning is formed providing a flow of negative charge to the earth.

3.10 (a) $\sigma_i = (\rho_0/5) a^2$, $\sigma_o = (\rho_0/5) \left( a^4/b^2 \right)$

3.12 $\varepsilon_0 E_0 \left[ 3 - \alpha r \right] \exp(-\alpha r)$

4.1 If the differential distance $d\ell$ is on the surface of constant potential then $d\varphi = 0$ and $qE \cdot d\ell = 0$. If $E \cdot d\ell = 0$ then $E \perp d\ell$.

4.2 $-4\varepsilon_0 e^{-r^2} \left( \sin(x + y) + x \cos(x + y) - x^2 \sin(x + y) \right)$

4.3 (a) Laplace’s Equation, Theorem II applies and the potential is zero inside the sphere. (b) $-V_R R/r$

4.5 $\left( Q / (2\pi \varepsilon_0 R^2) \right) \left( \sqrt{z^2 + R^2} - z \right)$

4.6 $\left( Q / (2\pi \varepsilon_0 R^2) \right) \left( 1 - z / \sqrt{z^2 + R^2} \right)$

4.7 $\left( \sigma_s R / (2\varepsilon_0) \right) \ln \left( \frac{2(z + \ell/2) + 2\sqrt{R^2 + (z + \ell/2)^2}}{2(z - \ell/2) + 2\sqrt{R^2 + (z - \ell/2)^2}} \right) + \left( \sigma_o / (2\varepsilon_0) \right) \left( \sqrt{(z - \ell/2)^2 + R^2} + \sqrt{(z + \ell/2)^2 + R^2} - (2z) \right)$

4.8 This is a Coulomb electrostatic field for a small very thin cylinder with negligible charge on the end plates, which is consistent with $z \pm \ell/2 \gg R$.

4.10 (a) $E_r = (q / (4\pi \varepsilon_0)) \left( 1 / (\ell r) + 1 / r^2 \right) \exp(-r/\ell)$ (b) $\rho = -\left( q / (4\pi \lambda^2 r) \right) \exp(-r/\lambda)$ (c) This is a negative charge density, which deceases rapidly away from the origin. The positive charge then has attracted negative charges to its vicinity. These negative charges shield the positive charge.

4.14 $C = \varepsilon_0 A/d$
4.16 $E(z) = \hat{e}_z (\rho \varepsilon / (2\varepsilon_0)) \left( 1 - z (z^2 + R^2)^{-1/2} \right)$ for $z > \varepsilon / 2$ and $E(z) = -\hat{e}_z (\rho \varepsilon / (2\varepsilon_0)) \left( 1 + z (z^2 + R^2)^{-1/2} \right)$ for $z < -\varepsilon / 2$

4.17 $E_0 = \hat{e}_z (\beta / (8\varepsilon_0)) \left[ -L^2 + L \sqrt{L^2 + 4R^2} + 2R^2 \ln \left( \left| -L + \sqrt{L^2 + 4R^2} \right| / \left( L + \sqrt{L^2 + 4R^2} \right) \right) \right]$

4.18 $\varphi = - (\rho / (4\varepsilon_0)) \left\{ \left( z - L \right) \sqrt{R^2 + (z - L)^2} + R^2 \ln \left( \left| z - L + \sqrt{R^2 + (z - L)^2} \right| / \left( z + \sqrt{R^2 + (z^2)} \right) \right) \right\} \mp (z - L)^2$

4.19 $U_E = (1.2 / 2) \left( Q^2 / (4\pi \varepsilon_0 R) \right)$

4.20 (a) $C = 55.6 \text{ pF}$ b) $U_E = 2.781 \times 10^{-13} \text{ J}$ c) $\frac{1}{2} \varepsilon_0 E^2 = 442.73 \frac{1}{\text{m}^2}$ d) $E = 1.0 \times 10^7 \frac{\text{V}}{\text{m}}$ e) $E / E_{\text{breakdown}} = 8.47$

4.21 (a) $C = 200 \text{ F}$ (b) $A = 225.88 \text{ m}^2$ c) This is somewhat large for microcircuits.

4.22 (d) $C^{(s)}_1 / C^{(p)}_1 = C_1 C_2 / (C_1 + C_2)^2 = C_1 C_2 / (C_1^2 + C_2^2 + 2C_1 C_2) < 1$

5.1 $(1/9 \times 10^{16}) \text{ m}^{-2} \text{ s}^2$

5.2 $B = \hat{e}_x B$

5.3 $B_x = -x \beta B / z$, $B_y = -y \beta B / z$, $B_z = B \exp (\beta z)$

5.4 (a) $V_{\text{Hall}} = a v_c B \beta$ b) $I_{\text{Hall}} = \sigma v_c B a \ell$ c) $P_{\text{MHD}} = \sigma \ell a^2 v_c^2 B^2$

5.11 (a) $\partial A_\theta / \partial z$ is small but $\neq 0$, $\partial A_x / \partial z = \partial A_y / \partial r b) - \frac{1}{r} (\partial / \partial r) (r \partial A_\theta / \partial z) + (1/r) (\partial / \partial r) (r \partial A_\theta / \partial z) = 0$

6.1 $A = \hat{e}_z \frac{I_{\mu_0}}{4\pi} \left\{ w + x \ln \left[ (x-w/2)^2 + y^2 \right] / \left( (x+w/2)^2 + y^2 \right) \right\}$

$- \frac{w}{2} \ln \left[ (x-w/2)^2 + y^2 \right] \left( (x+w/2)^2 + y^2 \right) + 2y \arctan \left[ (x-w/2) / y \right] - 2y \arctan \left[ (x+w/2) / y \right]$

6.2 $B = \left( I_0 \mu_0 / (2\pi r) \right) \hat{\theta}$ for $x, y \gg w$

6.3 $B = \hat{e}_z \mu_0 N \lambda I_0 \left( 1 + 4 (L / R)^2 \right)^{-1/2} \approx \hat{e}_z \mu_0 N \lambda I_0$ if $R / L < 1$

6.5 $J = -\hat{e}_\phi \left( \mu_0 N_\lambda I_0 / (2r) \right)$

6.10 $B = \left( \mu_0 I_0 N / (2a) \right) \hat{z}$

6.11 $B_x = \frac{3}{4} \mu_0 \mu_0 a / \left( a^2 + z^2 \right)^{3/2} \tau r$

6.13 $A = -\hat{e}_z \left( I_0 \mu_0 / (2\pi) \right) \{ \ln \left[ r_1 / r_2 \right] \}$

6.14 $B = - \left( I_0 \mu_0 \mu_0 / \pi \right) \left\{ a^4 - 2a^2 (x^2 - y^2) + (x^2 + y^2)^2 \right\}^{-1} \left\{ \hat{e}_x 2xy + \hat{e}_y \left[ a^2 - (x^2 - y^2) \right] \right\}$ As $a \to 0$ this becomes zero for all values of $(x, y)$. In the limit of small $a$. This is directly proportional to $a$. The magnetic field induction should become zero as the wires coalesce into a single wire.
6.15 \( B(r) = (\mu_0 J/2) [b^2/r - a^2/(r+s)] \hat{e}_\theta \)

7.3 (a) The canonical momenta are \( p_x = m\dot{x} - m(\Omega/2)y \) and \( p_y = m\dot{y} + m(\Omega/2)x \)

7.6 The canonical equations are
\[
\begin{align*}
\dot{x} &= (1/m) (p_x + m (\Omega/2)y), \\
\dot{y} &= (1/m) (p_y - m (\Omega/2)x), \\
\dot{\zeta} &= (1/m) p_z, \\
\dot{p}_x &= (\Omega/2) (p_y - m (\Omega/2)x), \\
\dot{p}_y &= -(\Omega/2) (p_x + m (\Omega/2)y), \\
\dot{p}_z &= 0
\end{align*}
\]

The result is motion in the \( z \)-direction

7.7 The canonical equations are
\[
\begin{align*}
\dot{x} &= (1/m) (p_x + m (\Omega/2)y) \\
\dot{y} &= (1/m) (p_y - m (\Omega/2)x) \\
\dot{\zeta} &= (1/m) p_z \\
\dot{p}_x &= (\Omega/2) (p_y - m (\Omega/2)x) \\
\dot{p}_y &= -(\Omega/2) (p_x + m (\Omega/2)y) \\
\dot{p}_z &= QE
\end{align*}
\]

7.8 The canonical equations are
\[
\begin{align*}
\dot{x} &= (1/m) (p_x + m (\Omega/2)y) \\
\dot{y} &= (1/m) (p_y - m (\Omega/2)x) \\
\dot{\zeta} &= \frac{p_z}{m} \\
\dot{p}_x &= (\Omega/2) (p_y - m (\Omega/2)x) \\
\dot{p}_y &= -(\Omega/2) (p_x + m (\Omega/2)y) + QE \gamma \\
\dot{p}_z &= QE_\gamma
\end{align*}
\]

7.9 (a) \( (E/B) < R (QB/m) \) (b) \( (E/B) = R (QB/m) \) (c) \( (E/B) > R (QB/m) \)

8.1 (a) \( \phi = - (\rho_0/4\epsilon_0) (2R^2 \ln R - R^2 + r^2) \) (b) \( \phi = - (\alpha/(16\epsilon_0)) (4R^4 \ln R - R^4 + r^4) \)

8.2 (a) \( \varphi = - (\rho_0/(2\epsilon_0)) R^2 \ln r \) \( \varphi = - (\alpha/(4\epsilon_0)) R^4 \ln r \)

8.3 \( \varphi = - (\lambda/(2\pi\epsilon_0)) \ln (r/R) \)

8.4 For \( a < r < R \), \( \varphi = (A/(2\epsilon_0r)) (r^2 - a^2) + (A/\epsilon_0) (R - r) \) For \( r > R \), \( \varphi = (A/(2\epsilon_0r)) (R^2 - a^2) \)

8.6 For \( a < r < R \), \( \varphi = -(A/\epsilon_0) (r - a) + (A/(2\epsilon_0r)) (r^2 - a^2) \) For \( R < r \), \( \varphi = -(A/(\epsilon_0)(R - a)) + (A/(2\epsilon_0r)) (R^2 - a^2) \)

8.7 \( A_\theta = -\mu_0 N_\lambda I_0 R \ln (R), \) \( B = -e_z \mu_0 N_\lambda I_0 R \ln (R) \frac{1}{r} \)

8.8 \( \phi = -(\sigma_0/\epsilon_0\sigma_0) x \) and \( E = \hat{e}_x (\sigma_0/\epsilon_0) \)

8.9 For \( (a + \epsilon) < r < b, \) \( \varphi = -(Q_a/(4\pi\epsilon_0r)) - (Q_b/(4\pi\epsilon_0b)) \) For \( b < r < (b + \epsilon) \), \( \varphi = -(Q_b + Q_a)/(4\pi\epsilon_0b) \) For \( b < r, \) \( \varphi = -(Q_a + Q_b)/(4\pi\epsilon_0r) \) \( Q_a = 4\pi\epsilon_0 (V_a - V_b) (ab/(a - b)) \) \( Q_b = -4\pi\epsilon_0 (b/(a - b)) (aV_b - bV_a) \)

8.10 For \( r < a \) \( A_z = -\mu_0 J_0 \frac{1}{2} b^2 \ln b - \frac{1}{2} a^2 \ln a - \frac{1}{4} (b^2 - a^2) \) For \( a \leq r \leq b \) \( A_z = \mu_0 J_0 (a^2/2) \ln (r) - \mu_0 J_0 (1/4) r^2 - \mu_0 J_0 (b^2/4) \ln b - (1/4) b^2 \), For \( b < r \) \( A_z = -\mu_0 J_0 (1/2) (b^2 - a^2) \ln (r) \) For \( r < a \)
\[ \mathbf{B} = -\hat{\mathbf{e}}_{\phi} dA_z / dr = 0 \text{. For } a \leq r \leq b \text{, } \mathbf{B} = \hat{\mathbf{e}}_{\phi} \left( \mu_0 J_0 / 2 \right) \left( (r^2 - a^2) / r \right) \text{. For } b < r \text{, } \mathbf{B} = -\hat{\mathbf{e}}_{\phi} \mu_0 J_0 \left( 1 / 2 \right) \left( b^2 - a^2 \right) \left( 1 / r \right) \]

10.1 (a) \( \mathcal{E} = \pi a^2 \mu_0 N_i^2 L dI / dt \) (b) \( I_0 \left( t \right) = \pi a^2 \mu_0 \left( V / \left( R_b L \right) \right) \exp \left( -Rt / L \right) \)

10.2 The current will decrease.

10.3 (a) \( \mathcal{E} = B \mathbf{v} \) (b) \( \mathcal{E} = B \mathbf{v} \)

10.4 If Faraday’s Law is actually a Law of physics, which it is, then this is an intolerable situation.

10.5 \( \mathcal{E} = BA_\omega \sin \omega t \)

10.6 \( I_t = \left( \mu_0 I / (4\pi R) \right) \omega \ln \left( b / a \right) \chi_0 \sin \left( \omega t \right) \)

10.7 (a) \( \mathcal{E} \left( t \right) = n \pi a^2 B_\omega \sin \omega t \) \( \) \( b) \mathcal{E} = \left( 1 / \left( 2n \pi a^2 \right) \right) \int_0^{t/2} \mathcal{E} \left( t \right) dt \)

10.8 \( I = \left( B \mathbf{a}_0 \right) \sin \left( \omega t \right) \)

10.11 (a) \( E_\phi = \left( 1 / 2 \right) \rho \mu_0 N_\lambda I_0 \sin \omega t \) \( b) \mathcal{I}_{\text{conductor}} = \left( \sigma / 4 \right) R^2 L \mu_0 N_\lambda I_0 \sin \omega t \)

10.13 In the case of the cylindrical solenoid there is an energy density. But the field energy is spread over larger regions of space and a density cannot be easily calculated.

The fact that we can, in each case, identify an inductance \( L \), which is a function only of the geometric properties of the solenoid is a result of Faraday’s Law and the Biot–Savart Law.

11.2 (a) \( y \)-axis and move in the negative direction \( b) \mathbf{E} = -\hat{\mathbf{e}}_z \)

11.4 (b) The Gaussian pulse is a laboratory reality. Our result in (a) indicates that it will propagate at the speed \( c \) into the space beyond the lamp as a function of \( p = \omega t - kz \).

11.9 The Fourier Transformation is a representation of the disturbance in terms of a particular set of basis functions, the complex exponentials \( \exp \left( i \omega t \pm ik \cdot r \right) \). In physical terms these are propagating plane waves.

12.1 Momentum density is \( \rho_p = 2\lambda_E / V^2 \) where \( \lambda_E \) is kinetic energy flux.

12.4 \( \mathbf{S} = -\left( 1 / \mu_0 \right) \left( J / \sigma \right) \left( \mu_0 \mathbf{J} R / 2 \right) \hat{e}_t \)

12.6 \( \mathbf{S} = \hat{e}_z E_0^2 \sqrt{\varepsilon_0 \mu_0} \)

12.7 \( Q^2 / 24\pi \varepsilon_0 R \)

13.9 \( m_\gamma c^2 = 0.520 \pm 75 \text{ MeV, } \beta_0 = 0.1927 \pm 0 \)

13.13 (a) \( \mathbf{F} = -q E_\gamma \mathbf{e}_y \) \( b) \mathcal{E} = \beta a B_\gamma \) Faraday’s Law then predicts the same emf as that resulting from the electric field \( E_\gamma \).

14.4 If we are in a region of space close to the moving charges the radiation gauge cannot be used and we must use the Lorentz Gauge.

14.5 It happened when \( \partial \mathbf{A} / \partial t = \left( \mu_0 / \left( 4\pi R \right) \right) \mathbf{E}_0 \hat{e}_a \)

14.6 The dominant emission is axial.

14.7 The orbiting electron will radiate energy and will eventually fall into the nucleus. The electron falling into the nucleus will radiate energy at the frequency with which it traverses the orbit. This will change continuously. There can be no line spectrum.

15.2 \( \left( C_T / C \right) = 2 \left( \varepsilon / \left( \varepsilon + \varepsilon_0 \right) \right) \)

15.3 (a) \( \sigma_1 = \left( 2Q / 4\pi \varepsilon_0 \right) \left( \varepsilon_0 + \varepsilon \right) \) \( \sigma_2 = \left( 2Q / 4\pi \varepsilon \right) \left( \varepsilon_0 + \varepsilon \right) \) \( b) E_1 = E_2 = \left( 2Q / 4\pi \right) / \left( \varepsilon_0 + \varepsilon \right) \) \( c) C_T = \left( A / \left( 2\gamma \right) \right) \left( \varepsilon_0 + \varepsilon \right) \)

15.4 (a) \( U = \varepsilon_0 L / \left( 2\gamma \right) V^2 \left( K - 1 \right) \) \( b) F = \varepsilon_0 L / \left( 2\gamma \right) V^2 \left( K - 1 \right) \)
15.5 (a) \( \rho_p = -2\alpha z \) b) \( \sigma_p (z = 0) = -\beta \), \( \sigma_p (z = L) = \alpha L^2 + \beta \) c) \( Q_p = \frac{Q}{p} \) (inside) + \( \frac{Q}{p} \) (end caps) = 0

15.6 (a) \( P (r) = (K - 1) \frac{Q}{(4\pi Kr^2)} \) e) \( \rho_p = 0 \) c) \( \sigma_p (a) = -(K - 1) \frac{Q}{(4\pi Ka^2)} \), \( \sigma_p (b) = (K - 1) \frac{Q}{(4\pi Kb^2)} \) d) \( Q_{p, \text{total}} = 4\pi a^2 \sigma_p (a) + 4\pi b^2 \sigma_p (b) = 0 \)

15.7 (a) \( \rho_p = -(Q / (4\pi r^2)) (\alpha / (\alpha r + 1)^2) \) b) \( \sigma_p (a) = -(\alpha a / (1 + \alpha a)) \)

\( (Q / (4\pi a^2)), \sigma(b) = (\alpha b / (1 + \alpha b)) (Q / (4\pi b^2)) \) c) \( Q_{p, \text{total}} = 0 \) (d) There has been no real charge transferred to the dielectric.

15.8 (a) \( \rho_p = 0 \) b) \( \sigma_p = p_0 \cos \phi \) c) \( Q_s = 0 \)

15.10 \( \varphi = D_{0}^{(in)} + D_{1}^{(in)} r \cos \phi, \varphi_{out} = D_{0}^{(out)} + \left[ D_{1}^{(out)} r + G_{1}^{(out)} (1/r^2) \right] \cos \phi \)

15.12 \( E_m = E + P / (3\varepsilon_0) \)

15.13 (a) \( \alpha = (3\varepsilon_0/n) (K + 1) / (K + 2) \) b) \( \chi = (\alpha n / \varepsilon_0) (1 - n\alpha / (3\varepsilon_0)) \)

15.14 (a) \( n\alpha = 3\varepsilon_0 \) b) \( \chi = (\alpha n / \varepsilon_0) / \xi \)

15.15 (a) \( \rho_p = 0 \) c) \( BL_\pi (a^2 - b^2) \)

15.16 (a) \( J_M = 0 \) b) \( J_{M}^{(s)} = -\hat{e}_z (K_M - 1) (1/2\pi r) \) for the outer surface and \( J_{M}^{(s)} = +\hat{e}_z (K_M - 1) (1/2\pi a) \) for the inner surface c) \( J_{M}^{(s)} \hat{e}_\lambda (K_M - 1) (1/2\pi r) \) for the top surface and \( J_{M}^{(s)} = -\hat{e}_\lambda (K_M - 1) (1/2\pi r) \) for the bottom surface

15.18 (b) yes (c) no

16.1 (a) \( \sigma = -i NQ^2 / (m\omega) \) b) \( \omega = \omega_{p,e}, \mathcal{E}_{\text{particle}} = \varepsilon_0 (\omega_{p,e}^2 / \omega^2) E^2, \mathcal{E}_{\text{wave}} = \varepsilon_0 \left( 1 + \omega_{p,e}^2 / \omega^2 \right) E^2 \)

\( \varepsilon_0 (\omega_{p,e}^2 / \omega^2) E^2 c) \omega = \sqrt{\omega_{p,e}^2 + k^2 c^2}, \partial \omega / \partial k = k c^2 / \sqrt{\omega_{p,e}^2 + k^2 c^2} < c, \mathcal{E}_{\text{particle}} = \varepsilon_0 (\omega_{p,e}^2 / \omega^2) E^2, \mathcal{E}_{\text{wave}} = 2\varepsilon_0 E^2 \)

16.2 (a) longitudinal: \( \sigma = -i \varepsilon_0 \omega_{p,e}^2 \omega / (\omega - k u)^2 \), transverse: \( \sigma = -i \varepsilon_0 \omega_{p,e}^2 / (\omega - k u) \) b) \( \omega = k u \pm \omega_{p,e}, \mathcal{E}_{\text{particle}} = \varepsilon_0 \omega_{p,e}^2 (\omega + k u) / (\omega - k u)^3 E^2, \mathcal{E}_{\text{wave}} = \varepsilon_0 \left( 1 + \omega_{p,e}^2 (\omega + k u) / (\omega - k u)^3 \right) E^2 c) \omega = \omega_{p,e}^2 + k^2 c^2 as u \rightarrow 0, \partial \omega / \partial k = k c^2 / \omega as u \rightarrow 0, \mathcal{E}_{\text{particle}} = \varepsilon_0 \left( \omega_{p,e}^2 / (\omega - k u)^2 \right) E^2, \mathcal{E}_{\text{wave}} = \left( 2\varepsilon_0 + \varepsilon_0 \omega_{p,e}^2 (k u) / (\omega - k u)^2 \right)^2 E^2 \)

16.3 (a) \( \sigma = -i \varepsilon_0 \omega_{p,e}^2 \omega / (\omega - k u)^2 b) \omega = k u \pm \omega_{p,e}, \mathcal{E}_{\text{particle}} = \varepsilon_0 \omega_{p,e}^2 (\omega + k u) / (\omega - k u)^3 \)

\( \left( \omega + k u / (\omega - k u)^3 \right)^2 E^2, \mathcal{E}_{\text{wave}} = \varepsilon_0 \left( 1 + \omega_{p,e}^2 (\omega + k u) / (\omega - k u)^3 \right)^2 E^2 \)
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