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DIMENSIONAL REDUCTION OF SYMMETRIES

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A B S T R A C T

It has been shown that ordinary $d-2$ dimensional quantum field theories are equivalent to the corresponding quantum field theories defined on a $d+2$ dimensional superspace with two anticommuting coordinates, order by order in perturbation theory. This dimensional reduction of the superspace quantum field theories is a consequence of the negative dimensionality of the anticommuting variables. In the present Letter we shall show that, as a consequence of the negative dimensionality of the anticommuting variables, all symmetries are also dimensionally reduced.

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Quantum field theories defined on a graded manifold

$$\Sigma = \{ \chi, \Theta, \bar{\Theta} \} \quad (1)$$

where $\{ \chi \}$ is a d-dimensional Euclidean (Minkowskian) space and $\Theta, \bar{\Theta}$ are anticommuting scalars, have recently been investigated /1 - 4/. Such theories are applicable to several problems including the theory of a ferromagnet in a quenched random magnetic field /1,3,4/, the problem of quark confinement /5/, study of dilute branched polymers /6/ and the unification of weak, electromagnetic and strong interactions /7/.

These superspace quantum field theories were introduced by Parisi and Sourlas /1/ to explain the equivalence between classical stochastic field theories in d dimensions and their quantum counterparts in d-2 dimensions /8/: Both of these theories are related to the corresponding superspace quantum field theories in the d+2 dimensional superspace Σ , and the dimensional reduction $d \rightarrow d-2$ is attributed to the negative dimensionality of the two anticommuting variables $\Theta, \bar{\Theta}$ /1,2/.

To be more specific, we define a superrotation to be a transformation that leaves the quadratic form

$$y^\alpha y_\alpha = \chi^2 + \frac{4}{\Delta} \bar{\Theta} \Theta = y^\alpha y_\alpha y^\beta y_\beta \quad (2)$$

invariant. A superfield $\Phi(\chi, \Theta, \bar{\Theta})$ is then a scalar under superrotations if

$$\delta_{\alpha\beta}^{\alpha\beta} \Phi = \frac{1}{2} (y^\alpha \partial^\beta - (-)^{\alpha\beta} y^\beta \partial^\alpha) \partial_{\beta\alpha} \Phi \quad (3)$$

where

$$\delta_{\alpha\beta} = -(-)^{\alpha\beta} \partial_{\beta\alpha} \quad (4)$$

defines the superrotation. Similarly one can also define supervector fields and also superspinors /2,7/. Obviously the superLaplacian

$$\square = \partial^\alpha \partial_\alpha = \square + \Delta \partial_{\bar{\Theta}} \partial_{\Theta} \quad (5)$$

is a superscalar and so is the superLagrangian

$$\mathcal{L}_m[\Phi] = \frac{1}{2} \Phi(-\square_m) \Phi + V[\Phi] \quad (6)$$

The superfield $\Phi(\chi, \Theta, \bar{\Theta})$ can be expanded in terms of its component fields,

$$\Phi(\chi, \Theta, \bar{\Theta}) = \phi(\chi) + \gamma(\chi) \bar{\Theta} + \bar{\gamma}(\chi) \Theta + \omega(\chi) \bar{\Theta} \Theta \quad (7)$$

If we substitute this component field expansion to (6) and integrate over Θ and $\bar{\Theta}$, we find the component field representation of the Lagrangian,

$$\int d\Theta d\bar{\Theta} \mathcal{L}_m[\Phi] = -\frac{\Delta}{2} \omega^2 + \omega(-\square + V'(\phi)) + \bar{\gamma}(-\square + V''(\phi)) \gamma \quad (8)$$

The effect of the superrotation (3) on the component fields (7) is given by a supersymmetry transformation, parametrized by the generators (4). As an example a superrotation parametrized by non vanishing $\delta_{\alpha\beta}$ and $\delta_{\alpha\bar{\beta}}$ leads to the supersymmetry transformation

$$\begin{aligned}
 \delta\phi &= -\frac{A}{2} x^\mu \partial_\mu \phi - \frac{A}{2} x^\mu \partial_\mu \bar{\psi} \\
 \delta\psi &= -\frac{A}{2} x^\mu \partial_\mu \omega - \partial^\mu \phi \partial_\mu \bar{\psi} \\
 \delta\bar{\psi} &= \frac{A}{2} x^\mu \partial_\mu \omega + \partial^\mu \phi \partial_\mu \bar{\psi} \\
 \delta\omega &= -\partial^\mu \phi \partial_\mu \omega - \partial^\mu \bar{\psi} \partial_\mu \bar{\psi}
 \end{aligned} \tag{9}$$

The invariance of the superLagrangian (6) under the superrotation is exposed by the invariance of the component field Lagrangian (8) under the corresponding supersymmetry transformation (9).

Recently, it has been pointed out /4,9/ that the Lagrangian (8) is also invariant under translations in the superspace,

$$\delta_T \bar{\Phi} = \partial^\alpha \bar{\Phi} \varepsilon_\alpha \tag{10}$$

This supertranslation invariance, if expressed in terms of the component fields (7), leads to a new supersymmetry that leaves the component field Lagrangian (8) invariant. This supersymmetry is

$$\begin{aligned}
 \delta_T \phi &= \varepsilon_0 \bar{\psi} - \varepsilon_0 \psi \\
 \delta_T \psi &= \varepsilon_0 \omega \\
 \delta_T \bar{\psi} &= \varepsilon_0 \omega
 \end{aligned} \tag{11}$$

In this Letter we generalize these observations: We address the question how symmetries are dimensionally reduced. We shall show that if there is an invariance of the d-2 dimensional Lagrangian, then the corresponding d+2 dimensional superspace Lagrangian is also invariant under the corresponding superspace transformation. As a consequence, the full symmetry structure of the superspace field theories ~~is~~ dimensionally reduced. For gauge transformations this has already been demonstrated /2/ and for global transformations there

is nothing to prove. Hence it is enough to discuss general conformal transformations in superspace/10/. (Such transformations form the symplectic conformal group $Osp(1,d/2)$.)

For simplicity we shall consider only scalar field theories (6) but the generalization to supergauge theories and superfermionic theories should be obvious /2/.

We define a superspace dilatation by

$$\delta_D \bar{\Phi} = (y^\alpha \partial_\alpha + D) \bar{\Phi} \tag{12}$$

where D is in general some matrix. In the case of the scalar field theory (6), D is a number that we call the scale dimension of the superfield $\bar{\Phi}$. Under the superspace dilatation the free Lagrangian

$$\mathcal{L}_m[\bar{\Phi}] = \frac{1}{2} \partial^\alpha \bar{\Phi} \partial_\alpha \bar{\Phi} \tag{13}$$

transforms in the following way

$$\delta_D \mathcal{L}_m[\bar{\Phi}] = y^\beta \partial_\beta \mathcal{L}_m[\bar{\Phi}] + 2(D+1) \mathcal{L}_m[\bar{\Phi}] \tag{14}$$

As a consequence of the grading,

$$\partial_\beta y^\beta = d+2 \quad \text{but} \quad \partial^\beta y_\beta = d-2 \tag{15}$$

we find that (14) is a total divergence in the superspace,

$$\delta_D \mathcal{L}_m[\bar{\Phi}] = \partial^\beta (y_\beta \mathcal{L}_m[\bar{\Phi}]) \tag{16}$$

if we choose the scale dimension D to be

$$D = \frac{1}{2} (d-4) \tag{17}$$

and consequently (13) is invariant under the superspace dilatation.

Recall, that in the case of conventional field theories the scale dimension of the scalar field is $\frac{1}{2}(d-2)$. Hence the scale dimension has been dimensionally reduced, $d+2 \rightarrow d-2$. It is straightforward to generalize this result: A field theory in the $d+2$ dimensional superspace is dilatation invariant if the corresponding $d-2$ dimensional ordinary field theory is. For example, the Lagrangian

$$\mathcal{L}_m[\Phi] = \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi + \frac{\lambda}{4!} \Phi^4 \quad (18)$$

is dilatation invariant in the $6+2$ dimensional superspace.

In the case of ordinary field theories the scale dimension of a field agrees with its canonical dimension. However, due to the negative dimensions of the anticommuting coordinates Θ and $\bar{\Theta}$ this is not true in the case of superspace field theories: In the $6+2$ dimensional space the canonical dimension of the component fields (7) is two. But if we express (12) in terms of the component fields (7), we find

$$\begin{aligned} \delta_D \phi &= (x^\mu \partial_\mu + 1) \phi \\ \delta_D \gamma &= (x^\mu \partial_\mu + 2) \gamma \\ \delta_D \bar{\gamma} &= (x^\mu \partial_\mu + 2) \bar{\gamma} \\ \delta_D \omega &= (x^\mu \partial_\mu + 3) \omega \end{aligned} \quad (19)$$

Hence only for the anticommuting fields γ , $\bar{\gamma}$ are the canonical and scale dimensions equal.

Under a special conformal transformation in the superspace a scalar field transforms

$$\delta_K \Phi = 2 y_\alpha y^\alpha \partial_\alpha \Phi - y^\alpha y_\alpha \partial_\alpha \Phi + 2 D y_\alpha \Phi \quad (20)$$

(For superspace vector fields and spinors the pertinent rotation terms must be added.) It is then straightforward to check that if a $d-2$ dimensional Lagrangian is invariant under the special conformal transformation, then the corresponding superspace

Lagrangian is also invariant under the pertinent superspace transformation.

As an example, one can verify that the Lagrangian (18) is invariant under the transformation (20) in $6+2$ dimensional superspace.

If we substitute the component field expansion (7) into (20), we find that corresponding to the conformal transformation there is a supersymmetry transformation of the component fields - just like in the case of superrotations (9) and supertranslations (11). As a consequence, the invariance of the Lagrangian under the superspace special conformal transformation is equivalent to the invariance of the corresponding component field Lagrangian under the invariance of the symmetry of the component fields. For example, if we parametrize the transformation (20) with the supervector

$$V_\alpha = (\vec{0}, a, 0) \quad (21)$$

in a $6+2$ dimensional space, where a is an anticommuting scalar, we find the supersymmetry

$$\begin{aligned} \delta_\epsilon \phi &= -\frac{a}{2} x^2 a \gamma \\ \delta_\epsilon \gamma &= 2 a x^\mu \partial_\mu \phi + 2 a \phi + \frac{a}{2} x^2 a \omega \\ \delta_\epsilon \omega &= 2 a x^\mu \partial_\mu \gamma + 6 a \gamma \end{aligned} \quad (22)$$

which leaves the component field representation of (18),

$$\mathcal{L} = -\frac{A}{2}\omega^2 + \omega(-\Box\phi + \frac{A}{3}\phi^3) + \bar{\psi}(-\Box + \frac{A}{2}\phi^2)\psi \quad (23)$$

invariant.

The finite special conformal transformation in the superspace is

$$y^\alpha \rightarrow \frac{y^\alpha + c^\alpha y^2}{1 + 2c \cdot y + c^2 y^2} \quad (24)$$

For large c^2 we find

$$y^\alpha \rightarrow \frac{c^\alpha}{c^2} + \frac{1}{c^2 y^2} \left[y^{\alpha\beta} - 2 \frac{c^\alpha c^\beta}{c^2} \right] y_\beta + \mathcal{O}\left(\frac{1}{c^3}\right) \quad (25)$$

Hence invariance under superspace inversions,

$$y^\alpha \rightarrow \frac{1}{y^2} y^\alpha \quad (26)$$

is a consequence of invariance under superrotations, supertranslations and finite superspace special conformal transformations. Notice that y^2 is indeed invertible,

$$y^2 \frac{1}{y^2} = (x^2 + \frac{4}{\Lambda} \bar{\theta}\theta) \left(\frac{1}{x^2} - \frac{4}{\Lambda} \frac{1}{x^4} \bar{\theta}\theta \right) = 1 \quad (27)$$

even though the coordinates θ and $\bar{\theta}$ are not.

As a conclusion we find that the inversion invariance is dimensionally reduced. This can also be verified directly: Consider first a free field theory,

$$\mathcal{L}_m[\Phi] = \frac{1}{2} \bar{\Phi} (-\Box_m) \Phi \quad (28)$$

The two-point function is $/2/$

$$\langle \bar{\Phi}(y) \Phi(0) \rangle = \frac{1}{(2\pi)^d} \int d^d k d\bar{\alpha} d\alpha \frac{\exp\{i y^\alpha p_\alpha\}}{k^2 + \Delta \bar{\alpha} \alpha} \quad (29)$$

where we have used the translation invariance of (28) in the superspace. Under the inversion (26) the r.h.s. of (29) transforms into

$$\frac{1}{(2\pi)^d} (y^2)^{d-2} \int d^d k d\bar{\alpha} d\alpha \frac{\exp\{i y^\alpha p_\alpha\}}{k^2 + \Delta \bar{\alpha} \alpha} \quad (30)$$

The dimensional reduction $d+2 \rightarrow d-2$ is now a consequence of the properties of the Jacobian determinant in the superspace. Hence the two-point function (29) and as a consequence, the Lagrangian (28) is inversion invariant in the superspace if we define the superfield to transform in the following way:

$$\bar{\Phi}(y^\alpha) \rightarrow \left(\frac{1}{y^2}\right)^{\frac{d-1}{2}} \bar{\Phi}\left(\frac{1}{y^2} y^\alpha\right) \quad (31)$$

This can also be verified by a direct calculation, by using the fact that under an inversion the integration measure transforms as follows:

$$d^p x d\bar{\theta} d\theta = d^{d+2} y \rightarrow \left(\frac{1}{y^2}\right)^{d-2} d^{d+2} y \quad (32)$$

Hence, if a Lagrangian in $d-2$ dimensional ordinary space is inversion invariant, then the corresponding superspace Lagrangian in $d+2$ dimensions is invariant under the superspace inversion. As an example, one can verify that the Lagrangian (18) is indeed inversion invariant in the $6+2$ dimensional superspace.

We have shown that for any invariance of the $d-2$ dimensional field theory, there is a corresponding invariance of the $d+2$ dimensional superspace field theory. Hence there is also an invariance of the corresponding stochastic differential equation in d dimensions $/2/$.

As a consequence the dimensional reduction $d+2 \rightarrow d-2$ goes far beyond the simple diagrammatic equivalence. In fact, all Noether conserved currents and charges defined for a $d-2$ dimensional field theory have their $d+2$ dimensional counterparts, hence also d dimensional stochastic counterparts exist. Thus the formulation of quantum field theories in terms of stochastic classical differential equations and Nicolai transformations /2,11/ is completely equivalent to conventional approaches.

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