A

Rotation matrices and Clebsch-Gordan coefficients

In this Appendix we collect some formulae that are useful for specific applications of the rotation group discussed in Chapter 2. First we give the explicit expressions of the functions \( d^j_{m'm} \) and of the spherical harmonics for the lowest angular momentum values. Then we report a few tables of Clebsch-Gordan coefficients which are used in the addition of angular momenta.

### A.1 Reduced rotation matrices and spherical harmonics

The reduced rotation matrix \( d^j_{m'm} (\beta) \) enter the rotation matrix \( D^{(j)} (\alpha, \beta, \gamma) \) according to (see Eq. (2.63))

\[
D^{(j)}_{m'm}(\alpha, \beta, \gamma) = \langle j, m' | e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} | j, m \rangle = e^{i\alpha m} d^j_{m'm}(\beta) e^{-i\gamma m},
\]

where one defines:

\[
d^j_{m'm}(\beta) = \langle j, m' | e^{-i\beta J_y} | j, m \rangle.
\]

The general expression proposed by Wigner for the \( d \)-functions\(^1\) is given in Eq. (2.65), that we report again here for the sake of completeness:

\[
d^j_{m'm}(\beta) = \sum_s (-1)^s [(j + m)!(j - m)!(j + m')!(j - m')!]!^{1/2} \\
\times \frac{s!(j - s - m')!(j + m - s)!(m' + s - m)!}{(2j + m - m' - 2s)} \left( \cos \frac{\beta}{2} \right)^{2j+m-m'+2s} \left( -\sin \frac{\beta}{2} \right)^{m'-m+2s},
\]

where the sum is over the values of the integer \( s \) for which the factorial arguments are equal or greater than zero.

We give here a few general properties:

\[
d^j_{m'm}(0) = \delta_{m',m},
\]
\[
d^j_{m'm}(\pi) = (-)^{j-m} \delta_{m',-m},
\]
\[
d^j_{m'm}(\beta) = d^j_{mm'}(-\beta),
\]
\[
d^j_{m'm}(-\beta) = (-)^{m'-m} d^j_{m'm}(\beta),
\]

and list the explicit expressions of the matrices for the lowest angular momenta\textsuperscript{2} \(d^\frac{3}{2}\) and \(d^1\):

\[
d^\frac{3}{2} = \begin{pmatrix}
\cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\
\sin \frac{\beta}{2} & \cos \frac{\beta}{2}
\end{pmatrix},
\]

\[
d^1 = \begin{pmatrix}
\frac{1}{2}(1 + \cos \beta) & -\frac{1}{\sqrt{2}} \sin \beta & \frac{1}{2}(1 - \cos \beta) \\
\frac{1}{\sqrt{2}} \sin \beta & \cos \beta & -\frac{1}{\sqrt{2}} \sin \beta \\
\frac{1}{2}(1 - \cos \beta) & \frac{1}{\sqrt{2}} \sin \beta & \frac{1}{2}(1 + \cos \beta)
\end{pmatrix},
\]

together with the most relevant matrix elements of the \(d^\frac{3}{2}\) (the other matrix elements being easily deduced on the basis of the symmetry properties given above):

\[
d^\frac{3}{2}_{3/2,3/2} = \frac{1}{2}(1 + \cos \beta) \cos \frac{\beta}{2}, \quad d^\frac{3}{2}_{3/2,1/2} = -\frac{\sqrt{3}}{2}(1 + \cos \beta) \sin \frac{\beta}{2},
\]
\[
d^\frac{3}{2}_{3/2,-1/2} = \frac{\sqrt{3}}{2}(1 - \cos \beta) \cos \frac{\beta}{2}, \quad d^\frac{3}{2}_{3/2,-3/2} = -\frac{1}{2}(1 - \cos \beta) \sin \frac{\beta}{2},
\]
\[
d^\frac{3}{2}_{1/2,1/2} = \frac{1}{2}(3 \cos \beta - 1) \cos \frac{\beta}{2}, \quad d^\frac{3}{2}_{1/2,-1/2} = \frac{1}{2}(3 \cos \beta + 1) \sin \frac{\beta}{2}.
\]

Finally we give the expressions of the spherical harmonics \(Y_\ell^m\) limiting ourselves to the case \(\ell = 1\textsuperscript{3}:

\[
Y^0_1 = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y^1_1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \quad Y^{-1}_1 = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}.
\]

\textsuperscript{2} A more complete list can be found in: \textit{The Review of Particle Physics}, C. Amsler \textit{et al.}, Physics Letters B 667, 1 (2008).
\textsuperscript{3} For \(\ell = 2\) see the reference quoted above.
A.2 Clebsch-Gordan coefficients

We report in the following the values of the Clebsch-Gordan coefficients $C(j_1, j_2, j; m_1, m_2, m)$ defined in Eq. (2.72), where $j$ is the eigenvalue of the total angular momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ and $m = m_1 + m_2$ the corresponding component along the $x^3$-axis. We limit ourselves only to the cases $j_2 = \frac{1}{2}$, very often useful in the calculations. The coefficients for higher values of $j_2$ can be found in the quoted references.

In Table A.1 we report the coefficients $C(j_1, j_2, j; m_1, m_2, m)$, for any value of $j_1$ and $j_2 = \frac{1}{2}$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m_2 = \frac{1}{2}$</th>
<th>$m_2 = -\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_1 + \frac{1}{2}$</td>
<td>$\sqrt{\frac{j_1 + m_1 + 1}{2j_1 + 1}}$</td>
<td>$\sqrt{\frac{j_1 - m_1 + 1}{2j_1 + 1}}$</td>
</tr>
<tr>
<td>$j_1 - \frac{1}{2}$</td>
<td>$-\sqrt{\frac{j_1 - m_1}{2j_1 + 1}}$</td>
<td>$\sqrt{\frac{j_1 + m_1}{2j_1 + 1}}$</td>
</tr>
</tbody>
</table>

The specific cases $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = \frac{1}{2})$ are given in the Tables A.2 and A.3, respectively.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m_2$</th>
<th>$\frac{1}{2}$</th>
<th>$-\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$1$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Table A.3. Clebsch-Gordan coefficients $C(1, \frac{1}{2}, j; m_1, m_2, m)$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m_2 = \frac{1}{2}$</th>
<th>$m_2 = -\frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m_1 = \frac{1}{2}$</td>
<td>$m_1 = -\frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\sqrt{\frac{7}{3}}$</td>
</tr>
<tr>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>$\sqrt{\frac{7}{3}}$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\frac{7}{3}}$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\sqrt{\frac{7}{3}}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$-\sqrt{\frac{7}{3}}$</td>
</tr>
<tr>
<td></td>
<td>$-\sqrt{\frac{7}{3}}$</td>
<td>0</td>
</tr>
</tbody>
</table>

In Table A.4 we report the coefficients $C(j_1, 1, j; m_1, m_2, m)$, for any value of $j_1$ and $j_2 = 1$.

Table A.4. Clebsch-Gordan coefficients $C(j_1, 1, j; m_1, m_2, m)$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m_2 = 1$</th>
<th>$m_2 = 0$</th>
<th>$m_2 = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j_1 + 1$</td>
<td>$\sqrt{(j_1 + m_1 + 1)(j_1 + m_1 + 2)}$</td>
<td>$\sqrt{(j_1 - m_1 + 1)(j_1 + m_1 + 1)}$</td>
<td>$\sqrt{(j_1 - m_1 + 1)(j_1 - m_1 + 2)}$</td>
</tr>
<tr>
<td>$j_1$</td>
<td>$-\sqrt{(j_1 + m_1 + 1)(j_1 - m_1)}$</td>
<td>$\sqrt{m_1}$</td>
<td>$\sqrt{(j_1 - m_1 + 1)(j_1 + m_1)}$</td>
</tr>
<tr>
<td>$j_1 - 1$</td>
<td>$\sqrt{(j_1 - m_1 - 1)(j_1 - m_1)}$</td>
<td>$-\sqrt{(j_1 - m_1 + 1)(j_1 + m_1)}$</td>
<td>$\sqrt{(j_1 + m_1)(j_1 + m_1 - 1)}$</td>
</tr>
</tbody>
</table>

Finally, the specific case $(j_1 = 1, j_2 = 1)$ is given in Tables A.5.
### Table A.5. Clebsch-Gordan coefficients $C(1, 1, j; m_1, m_2, m)$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>1</th>
<th>0</th>
<th>$-1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\sqrt{\frac{j}{2}}$</td>
<td>$\sqrt{\frac{j}{6}}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$\sqrt{\frac{j}{2}}$</td>
<td>$\sqrt{\frac{j}{3}}$</td>
<td>$\sqrt{\frac{j}{2}}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$\sqrt{\frac{j}{6}}$</td>
<td>$\sqrt{\frac{j}{2}}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\sqrt{\frac{j}{2}}$</td>
<td>0</td>
<td>$\sqrt{\frac{j}{2}}$</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$-\sqrt{\frac{j}{2}}$</td>
<td>$-\sqrt{\frac{j}{2}}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\sqrt{\frac{j}{3}}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$-1$</td>
<td>$\sqrt{\frac{j}{3}}$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
B

Symmetric group and identical particles

In this Appendix we examine briefly the symmetry properties of identical particles, which have to be taken into account when dealing with more particle states. This symmetry appears only in quantum mechanics, and it is related to the fact that particles of the same kind are to be considered absolutely indistinguishable from one another.

B.1 Identical particles

Suppose we have a system of \( n \) particles of the same kind, e.g. protons, not interacting among each other. A state of the system could be represented in terms of one-particle states:

\[
|a_1 b_2 c_3 \ldots z_n> = |a_1> |b_2> |c_3> \ldots |z_n>,
\]

(B.1)

where \( a_1, b_2, \ldots \) correspond to the dynamical variables of the first, second, \( \ldots \) particle, respectively. The fact that the \( n \) particles are identical has as a consequence that a transition from the above state to the state e.g.

\[
|a_2 b_1 c_3 \ldots z_n> = |a_2> |b_1> |c_3> \ldots |z_n>,
\]

(B.2)

obtained by interchanging particles 1 and 2, could not be observable by any means. It is then necessary to describe the situation represented by Eqs. (B.1) and (B.2) in terms of the same state.

More generally, we want to have a situation in which any permutation (built by repeated interchanges of two particles) among the \( n \) identical particles leads essentially to the same state of the system. The process of permuting the particles will be represented by a linear operator \( P \) in the Hilbert space.

Since the \( n \) particles are indistinguishable, the Hamiltonian \( H \) of the system will be a symmetrical function of the dynamical variables of the \( n \) particles, and then it will commute with the permutation operator.
This means that a state which has initially some symmetry property (e.g. it is totally symmetric under $P$) will always conserve this symmetry.

### B.2 Symmetric group and Young tableaux

In order to study in general the possible symmetry properties (under $P$) of the quantum mechanical states, it is convenient to make use of the *symmetric group*. We summarize briefly here the relevant properties of this group.

As defined in Section 1.1, the symmetric group $S_n$ is the group of permutations of $n$ objects. Each permutation can be decomposed into a product of transpositions (i.e. permutations in which only two elements are interchanged).

The order of the group is $n!$. We recall a useful theorem of finite groups: the sum of the squares of the dimensions of the IR’s equals the order of the group.

We want now to make a correspondence between the IR’s of the group $S_n$ and the so-called *Young tableaux*.

Let us start with the states of two identical particles. The only possible permutation is the transposition $P_{12}$ and one can build two independent states, respectively *symmetrical* and *antisymmetrical*:\footnote{In the following the compact notation $|12\ldots\rangle = |a_1 b_2 c_3 \ldots\rangle$ will be used.}

\begin{equation}
|\Phi_s\rangle = |12\rangle + |21\rangle = (1 + P_{12})|12\rangle = S_{12}|12\rangle, \\
|\Phi_a\rangle = |12\rangle - |21\rangle = (1 - P_{12})|12\rangle = A_{12}|12\rangle,
\end{equation}

where we have introduced the symmetrizing and antisymmetrizing operators

\begin{equation}
S_{12} = (1 + P_{12}), \\
A_{12} = (1 - P_{12}).
\end{equation}

We can take $|\Phi_s\rangle$ and $|\Phi_a\rangle$ as bases of the two non equivalent (one-dimensional) IR’s of $S_2$, which are $(1, 1)$ and $(1, -1)$. A convenient notation is given in terms of the diagrams

```
    \[
    \begin{array}{c}
    \square \\
    \square \\
    \end{array}
    \quad \text{for } (1, 1) \\
    \begin{array}{c}
    \square \\
    \end{array}
    \quad \text{for } (1, -1)
    \end{equation}
```

which are particular Young tableaux, whose general definition will be given later.

Before giving the rules for the general case, let us consider explicitly the case of three identical particles. In principle we have 6 different states
\[ |\Phi_1 \rangle = S_{123}|123\rangle, \quad |\Phi_4 \rangle = A_{23}S_{12}|123\rangle, \]
\[ |\Phi_2 \rangle = A_{123}|123\rangle, \quad |\Phi_5 \rangle = A_{23}S_{13}|123\rangle, \]
\[ |\Phi_3 \rangle = A_{13}S_{12}|123\rangle, \quad |\Phi_6 \rangle = A_{12}S_{13}|123\rangle, \]  \hspace{1cm} \text{(B.6)}

where
\[ S_{123} = 1 + P_{12} + P_{13} + P_{23} + P_{13}P_{12} + P_{12}P_{13}, \]
\[ A_{123} = 1 - P_{12} - P_{13} - P_{23} + P_{13}P_{12} + P_{12}P_{13}. \] \hspace{1cm} \text{(B.7)}

In fact, there are only 4 independent states: one is totally symmetric, one totally antisymmetric, and two have mixed symmetry. They correspond to the (3 box) Young tableaux in Table B.1. To each Young tableau one can associate an IR of \( S_3 \). Moreover, one can also know the dimension of each IR, looking at the "standard arrangement of a tableau" or "standard Young tableau". Let us label each box with the numbers 1, 2, 3 in such a way that the numbers increase in a row from left to right and in a column from top to bottom. We see from the Table that there is only a standard tableau for each of the first two patterns, and two different standard tableaux for the last one. The number of different standard arrangements for a given pattern gives the dimension of the corresponding IR.

<table>
<thead>
<tr>
<th>Young tableaux</th>
<th>Symmetry</th>
<th>Standard tableaux</th>
<th>IR dimension</th>
<th>Young operator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>totally symmetric</td>
<td>[ 1 \ 2 \ 3 ]</td>
<td>1</td>
<td>( S_{123} )</td>
</tr>
<tr>
<td></td>
<td>totally antisymmetric</td>
<td>[ 1 \ 2 \ 3 ]</td>
<td>1</td>
<td>( A_{123} )</td>
</tr>
<tr>
<td></td>
<td>mixed symmetry</td>
<td>[ 1 \ 2 \ 3 ]</td>
<td>2</td>
<td>( A_{13}S_{12}, A_{12}S_{13} )</td>
</tr>
</tbody>
</table>

To each standard tableau we can associate one of the symmetry operators (or Young operators) used in Eqs. (B.6), (B.7). We note that we have one-dimensional IR’s for the totally symmetric and antisymmetric cases, which means that the corresponding states are singlets, and a two-dimensional IR in the case of mixed symmetry, which corresponds to a doublet of degenerate...
states\(^2\). One can check that this result agrees with the rule of finite groups mentioned above: in fact \(3! = 1 + 1 + 2\).

The above example can be easily extended to the case of \(n\) identical particles. In this case the group is \(S_n\) and one has to draw all Young tableaux consisting of \(n\) boxes. Each tableau is identified with a given partition \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) of the number \(n\) (i.e. \(\sum_i \lambda_i = n\)), ordering the \(\lambda_i\) in such a way that
\[
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n . \tag{B.8}
\]

The corresponding Young tableau

![Young tableau](image)

has \(\lambda_i\) boxes in the \(i\)-th row. Then for each tableau one finds all standard arrangements. Each Young tableau corresponds to an IR of \(S_n\), and the dimension of the IR is equal to the number of different standard arrangements.

To each IR there corresponds a \(n\)-particle state with a different symmetry property, and the dimension of the IR gives the degeneracy of the state.

In order to obtain the state with the required symmetry property from the state (B.1), we introduce a Young operator for each standard Young tableau \(\tau\):
\[
Y_\tau = \left( \sum_{\text{col}} A_\nu \right) \left( \sum_{\text{rows}} S_\lambda \right), \tag{B.9}
\]

where \(S_\lambda\) is the symmetrizer corresponding to \(\lambda\) boxes in a row and \(A_\nu\) the antisymmetrizer corresponding to \(\nu\) boxes in a column, the two sums being taken for each tableau over all rows and columns, respectively.

The explicit expressions for \(S_\lambda\) and \(A_\nu\) have already been given by (B.5) and (B.7) in the simplest cases \(\lambda \leq 3, \nu \leq 3\). In general, they are defined by
\[
S_\lambda = \sum_p \binom{1 \ 2 \ \ldots \ \lambda}{p_1 \ p_2 \ \ldots \ p_{\lambda}} , \tag{B.10}
\]
\[
A_\nu = \sum_p \epsilon_p \binom{1 \ 2 \ \ldots \ \nu}{p_1 \ p_2 \ \ldots \ p_{\nu}} , \tag{B.11}
\]

\(^2\) According to Eq. (B.6), one would expect two distinct two-dimensional IR’s, corresponding to the mixed symmetry states \(|\Phi_3\rangle, |\Phi_4\rangle, |\Phi_5\rangle, |\Phi_6\rangle\). However, the two IR’s can be related to one another by a similarity transformations, i.e. they are equivalent.
where $\sum_p$ indicates the sum over all permutations and $\epsilon_p = +$ or $-$ in correspondence to an even or odd permutation (a permutation is even or odd if it contains an even or odd number of transpositions).

The state with the symmetry of the $\tau$ standard Young tableau is obtained by means of

$$|\Phi_{\tau}\rangle = Y_{\tau}|a_1 b_2 \ldots z_n\rangle . \quad \text{(B.12)}$$

Particular importance have the states which are totally symmetrical or totally antisymmetrical, because they are those occurring in nature. They correspond to the Young tableaux consisting of only one row or one column, so that they are simply given by

$$|\Phi_s\rangle = S_n|a_1 b_2 \ldots z_n\rangle , \quad \text{(B.13)}$$

$$|\Phi_a\rangle = A_n|a_1 b_2 \ldots z_n\rangle . \quad \text{(B.14)}$$

An assembly of particles which occurs only in symmetrical states (B.13) is described by the Bose-Einstein statistics. Such particles occur in nature and are called bosons. A comparison between the Bose statistics and the usual classical (Boltzmann) statistics (which considers particles distinguishable) shows e.g. that the probability of two particles being in the same state is greater in Bose statistics than in the classical one.

Let us examine the antisymmetrical state (B.14). It can be written in the form of a determinant

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \\ \ldots \ldots \ldots \\ z_1 & z_2 & \ldots & z_n \end{pmatrix} . \quad \text{(B.15)}$$

It then appears that if two of the states $|a\rangle, |b\rangle, \ldots, |z_n\rangle$ are the same, the state (B.14) vanishes. This means that two particles cannot occupy the same state; this rule corresponds to the Pauli exclusion principle. In general, the occupied states must be all independent, otherwise (B.15) vanishes. An assembly of particles which occur only in the antisymmetric states (B.14) is described by the Fermi-Dirac statistics. Such particles occur in nature: they are called fermions.

Other quantum statistics exist which allow states with more complicated symmetries than the complete symmetry or antisymmetry. Until now, however, all experimental evidence indicates that only the Bose and the Fermi statistics occur in nature; moreover, systems with integer spin obey the Bose statistics, and systems with half-integer spin the Fermi statistics. This connection of spin with statistics is shown to hold in quantum field theory (spin-statistics theorem), in which bosons and fermions are described by fields which commute and anticommute, respectively, for space-like separations.

Then, unless new particles obeying new statistics are experimentally detected, all known particles states belong to the one-dimensional representation of the symmetric group. We note, however, that the states of $n$-particles have
to be singlet (totally symmetric or totally antisymmetric) under permutations of all variables (space-time coordinates, spin, internal quantum numbers); if one takes into account only a subset of variables, e.g. the internal ones, a state can have mixed symmetry. In this case, one has to make use of the other – in general higher dimensional – IR’s of the symmetric group $S_n$. 
C

Young tableaux and irreducible representations of the unitary groups

In this Appendix we illustrate the use of the Young tableaux in the construction of tensors which are irreducible with respect to the group $U(n)$ of unitary transformations.

The main property on which the method is based, and which will not be demonstrated here, is that the irreducible tensors can be put in one-to-one correspondence with the IR’s of the symmetric group; these are associated, as pointed out in Appendix B, to the different Young tableaux.

On the other hand, since the irreducible tensors are bases of the IR’s of the group $U(n)$, one can go from the Young tableaux to the IR’s of the group $U(n)$ itself.

The method can be used, in general, for the group $GL(n, C)$ of linear transformations, but we shall limit here to the groups $U(n)$ and $SU(n)$ which are those considered in Chapter 8$^1$. In the following, the main properties will be illustrated by examples which have physical interest.

C.1 Irreducible tensors with respect to $U(n)$

Let us consider a $n$-dimensional linear vector space; its basic elements are contravariants vectors defined by sets of $n$ complex numbers $\xi^\alpha (\alpha = 1, 2, \ldots, n)$ and denoted by:

$$\xi \equiv \begin{pmatrix} \xi^1 \\ \xi^2 \\ \vdots \\ \xi^n \end{pmatrix}, \quad (C.1)$$

The group $U(n)$ can be defined in terms of unitary transformations

---

$^1$ For the extension of the method to $GL(n, C)$ see e.g. M. Hamermesh ”Group theory and its applications to physical problems”, Addison-Wesley (1954).
\[ \xi \to \xi' = U \xi \quad \text{i.e.} \quad \xi'^\alpha = U^\alpha_\beta \xi^\beta , \] (C.2)

where
\[ U^\dagger U = I \quad \text{i.e.} \quad (U^\dagger)^\alpha_\beta U^\beta_\gamma = \delta^\alpha_\gamma . \] (C.3)

We define covariant vector the set of \( n \) complex numbers \( \eta_\alpha \) (\( \alpha = 1, 2, \ldots, n \))
\[ \eta \equiv (\eta_1 \, \eta_2 \ldots \, \eta_n) , \] (C.4)
which transforms according to
\[ \eta \to \eta' = \eta U^\dagger \quad \text{i.e.} \quad \eta'_\alpha = \eta_\beta (U^\dagger)^\beta_\alpha . \] (C.5)

It follows that the quantity
\[ \xi^\dagger = (\xi_1 \, \xi_2 \ldots \, \xi_n) , \] (C.6)
where
\[ \xi_\alpha \equiv (\xi^\alpha)^* , \] (C.7)
transforms as a covariant vector.

It is also immediate to check that the scalar product
\[ \eta \xi = \eta_\alpha \xi^\alpha \] (C.8)
is invariant under the group transformations.

A general mixed tensor \( \zeta^\alpha_\beta \ldots \) where \( \alpha, \beta, \ldots \) are controvariant and \( \mu, \nu, \ldots \) covariant indeces, is defined by the transformation property
\[ \zeta'^\alpha_\beta \ldots = U^\alpha_\alpha' U^\beta_\beta' \cdots \zeta'^\alpha'_{\mu' \nu' \ldots} (U^\dagger)^\mu'_\mu (U^\dagger)^\nu'_\nu \cdots . \] (C.9)
We note, in passing, that the tensor \( \delta^\alpha_\beta \) is invariant under \( U(n) \).

A general tensor of the type given above is, in general, reducible. Let us first consider, for the sake of simplicity, a tensor of the type \( \zeta^\alpha_\beta \gamma \ldots \) with \( r \) indeces of controvariant type only: clearly, it corresponds to the basis of the direct product representation
\[ U \otimes U \otimes \ldots \otimes U , \] (C.10)
where \( U \) appears \( r \) times. In order to reduce \( \zeta^\alpha_\beta \gamma \ldots \) into the irreducible tensors which are bases of the IR's of \( U(n) \) we make use of the Young tableaux.

We build all the different standard Young tableaux with \( r \) boxes, as defined in Appendix B. To each standard tableau \( \tau \) we associate a Young operator \( Y_\tau \) as defined in (B.9).

By applying the operator \( Y_\tau \) to the tensor \( \zeta = \zeta^\alpha_\beta \ldots \), one gets a tensor \( \theta = \theta^\alpha_\beta \ldots \) which has the permutation symmetry of the corresponding standard tableau:
\[ \theta = Y_\tau \zeta . \] (C.11)
One can show, in general, that the Young operators $Y_\tau$ commute with the transformations of the group $U(n)$. Then one can write

$$(U \otimes U \otimes \ldots)\theta = (U \otimes U \otimes \ldots)Y_\tau \zeta = Y_\tau(U \otimes U \otimes \ldots)\zeta = Y_\tau\zeta' = \theta'. \quad (C.12)$$

This result can be understood if one considers a tensor as product of basic vectors $\zeta^{\alpha_1 \alpha_2 \ldots} = \xi^\alpha \xi^\beta \ldots$. Now the transformations of the product do not depend on the order in which the vectors are taken. This means that the transformations of $U(n)$ commute with the operations of permutations of the individual vectors, and therefore that they do not change the symmetry character of a tensor.

The meaning of Eq. (C.12) is that the subspace spanned by the tensor $\theta$ is invariant under the transformations of $U(n)$. Therefore, the tensor $\theta$ can be taken as the basis of a IR of the group $U(n)$.

Taking into account all the possible Young operators $Y_\tau$, i.e. all possible standard tableaux, one can then decompose the tensor $\zeta$ into the irreducible tensors $Y_\tau\zeta$ and, therefore, the reducible representation (C.10) into the IR’s contained in it, the dimension of each IR being equal to the number of independent components of the irreducible tensor.

**Example 1**

Let us consider the third order tensor

$$\zeta^{\alpha_1 \alpha_2 \alpha_3} = \xi^{\alpha_1} \xi^{\alpha_2} \xi^{\alpha_3}, \quad (C.13)$$

where $\xi^{\alpha_i}$ is a generic component of the vector $\xi$ defined by Eq. (C.2). Taking into account the standard Young tableaux with three boxes (see Appendix B), we obtain easily the following decomposition

$$\zeta^{\alpha_1 \alpha_2 \alpha_3} = \zeta^{(\alpha_1 \alpha_2 \alpha_3)} \oplus \zeta^{(\alpha_1 \alpha_3)\alpha_2} \oplus \zeta^{(\alpha_1 \alpha_2)\alpha_3} \oplus \zeta^{[\alpha_1 \alpha_2 \alpha_3]}, \quad (C.14)$$

i.e.

$$1 \otimes 2 \otimes 3 = 1 2 3 \oplus 1 2 3 \oplus 1 3 2 \oplus 2 1 3 \oplus 2 3 1$$

Each index $\alpha_i$ goes from 1 to $n$; the number of components of a tensor corresponds to the different ways of taking all the independent sets $(\alpha_1, \alpha_2, \alpha_3)$ with given permutation properties. With the help of a little bit of combinatorics, one can determine the dimension $N$ of the different kinds of tensors in (C.14):

$$\zeta^{\{\alpha \beta \gamma\}} \quad N = \binom{n+3-1}{3} = \frac{1}{6}n(n+1)(n+2), \quad (C.15)$$

$$\zeta^{\{\alpha \beta\} \gamma} \quad N = \frac{1}{3}n(n^2-1), \quad (C.16)$$
\[ \zeta^{[\alpha\beta\gamma]} \]

\[ N = \binom{n}{3} = \frac{1}{6}n(n-1)(n-2). \quad (C.17) \]

Let us now consider a mixed tensor, such as \( \zeta^{\alpha\beta\ldots} \) defined by Eq. (C.9). In this case one has to apply the above procedure independently to the upper (contravariant) and lower (covariant) indices. In other words, one has to symmetrize the tensor, according to all possible Young standard tableaux, in both the upper and lower indices, independently. Moreover, one has to take into account that a further reduction occurs, for a mixed tensor, due to the invariance of the tensor \( \delta^{\alpha}_{\beta} \). In fact, a tensor \( \zeta^{\alpha\beta\ldots} \), already symmetrized, can be further reduced according to

\[ \delta^{\mu}_{\alpha} \zeta^{\alpha\beta\ldots} = \zeta^{\alpha\beta\ldots}, \quad (C.18) \]

and in order to get irreducible tensors, one has to contract all possible pairs of upper and lower indices, and separate the "trace" from the traceless tensors. An example will suffice, for our purpose, to clarify the situation.

**Example 2**

Let us consider the \((n \times n)\)-component mixed tensor \( \zeta^{\alpha}_{\beta} = \xi^{\alpha} \xi_{\beta} \). It can be decomposed in the form

\[ \zeta^{\alpha}_{\beta} = \hat{\zeta}^{\alpha}_{\beta} + \frac{1}{n} \delta^{\alpha}_{\beta} \xi^{\gamma} \xi_{\gamma}, \quad (C.19) \]

i.e. in the trace \( \text{Tr}\zeta = \xi^{\gamma} \xi_{\gamma} \) and the traceless tensor

\[ \hat{\zeta}^{\alpha}_{\beta} = \zeta^{\alpha}_{\beta} - \frac{1}{n} \delta^{\alpha}_{\beta} \xi^{\gamma} \xi_{\gamma}, \quad (C.20) \]

which has \( n^{2} - 1 \) components.

**C.2 Irreducible tensors with respect to \( SU(n) \)**

The matrices \( U \) of the group \( SU(n) \) satisfy the further condition

\[ \det U = 1, \quad (C.21) \]

which can be written also in the form

\[ \epsilon_{\beta_1\ldots\beta_n} U^{\beta_1}_{\alpha_1} \ldots U^{\beta_n}_{\alpha_n} = \epsilon_{\alpha_1\ldots\alpha_n}, \quad (C.22) \]

with the introduction of the completely antisymmetric tensor

\[ \epsilon_{\alpha_1\alpha_2\ldots\alpha_n} = \epsilon^{\alpha_1\alpha_2\ldots\alpha_n}, \quad (C.23) \]

the only components different from zero being those obtained by permutations from \( \epsilon_{12\ldots n} = +1 \), equal to \( +1 \) or \( -1 \) for even and odd permutations, respectively.
Eq. (C.22) shows that the tensor $\epsilon$ is invariant under the group $SU(n)$. This means that the corresponding IR is the identity (one-dimensional) representation. It is interesting to note that the corresponding standard Young tableau consists of a column of $n$ boxes:

$$\epsilon_{\alpha_1\alpha_2...\alpha_n} \rightarrow \left(\begin{array}{c}
\vdots \\
\end{array}\right) = \bullet$$

From this fact, useful important properties are derived for the irreducible tensors of $SU(n)$ (which hold, in general, for the unimodular group $SL(n, C)$). Making use of the antisymmetric tensor $\epsilon$, one can transform covariant indeces into controvariant ones, and viceversa; for example one can write

$$\epsilon^{\alpha_1\alpha_2...\alpha_n} \zeta_{\beta_1\alpha_1} = \theta^{\beta_1[\alpha_2...\alpha_n]} . \quad (C.24)$$

In this way, one can transform all mixed tensors into one kind of tensors, say controvariant. In fact, one can show that all the IR’s of $SU(n)$ can be obtained starting only from one kind of tensors, e.g. $\zeta^{\alpha\beta...}$.

In particular, the covariant vector $\xi_\alpha$ is transformed through

$$\xi_{\alpha_1} = \epsilon_{\alpha_1\alpha_2...\alpha_n} \zeta^{[\alpha_2...\alpha_n]} , \quad (C.25)$$

i.e. into the $(n - 1)$-component tensor $\zeta^{[\alpha_2...\alpha_n]}$, which corresponds to a one-column Young tableau with $(n - 1)$ boxes.

A Young tableau (with $r$ boxes) employed for the IR’s of $U(n)$ or $SU(n)$ is identified by a set of $n$ integer $(\lambda_1, \lambda_2 ... \lambda_n)$ with the conditions $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, $\sum_{i=1}^{n} \lambda_i = r$.

In fact, the maximum number of boxes in a column is $n$, since column means complete antisymmetrization and each index $\alpha$ of a tensor can go from 1 to $n$.

However, in the case of $SU(n)$ the use of the invariant tensor $\epsilon_{\alpha_1\alpha_2...\alpha_n}$ shows that, from the point of view of the IR’s and irreducible tensors, each Young tableau can be replaced by one in which all complete $(n$ boxes) columns are erased.
Each IR of $SU(n)$ is then specified by a set of $(n - 1)$ integers $(p_1, p_2 \ldots p_{n-1})$, the relation between the two sets $\lambda_i$ and $p_i$ being
\[
\begin{align*}
p_1 &= \lambda_1 - \lambda_2, \\
p_2 &= \lambda_2 - \lambda_3, \\
&\vdots \\
p_{n-1} &= \lambda_{n-1} - \lambda_n.
\end{align*}
\] (C.26)

It can be shown that the dimension of the IR of $SU(n)$ identified by the set $p_i$ is given by
\[
N = \frac{1}{1!2! \ldots (n-1)!} (p_1 + 1)(p_1 + p_2 + 2) \ldots (p_1 + \ldots + p_{n-1} + n - 1) \cdot (p_2 + 1)(p_2 + p_3 + 2) \ldots (p_2 + \ldots + p_{n-1} + n - 2) \cdot \ldots \ldots \cdot (p_{n-2} + 1)(p_{n-2} + p_{n-1} + 2) \cdot (p_{n-1} + 1).
\] (C.27)

In particular, one gets for $SU(2)$
\[
N = p_1 + 1,
\] (C.28)
and for $SU(3)$
\[
N = \frac{1}{2} (p_1 + 1)(p_1 + p_2 + 2)(p_2 + 1).
\] (C.29)

We point out that in the case of $U(n)$, while the dimension of any IR is still given by Eq. (C.27), the set $(p_1, p_2 \ldots p_{n-1})$ is no longer sufficient to identify completely the IR. For $U(n)$ one needs to know also the number of boxes of the tableau corresponding to the set $(\lambda_1, \lambda_2 \ldots \lambda_n)$, i.e. the integer $r = \sum_i \lambda_i$. This fact can be interpreted in the following way: the groups $U(n)$ and $SU(n)$ are related by
\[
U(n) = SU(n) \otimes U(1),
\] (C.30)
and one can associate to the group $U(1)$ an additive quantum number. Fixing this quantum number for the vector $\xi^\alpha$ to a given value, say $\rho$, the value for a controvariant tensor $\xi^{\alpha_1 \alpha_2 \ldots \alpha_r}$ is $r \rho$. The value for the covariant vector, according to Eq. (C.7) is $-\rho$.

Moreover, in the case of $U(n)$, controvariant and covariant indeces cannot be transformed into each others, and, in general, irreducible tensors which are equivalent with respect to $SU(n)$ will not be equivalent with respect to $U(n)$. The same will occur for the corresponding IR’s.

Given a Young tableau defined by the set $(p_1, p_2 \ldots p_{n-1})$, it is useful to introduce the adjoint Young tableau defined by $(p_{n-1}, p_{n-2} \ldots p_1)$: it corresponds to the pattern of boxes which, together with the pattern of the first tableau, form a rectangle of $n$ rows. For example, for $SU(3)$, the following tableaux are adjoint to one another since
Given an IR of $SU(n)$ corresponding to a Young tableau, the adjoint IR is identified by the adjoint tableau; the two IR’s have the same dimensionality (see Eq. (C.27)), but are in general inequivalent (they are equivalent only for $SU(2)$).

A Young tableau is self-adjoint if it coincides with its adjoint, and the corresponding IR is called also self-adjoint. For $SU(3)$, e.g., the octet, corresponding to the tableau

$$
\begin{array}{c}
\hline
\hline
\hline
\end{array}
$$

is self-adjoint.

Fundamental representations are said the IR’s which correspond to set $(p_1, p_2 \ldots p_{n−1})$ of the type

$$
\begin{align*}
\begin{cases}
p_i = 1 \\
p_j = 0 \quad (j \neq i)
\end{cases}
\end{align*}
$$

$SU(n)$ then admits $n − 1$ fundamental representations, identified by the tableaux with only one column and number of boxes going from 1 to $n − 1$.

It is useful to distinguish different classes of IR’s for the group $SU(n)$. In the case of $SU(2)$, as we know already from the study of the rotation group in Chapter 2, there are two kinds of IR’s, the integral and half-integral representations, corresponding to even and odd values of $p_1$ ($p_1$ gives, in this case, just the number of boxes in the Young tableaux; in terms of the index $j$ of the $D^{(j)}$ it is $p_1 = 2j$).

For $n > 2$, given the total number $r$ of boxes in a tableau, we define the number $k$:

$$
k = r − ℓn ,
$$

where $ℓ$ and $k$ are non-negative integers such that

$$
0 \leq k \leq n − 1 .
$$

For $SU(n)$, one can then distinguish $n$ classes of IR’s, corresponding to the values $k = 0, 1, \ldots n − 1$. Each fundamental IR identifies a different class.

It is interesting to note that the direct product of two IR’s of classes $k_1$ and $k_2$ can be decomposed into IR’s which belong all to the class $k = k_1 + k_2$ (modulo $n$). In particular, product representations of class $k = 0$ contain only IR’s of the same class $k = 0$.

This fact is related to the following circumstance: the group $SU(n)$ contains as invariant subgroup the abelian group $Z_n$ (of order $n$) which consists in the $n$-th roots of unity. With respect to the factor group $SU(n)/Z(n)$, the only IR’s which are single-valued are those of class $k = 0$; the other IR’s of $SU(n)$
(class $k \neq 0$) are multi-valued representations. This is clearly a generalization of what happens for $SU(2)$ and $SO(3)$.

### C.3 Reduction of products of irreducible representations

It is useful to give a general recipe for the reduction of the direct product of IR’s of $SU(n)$. A simple example is provided already by Eq. (C.14).

The recipe is based on the analysis of the construction of irreducible tensors and can be expressed as follows. Given two Young tableaux, insert in one of them the integer $k$, $(k = 0, 1, \ldots n - 1)$ in all the boxes of the $k$-th row, e.g.

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
\vdots & \vdots & \vdots \\
k & k & k \\
\vdots & \vdots & \vdots
\end{array}
\]

Then enlarge the other tableau by attaching in all ”allowed” ways successively the boxes of type 1, those of type 2, etc. In each step, the following conditions have to be fulfilled:

a) Each tableau must be a proper tableau: no row can be longer than any row above it, and no column exists with a number of boxes $> n$.

b) The numbers in a row must not decrease from left to right: only different numbers are allowed in a column and they must increase from top to bottom.

c) Counting the numbers $n_1$, $n_2$, $\ldots n_k$ of the boxes of type 1, 2, $\ldots k$ row by row from the top, and from right to left in a row, the condition $n_1 \geq n_2 \ldots \geq n_k$ must be satisfied.

The above procedure is illustrated by a simple example.

**Example 3**

Suppose we have to decompose the direct product $8 \otimes 8$, where 8 is the eight-dimensional IR of $SU(3)$. Making use of the Young tableaux, we can write

\[
\begin{array}{cccccc}
\otimes & 1 & 1 \\
2 & 1 & 2 \\
12 & 1 & 2
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 + 2 \\
1 & 2 \\
1 & 1
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 + 2 \\
1 & 2 \\
1 & 1
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 \\
12 & 1 \\
2 & 2
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 \\
12 & 1 \\
2 & 2
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 \\
12 & 1 \\
2 & 2
\end{array}
\oplus
\begin{array}{cccccc}
1 & 1 \\
12 & 1 \\
2 & 2
\end{array}
\]

In the tableaux on the r.h.s., we can get rid of the complete (3 boxes) columns, and the numbers $(p_1, p_2)$ of boxes in the remaining two rows identify completely the IR’s into which the direct product is decomposed.
C.4 Decomposition of the IR’s of $SU(n)$ with respect to given subgroups

Labelling each IR by the dimension $N$ (see Eq. (C.29)), and the corresponding adjoint IR by $\overline{N}$, one has:

$$\begin{array}{ccccccccc}
\tiny \begin{array}{ccc}
\tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tiny \tin...
where the tableaux in parenthesis refer to $SU(\ell)$ and $SU(m)$, respectively.

For a general tensor, we can proceed in the following way. We split the corresponding Young tableau into two pieces in such a way that each piece is an allowed Young tableau relative to the subgroups $SU(\ell)$ or $SU(m)$. We perform all allowed splittings: all pairs of the sub-Young tableaux so obtained correspond to irreducible tensors and then to IR’s of the subgroup $SU(\ell) \otimes SU(m)$.

The procedure is illustrated by the two following examples:

**Example 4.**
Let us consider the IR’s of $SU(6)$ corresponding to the Young tableaux

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}
, \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}
\end{array}
\]

whose dimensions, according to Eqs. (C.15) and (C.17), are 56 and 20, respectively. Their decomposition with respect to the subgroup $SU(4) \otimes SU(2)$ is obtained by writing

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}
\end{array} = \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right) \oplus \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right) \oplus \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right) \oplus \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right)
\]

\[
56 = (20, 1) \oplus (10, 2) \oplus (4, 3) \oplus (1, 4)
\]

where the first tableau in parenthesis refers to $SU(4)$ and the second to $SU(2)$. In a similar way

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}
\end{array} = \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right) \oplus \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right) \oplus \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\cdot
\end{array}
\end{array}
\end{array}\right)
\]

\[
20 = (4, 1) \oplus (6, 2) \oplus (4, 1)
\]

Under the tableaux we have written the dimension of the corresponding IR. We note that, in the case of $SU(2)$, the Young tableau consisting of a column of two boxes corresponds to the identity, and a column of three boxes is not allowed.

**Example 5**
Let us consider now the same IR’s of $SU(6)$ and their decomposition with respect to the group $SU(3) \otimes SU(3)$. One obtains in this case:
\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
&= (\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}, \cdot) \oplus (\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}, \boxempty) \oplus (\boxempty, \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}) \oplus (\cdot, \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}) \\
56 &= (10, 1) \oplus (6, 3) \oplus (3, 6) \oplus (1, 10)
\end{align*}
\]

and
\[
\begin{align*}
\begin{bmatrix}
1 & 2
\end{bmatrix}
&= (\begin{bmatrix}
1 & 2
\end{bmatrix}, \cdot) \oplus (\begin{bmatrix}
1 & 2
\end{bmatrix}, \boxempty) \oplus (\boxempty, \begin{bmatrix}
1 & 2
\end{bmatrix}) \oplus (\cdot, \begin{bmatrix}
1 & 2
\end{bmatrix}) \\
20 &= (1, 1) \oplus (3, 3) \oplus (3, 3) \oplus (1, 1)
\end{align*}
\]

As a byproduct of the above recipe, taking \(m = 1\) \((\ell = n - 1)\), one can obtain immediately the content of an IR of \(SU(n)\) in terms of the IR’s of \(SU(n - 1)\).

b) \(\ell \cdot m = n\)

In this case the index \(A\) of the vector component \(\xi^A\) can be put in the one-to-one correspondence with a pair of indeces \((\alpha, a)\) \((\alpha = 1, \ldots, \ell; a = \ell + 1, \ldots, n)\), i.e.
\[
\xi^A = x^\alpha y^a \quad \text{or} \quad \xi = x \otimes y,
\]
where
\[
x = \begin{pmatrix}
x^1 \\ x^2 \\ \vdots \\ x^\ell
\end{pmatrix}, \quad y = \begin{pmatrix}
y^1 \\ y^2 \\ \vdots \\ y^n
\end{pmatrix},
\]
and in terms of the corresponding Young tableaux
\[
\begin{align*}
\begin{bmatrix}
\end{bmatrix}
&= (\begin{bmatrix}
\end{bmatrix}, \begin{bmatrix}
\end{bmatrix}) \\
n &= (\ell, m)
\end{align*}
\]

For a general tensor \(\xi^{ABC\ldots}\) we consider the corresponding Young tableau, which specifies the symmetry properties of the indeces \(A, B, C\ldots\). Each index is split into a pair of indeces according to the rule (C.36): \(A = (\alpha, a)\), \(B = (\beta, b)\), \(C = (\gamma, c)\), \ldots Then we consider pairs of Young tableaux which refer independently to symmetry properties of the sets of indeces \((\alpha, \beta, \gamma, \ldots)\) and \((a, b, c, \ldots)\). However, one has to keep only those pairs of Young tableaux such that the global symmetry in \((A, B, C, \ldots)\) corresponding to the original tableau is preserved.
Example 6

Let us consider again the IR’s 56 and 20 of $SU(6)$ and their decomposition with respect to the group $SU(3) \otimes SU(2)$. We can write

$$
\begin{align*}
A & B & C = \left( \begin{array}{ccc}
\alpha & \beta & \gamma \\
\alpha & \beta & \gamma \\
\end{array} \right) & \oplus & \left( \begin{array}{cc}
a & b \\
c & \\
\end{array} \right) \\
56 &= (10, 4) & \oplus & (8, 2) \\
\end{align*}
$$

and

$$
\begin{align*}
A & B & C = \left( \begin{array}{ccc}
\alpha & \\
\beta & & \\
\gamma & & \\
\end{array} \right) & \oplus & \left( \begin{array}{cc}
a & b \\
& c \\
\end{array} \right) \\
20 &= (1, 4) & \oplus & (8, 2) \\
\end{align*}
$$

For the sake of convenience the boxes in the tableaux have been labelled with different letters. The order in which they are written is immaterial, provided the symmetry is preserved by the correspondence $A \leftrightarrow (\alpha, a)$, $B \leftrightarrow (\beta, b)$, $C \leftrightarrow (\gamma, c)$. Moreover, each pair of tableaux must appear only once.

Finally we remind that only allowed Young tableaux have to be included. For instance, in the general case $SU(\ell) \otimes SU(m)$, with $\ell \geq 3$, $m \geq 3$, the previous decompositions would be replaced by:

$$
\begin{align*}
\begin{array}{ccc}
\cline{1-3}
\cline{2-2}
\end{array} &= \left( \begin{array}{ccc}
\begin{array}{ccc}
\alpha & \\
\beta & & \\
\gamma & & \\
\end{array} & \oplus & \left( \begin{array}{cc}
\begin{array}{cc}
a & b \\
& c \\
\end{array} & \oplus & \left( \begin{array}{cc}
\begin{array}{c}
\end{array} & \right) \\
\end{array} \\
\end{array} \right) \\
\end{align*}
$$

$$
\begin{align*}
\begin{array}{ccc}
\cline{1-3}
\cline{2-2}
\end{array} &= \left( \begin{array}{ccc}
\begin{array}{ccc}
\alpha & \\
\beta & & \\
\gamma & & \\
\end{array} & \oplus & \left( \begin{array}{cc}
\begin{array}{cc}
a & b \\
& c \\
\end{array} & \oplus & \left( \begin{array}{cc}
\begin{array}{c}
\end{array} & \right) \\
\end{array} \\
\end{array} \right) \\
\end{align*}
$$

Obviously, the last term in each decomposition is not allowed in the case $m = 2$. 

Solutions

Problems of Chapter 2

2.1 Making use of the definition (2.20), Eq. (2.21) can be written as

\[ \sum_k \sigma_k x'_k = \sum_k \sigma_k R_{kj} x_j = \sum_j u \sigma_j u^\dagger x_j. \]

Multiplying the above expression on the left by \( \sigma_i \) and taking the trace, one gets:

\[ R_{ij} = \frac{1}{2} \text{Tr}(\sigma_i u \sigma_j u^\dagger). \]

Finally, inserting the expressions of the Pauli matrices (2.22) and of the \( u \) matrix (2.15), one obtains:

\[
R = \begin{pmatrix}
\Re(a^2 - b^2) & -\Im(a^2 + b^2) & -2 \Re(ab) \\
\Im(a^2 - b^2) & \Re(a^2 + b^2) & -2 \Im(ab) \\
2 \Re(ab) & 0 & |a|^2 - |b|^2
\end{pmatrix}.
\]

2.2 Let us denote by \( U(R) \) the unitary operator in the Hilbert space corresponding to the rotation \( R \). If \( H \) is invariant under rotations and \( |\psi> \) is a solution of the Schrödinger equation, also \( U(R)|\psi> \) is a solution and \( H \) commutes with \( U(R) \). Then, according to Eq. (2.29), an infinitesimal rotation is given by \( R \simeq 1 - i \phi \mathbf{J} \cdot \mathbf{n} \) and we get: \([H, J_k] = 0\), with \( k = 1, 2, 3\).

2.3 The \( \pi N \) states corresponding to the resonant states with \( J = \frac{3}{2} \) and \( J_z = \pm \frac{1}{2} \), making use of the Eq. (2.71), can be written as:

\[ |1, \frac{1}{2}, \frac{3}{2}, +\frac{1}{2}> = C(1, \frac{1}{2}, \frac{3}{2}; 0, \frac{1}{2}, \frac{1}{2}) |0, \frac{1}{2}> + C(1, \frac{1}{2}, \frac{3}{2}; 1, -\frac{1}{2}, \frac{1}{2}) |1, -\frac{1}{2}> \]

and
\[ |1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}> = C(1, \frac{1}{2}, \frac{3}{2}; 0, -\frac{1}{2}, -\frac{1}{2}) |0, -\frac{1}{2}> + C(1, \frac{1}{2}, \frac{3}{2}; 1, -\frac{1}{2}, \frac{1}{2}) | -1, \frac{1}{2} > \]

where all the states on the r.h.s. are understood to be referred to the case \( j_1 = \ell = 1, j_2 = s = \frac{1}{2} \).

Inserting the explicit values of the CG-coefficients and the expressions for the spherical harmonics \( Y_{1}^{0} \) and \( Y_{1}^{-1} \) (see Table A.3 and Eq. (A.11) in Appendix A), the final \( \pi N \) states can be represented by

\[ |1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2} > \sim \sqrt{2} \cos \theta | \alpha > + \frac{1}{\sqrt{2}} e^{i\phi} \sin \theta |\beta > , \]

and

\[ |1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2} > \sim \sqrt{2} \cos \theta | \beta > + \frac{1}{\sqrt{2}} e^{-i\phi} \sin \theta | \alpha > . \]

where \( | \alpha >, | \beta > \) stand for the spin \( \frac{1}{2} \) states with \( s_z = +\frac{1}{2} \) and \( -\frac{1}{2} \), respectively.

Projecting each state onto itself, one gets the angular distribution, which appears in the differential cross-section

\[ \frac{d\sigma}{d\Omega} \sim |A_{\frac{3}{2}}|^2 \{ 1 + 3 \cos^2 \theta \} , \]

where \( A_{\frac{3}{2}} \) is the \( \pi N \) scattering amplitude in the \( J = \frac{3}{2} \) state.

2.4 We start from Eq. (2.60), which we re-write here

\[ R = R''_{\gamma} R'_{\beta} R_{\alpha} = e^{-i\gamma J_z''} e^{-i\beta J_y'} e^{-i\alpha J_z} , \]

and make use of the appropriate unitary transformations \( R_{\kappa} = U R_{\kappa} U^{-1} \) that express each of the first two rotations in terms of the same rotation as seen in the previous coordinate system. Specifically,

\[ e^{-i\gamma J_z''} = e^{-i\beta J_y'} e^{-i\gamma J_z'} e^{i\beta J_y'} , \]

and

\[ e^{-i\beta J_y'} = e^{-i\alpha J_z} e^{-i\beta J_y} e^{i\alpha J_z} . \]

By inserting the above expressions in the first relation and taking into account a similar expression for \( e^{-i\gamma J_z'} \), one gets Eq. (2.61).

2.5 We denote by

\[ \zeta_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

the two eigenstates \( | \frac{1}{2}, +\frac{1}{2} > \) and \( | \frac{1}{2}, -\frac{1}{2} > \) and apply to them the rotation

\[ \exp \left\{ i \frac{\pi \sigma_1}{2} \right\} = 1 \cos \left( \frac{\pi}{4} \right) + i \sigma_1 \sin \left( \frac{\pi}{4} \right) = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} . \]
Then we obtain the requested (normalized) eigenstates:

\[ \xi_+ = \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \xi_- = \sqrt{\frac{1}{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}. \]

## Problems of Chapter 3

### 3.1

Starting from the general rotation (2.9), one can derive the rotation \( R_n \) which transforms the unit vector along the \( x^3 \)-axis into the generic vector \( \mathbf{n} \) of components \( (n_1, n_2, n_3) \). Written in terms of the components of \( \mathbf{n} \), it reads

\[
R_n = \begin{pmatrix}
1 - \frac{(n_1)^2}{1 + n_3} & \frac{-n_1 n_2}{1 + n_3} & n_1 \\
\frac{-n_1 n_2}{1 + n_3} & 1 - \frac{(n_2)^2}{1 + n_3} & n_2 \\
-n_1 & -n_2 & n_3
\end{pmatrix}.
\]

By assuming \( \mathbf{n} = \beta/|\beta| \) and introducing the rotation \( R_n \) as a \( 4 \times 4 \) matrix in Eq. (3.28), one can easily derive Eq. (3.27).

### 3.2

Making use of Eq. (3.42), one gets

\[
A^\mu_{\nu}(AB) = \frac{1}{2} \text{Tr} \left[ \sigma^\mu A B \sigma_{\nu}(AB)^\dagger \right] = \frac{1}{2} \text{Tr} \left[ A^\dagger \sigma^\mu A B \sigma_{\nu} B^\dagger \right]
\]

and

\[
[A(A)B]^\mu_{\nu} = A^\mu_{\rho}(A)A^\rho_{\nu}(B) = \frac{1}{4} \text{Tr} \left[ A^\dagger \sigma^\mu A \sigma_{\rho} \right] \text{Tr} \left[ \sigma^\rho B \sigma_{\nu} B^\dagger \right]
\]

The identity \( A(AB) = A(A)A(B) \) follows immediately from the property of the trace

\[
\sum_\mu \text{Tr} [G\sigma_\mu] \text{Tr} [\sigma^\mu H] = 2 \text{Tr} [GH]
\]

valid for two arbitrary matrices \( G \) and \( H \), and which can be easily checked. As an immediate consequence of the proved identity, one gets \( [A(A)]^{-1} = A(A^{-1}) \).

### 3.3

By inserting (3.39) into Eq. (3.42) one obtains the explicit expression
From the previous problem one gets
\[
A(A) = \frac{1}{2} \begin{pmatrix}
|\alpha|^2 + |\beta|^2 & \alpha \beta^* + \beta \alpha^* & i(-\alpha \beta^* + \beta \alpha^*) & |\alpha|^2 - |\beta|^2 \\
+|\gamma|^2 + |\delta|^2 & +\gamma \delta^* + \delta \gamma^* & -\gamma \delta^* + \delta \gamma^* & +|\gamma|^2 - |\delta|^2 \\
\alpha \gamma^* + \beta \delta^* & \alpha \delta^* + \beta \gamma^* & i(-\alpha \delta^* + \beta \gamma^*) & \alpha \gamma^* - \beta \delta^* \\
+\gamma \alpha^* + \delta \beta^* & +\gamma \beta^* + \delta \alpha^* & -\gamma \beta^* + \delta \alpha^* & +\gamma \alpha^* - \delta \beta^* \\
i(\alpha \gamma^* + \beta \delta^*) & i(\alpha \delta^* + \beta \gamma^*) & \alpha \delta^* - \beta \gamma^* & i(\alpha \gamma^* - \beta \delta^*) \\
-\gamma \alpha^* - \delta \beta^* & -\gamma \beta^* - \delta \alpha^* & -\gamma \beta^* + \delta \alpha^* & -\gamma \alpha^* + \delta \beta^* \\
|\alpha|^2 + |\beta|^2 & \alpha \beta^* + \beta \alpha^* & i(-\alpha \beta^* + \beta \alpha^*) & |\alpha|^2 - |\beta|^2 \\
-|\gamma|^2 - |\delta|^2 & -\gamma \delta^* - \delta \gamma^* & +\gamma \delta^* - \delta \gamma^* & -|\gamma|^2 + |\delta|^2 \\
\end{pmatrix}
\]
which can be written as the product of two matrices, as follows

\[
A(A) = \frac{1}{2} \begin{pmatrix}
\alpha & \beta & \gamma & \delta \\
\alpha & \beta & \gamma & \delta \\
-i\gamma & -i\delta & i\alpha & i\beta \\
\alpha & \beta & -\gamma & -\delta \\
\end{pmatrix}
\begin{pmatrix}
\alpha^* & \beta^* & -i\beta^* & \alpha^* \\
\beta^* & \alpha^* & -i\alpha^* & -\beta^* \\
\gamma^* & \delta^* & -i\delta^* & \gamma^* \\
\delta^* & \gamma^* & i\gamma^* & -\delta^* \\
\end{pmatrix}
\]

The calculation of the determinant (making use of the minors of the second order) gives, since det \( A = 1 \),

\[
\det A(A) = \frac{1}{16} \left[ -4i(\det A)^2 \right] = 1
\]

3.4 From the previous problem one gets

\[
A^0_0(A) = \frac{1}{2} \left( |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 \right)
\]

and one can write

\[
A^0_0(A) = \frac{1}{4} \left( |\alpha + \delta^*|^2 + |\alpha - \delta^*|^2 + |\beta + \gamma^*|^2 + |\beta - \gamma^*|^2 \right) \geq \\
\geq \frac{1}{4} \left( |\alpha + \delta^*|^2 - |\alpha - \delta^*|^2 + |\beta + \gamma^*|^2 - |\beta - \gamma^*|^2 \right) = \\
= \text{Re}(\alpha \delta^* - \beta \gamma^*)
\]

Taking into account that

\[
\det A = \alpha \delta^* - \beta \gamma^* = 1,
\]

one has \( A^0_0(A) \geq 1 \).

3.5 If one takes the matrix \( A \) of the particular form

\[
U = \cos \frac{1}{2} \phi - i \sigma \cdot n \sin \frac{1}{2} \phi ,
\]

Eq. (3.38) becomes
\[ X' = \sigma_\mu x'^\mu = (\cos \frac{i}{2} \phi - i \sigma \cdot n \sin \frac{i}{2} \phi) \sigma_\mu x'^\mu \left( \cos \frac{i}{2} \phi + i \sigma \cdot n \sin \frac{i}{2} \phi \right) = \\
= x^0 + \sigma \cdot x \cos^2 \frac{i}{2} \phi + (\sigma \cdot n)(\sigma \cdot x)(\sigma \cdot n) \sin^2 \frac{i}{2} \phi - \\
- i \sin \frac{i}{2} \phi \cos \frac{i}{2} \phi \left[ (\sigma \cdot n)(\sigma \cdot x) - (\sigma \cdot x)(\sigma \cdot n) \right]. \]

Making use of the well-known identity

\[(\sigma \cdot a)(\sigma \cdot b) = a \cdot b + i \sigma \cdot (a \times b),\]

the above relation becomes

\[x^0' + (\sigma \cdot x') = x^0 + \sigma \cdot \left\{ (n \cdot x)n + \cos \phi [x - (n \cdot x)n] + \sin \phi (n \times x) \right\},\]

which clearly represents the application of the rotation matrix given by the Eqs. (3.23), (2.9) to the four-vector \( x = (x^0, x) \).

### 3.6 Taking for \( A \) the matrix

\[ H = \cosh \frac{1}{2} \psi - \sigma \cdot n \sinh \frac{1}{2} \psi, \]

Eq. (3.38) becomes

\[ X' = \sigma_\mu x'^\mu = (\cosh \frac{1}{2} \psi - \sigma \cdot n \sinh \frac{1}{2} \psi) \sigma_\mu x'^\mu \left( \cosh \frac{1}{2} \psi - \sigma \cdot n \sinh \frac{1}{2} \psi \right) = \\
= x^0 \cosh \psi - n \cdot x \sinh \psi + \\
+ \sigma \cdot \left\{ x - (n \cdot x)n + n \left[ (n \cdot x) \cosh \psi - x^0 \sinh \psi \right] \right\}. \]

The above relation corresponds to the transformations

\[ x^0' = x^0 \cosh \psi - n \cdot x \sinh \psi, \]

\[ x' = x - (n \cdot x)n + n \left[ (n \cdot x) \cosh \psi - x^0 \sinh \psi \right]. \]

which are immediately identified with the application of the pure Lorentz matrix (3.27) to the four-vector \( x \).

### 3.7 First let us check that an infinitesimal Lorentz transformation can be written in the form

\[ \Lambda^\rho_\sigma = g^\rho_\sigma + \delta \omega^\rho_\sigma \quad \text{with} \quad \delta \omega^\rho_\sigma = -\delta \omega^\sigma_\rho. \]

The condition (3.7) gives

\[ (g^\mu_\rho + \delta \omega^\mu_\rho) g_{\mu\nu} (g^\nu_\sigma + \delta \omega^\nu_\sigma) = g_{\rho\sigma}, \]

i.e. \( \delta \omega^\rho_\sigma = -\delta \omega^\sigma_\rho \). Next, let us consider an infinitesimal transformation obtained from (3.61):
\[ \Lambda^\rho{}_{\sigma} = g^\rho{}_{\sigma} - \frac{i}{2} \delta \omega^{\mu\nu} (M_{\mu\nu})^\rho{}_{\sigma} . \]

If we compare this expression of \( \Lambda^\rho{}_{\sigma} \) with the one given above, we get immediately
\[ (M_{\mu\nu})^\rho{}_{\sigma} = i (g_\mu{}^\rho g_\nu{}^\sigma - g_\nu{}^\rho g_\mu{}^\sigma) . \]

Writing explicitly \( M_{\mu\nu} \) in matrix form, one gets the six independent matrices \( J_1, K_i \) given in (3.50) and (3.54).

3.8 According to the definition given in Section 1.2, \( SO(4) \) is the group of four-dimensional real matrices \( \alpha \) satisfying the condition:
\[ \alpha \tilde{\alpha} = 1 , \quad \det \alpha = 1 . \]

The group leaves invariant the length of a four-vector \( x^\mu \) \((\mu = 1, 2, 3, 4)\) in a four-dimensional euclidean space. Using the notation
\[ x^2 = \delta_{\mu\nu} x^\mu x^\nu , \]
where \( \delta_{\mu\nu} \) is the Kronecker symbol, one gets in fact:
\[ (x')^2 = \delta_{\mu\nu} x'^\mu x'^\nu = \delta_{\mu\nu} \alpha^\mu{}_{\rho} \alpha^\nu{}_{\sigma} x^\rho x^\sigma = (x)^2 . \]

For this reason, \( SO(4) \) can be regarded as the group of rotations in a four-dimensional space; they are proper rotations since the group contains only unimodular matrices (\( \det \alpha = 1 \)).

In analogy with Problem 3.7, the infinitesimal transformation of \( SO(4) \) can be written as
\[ \alpha^\mu{}_{\nu} = \delta^\mu{}_{\nu} + \epsilon^\mu{}_{\nu} . \]
where \( \epsilon^\mu{}_{\nu} = -\epsilon^\nu{}_{\mu} \) (the group is characterized by six parameters, since its order is 6), as can be easily checked from the condition \( \alpha \tilde{\alpha} = 1 \).

One can find the infinitesimal generators \( J_{\mu\nu} \) by writing
\[ \alpha^\rho{}_{\sigma} = \delta^\rho{}_{\sigma} - \frac{i}{2} \epsilon^{\mu\nu} (J_{\mu\nu})^\rho{}_{\sigma} , \]
so that:
\[ (J_{\mu\nu})^\rho{}_{\sigma} = i (\delta^\rho{}_{\mu} \delta_{\nu\sigma} - \delta^\rho{}_{\nu} \delta_{\mu\sigma}) . \]

From these, one can obtain the commutation relations
\[ [J_{\mu\nu}, J_{\rho\sigma}] = -i (\delta_{\mu\rho} J_{\nu\sigma} - \delta_{\mu\sigma} J_{\nu\rho} - \delta_{\nu\rho} J_{\mu\sigma} + \delta_{\nu\sigma} J_{\mu\rho}) . \]

Let us now introduce the following linear combinations
\[ M_i = \frac{1}{2} (J_{i4} + J_{jk}) , \quad N_i = -\frac{1}{2} (J_{i4} - J_{jk}) . \]
Since these combinations are real, also the quantities $M_i$, $N_i$, as $J_{\mu\nu}$, can be taken as basic elements of the Lie algebra of $SO(4)$. It is easy to verify that the commutators of $J_{\mu\nu}$, expressed in terms of $M_i$, $N_i$, as $J_{\mu\nu}$, can be taken as basic elements of the Lie algebra of $SO(4)$. It is easy to verify that the commutators of $J_{\mu\nu}$, expressed in terms of $M_i$, $N_i$, as $J_{\mu\nu}$, become:

\[
\begin{align*}
[M_i, M_j] &= i\epsilon_{ijk} M_k, \\
[N_i, N_j] &= i\epsilon_{ijk} N_k, \\
[M_i, N_j] &= 0,
\end{align*}
\]

The quantities $M_i$ and $N_i$ can be considered as the components of two independent angular momentum operators $M$ and $N$; each of them generates a group $SO(3)$. Then the group $SO(4)$ corresponds to the direct product $SO(4) \sim SO(3) \otimes SO(3)$.

This correspondence is not strictly an isomorphism, due to the arbitrariness in the choice of the signs for the two subgroups $SO(3)$. It is instructive to compare the above properties of $SO(4)$ with those given for $L_\uparrow$. In particular, the IR’s of $SO(4)$ can still be labelled by the eigenvalues of the Casimir operators $M_2$ and $N_2$ (see Section 3.4); however, the IR $D(j,j')_\alpha$ of $SO(4)$, which is of order $(2j+1)(2j'+1)$, is now unitary since the group $SO(4)$ is compact, in contrast with the finite IR’s of the non-compact group $L_\uparrow$, which are not unitary.

3.9 From Eq. (3.70) one gets the following representation for a rotation about $x^3$ and a pure Lorentz transformation along $x^3$

\[
A(R_3) = e^{-\frac{i}{2} \phi \sigma_3} = \begin{pmatrix} e^{-\frac{1}{2} \phi} & 0 \\ 0 & e^{\frac{1}{2} \phi} \end{pmatrix}, \quad A(L_3) = e^{-\frac{i}{2} \psi \sigma_3} = \begin{pmatrix} e^{-\frac{1}{2} \psi} & 0 \\ 0 & e^{\frac{1}{2} \psi} \end{pmatrix}.
\]

The relative transformations of the spinor $\xi$ are given from (3.68) and are

\[
\begin{align*}
\xi'^1 &= e^{-\frac{1}{2} i \psi} \xi^1, \\
\xi'^2 &= e^{\frac{1}{2} i \psi} \xi^2,
\end{align*}
\]

respectively. The transformations of the spinor $\xi^*$, according to (3.69), are given by the complex conjugate relations.

3.10 Any antisymmetric tensor can always be decomposed as

\[
A_{\mu\nu} = \frac{1}{2} \left( A_{\mu\nu} + A^D_{\mu\nu} \right) + \frac{1}{2} \left( A_{\mu\nu} - A^D_{\mu\nu} \right),
\]

where the dual tensor $A^D_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\sigma\tau} A^{\sigma\tau}$ is clearly anti-symmetric. The two tensor $(A_{\mu\nu} + A^D_{\mu\nu})$ and $(A_{\mu\nu} - A^D_{\mu\nu})$ are selfdual and anti-selfdual, respectively, since $(A^D_{\mu\nu})^D = A_{\mu\nu}$ (remember that $\frac{i}{2} \epsilon^{\mu\nu\sigma\tau} \epsilon_{\mu\nu\alpha\beta} = g^{\sigma\alpha} g^{\tau\beta} - g^{\sigma\beta} g^{\tau\alpha}$).
If $A'_{\mu\nu}$ is the transformed of $A_{\mu\nu}$ under an element of $L^\uparrow_+$, the transformed of $A^D_{\mu\nu}$ is $A'^D_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\tau}A'^{\sigma\tau}$; so that the selfdual and anti-selfdual tensors are irreducible under $L^\uparrow_+$.

The electromagnetic field tensor $F^{\mu\nu}$ and its dual $F^D{\mu\nu}$ are given in terms of the field components $E^i$, $B^i$ by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 - B^2 \\ -E^2 - B^3 & 0 & B^1 \\ -E^3 & B^2 - B^1 & 0 \end{pmatrix}, \quad F^D{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & -E^3 & E^2 \\ B^2 & E^3 & 0 & -E^1 \\ B^3 & -E^2 & E^1 & 0 \end{pmatrix},$$

We see that the selfdual and anti-selfdual tensors correspond to the field combinations $E^i - B^i$ and $E^i + B^i$, respectively.

### 3.11
As seen in Section 3.4, the four-vector $x^\mu$ is the basis of the IR $D^{(\frac{1}{2}, \frac{1}{2})}$ of $L^\uparrow_+$. The basis of the direct product representation $D^{(\frac{1}{2}, \frac{1}{2})} \otimes D^{(\frac{1}{2}, \frac{1}{2})}$ can then be taken to be the tensor $x^\mu y_\nu$. This tensor is not irreducible and one can decompose it in the following way:

$$x^\mu y_\nu = \left\{ \frac{1}{2} (x^\mu y_\nu + y^\mu x_\nu) - \frac{1}{4} (x \cdot y) \right\} + \frac{1}{2} (x^\mu y_\nu - y^\mu x_\nu) + \frac{1}{4} (x \cdot y).$$

It is easy to check that each of the terms on the r.h.s (the 9-component traceless symmetric tensor, the 6-component anti-symmetric tensor and the trace) transform into itself under $L^\uparrow_+$. Moreover, we learnt from Problem 3.10 that the anti-symmetric tensor can be decomposed further into two irreducible 3-component tensors. One gets in this way four irreducible tensors. It is natural to take them as the bases of the IR’s $D^{(1,1)}$, $D^{(0,0)}$, $D^{(1,0)}$ and $D^{(0,1)}$, which have, respectively, dimensions 9, 1, 3, and 3.

### Problems of Chapter 4

#### 4.1
The composition law of the Poincaré group is given by Eq. (4.2). One must show that the group properties given in Section 1.1 are satisfied. The multiplication is associative:

$$(a_3, A_3) [(a_2, A_2)(a_1, A_1)] = [a_3 + A_3(a_2 + A_2a_1), A_3A_2A_1] =$$

$$= (a_3 + A_3a_2, A_3A_2) (a_1, A_1) =$$

$$= [(a_3, A_3)(a_2, A_2)] (a_1, A_1).$$

The identity is $(0, I)$ and the inverse of $(a, A)$ is $(-A^{-1}a, A^{-1})$. In fact:
\[ (a, A)(-A^{-1}a, A^{-1}) = (a - AA^{-1}a, AA^{-1}) = (0, I), \]
\[ (-A^{-1}a, A^{-1})(a, A) = (-A^{-1}a + A^{-1}a, A^{-1}A) = (0, I). \]

In order to show that the translation group \( S \) is an invariant subgroup, it is required that, for any translation \((b, I)\) and any element \((a, A)\) of \( P_+^1 \), the product
\[ (a, A)(b, I)(a, A)^{-1}, \]
is a translation. In fact, one gets immediately:
\[ (a, A)(b, I)(-\Lambda^{-1}a, \Lambda^{-1}) = (a + Ab, \Lambda)(-\Lambda^{-1}a, \Lambda^{-1}) = (Ab, I). \]

4.2 We have to show that \( P^2 \) and \( W^2 \) commute with all the generators of \( P_+^1 \), i.e. with \( P^\mu \) and \( M^\mu\nu \). First, making use of the commutation relations (4.16) and (4.17), we get
\[ [P^2, P^\nu] = [P_\mu P^\mu, P^\nu] = 0, \]
\[ [P^2, M^\mu\nu] = g_{\sigma\tau}[P^\sigma P^\tau, M^\mu\nu] = g_{\sigma\tau}[P^\sigma, M^\mu\nu]P^\tau + g_{\sigma\tau}P^\sigma[P^\tau, M^\mu\nu] = \]
\[ = ig_{\sigma\tau}(g^{\mu\sigma}P^\nu - g^{\nu\sigma}P^\mu)P^\tau + ig_{\sigma\tau}P^\sigma(g^{\mu\tau}P^\nu - g^{\nu\tau}P^\mu) = \]
\[ = 2i[P^\mu, P^\nu] = 0. \]

Then, making use of (4.21), we obtain
\[ [W^2, P^\mu] = [W_\nu W^\nu, P^\mu] = 0, \]
\[ [W^2, M^\mu\nu] = g_{\sigma\tau}[W^\sigma, M^\mu\nu]W^\tau + g_{\sigma\tau}W^\sigma[W^\tau, M^\mu\nu] = -2i[W^\mu, W^\nu] = 0. \]

4.3 Writing explicitly the translation in (4.8), one gets
\[ U^{-1}(A)e^{-ia^\mu P_\mu}U(A) = e^{-i(A^{-1})^\nu a^\nu P_\nu} = e^{-ia^\mu A_\mu^\nu P_\nu}, \]
where \( U(A) \) stands for \( U(0, A) \). In the case of an infinitesimal translation, one obtains
\[ U^{-1}(A)P_\mu U(A) = A_\mu^\nu P_\nu, \]
which coincides with Eq. (4.35). Making use explicitly of the Lorentz transformation (3.61), one gets for \( \omega^\mu\nu \) infinitesimal
\[ [M_{\mu\nu}, P_\rho] = -(M_{\mu\nu})_{\rho}^\sigma P_\sigma, \]
which are equivalent to the commutation relations (4.17) when one takes into account the expression of \((M_{\mu\nu})_{\rho}^\sigma\) derived in Problem 3.7.
4.4 Starting from

\[ U(A') = e^{-\frac{i}{2} \omega_{\mu\nu} M_{\mu\nu}}, \]

and assuming \( \omega_{\mu\nu} \) infinitesimal, one gets

\[ U(\Lambda) U(A') U(\Lambda^{-1}) = I - \frac{i}{2} \omega_{\mu\nu} U(\Lambda) M_{\mu\nu} U^{-1}(\Lambda) = I - \frac{i}{2} \overline{\omega}^{\alpha\beta} M_{\alpha\beta}, \]

where \( \overline{\omega}^{\alpha\beta} \) is given by

\[ g^{\alpha\beta} + \overline{\omega}^{\alpha\beta} = (\Lambda A' \Lambda^{-1})^{\alpha\beta} = \Lambda^{\alpha\mu} (g^{\mu\nu} + \omega^{\mu\nu}) \Lambda^{\nu\beta} = g^{\alpha\beta} + \Lambda^{\alpha\mu} \Lambda^{\nu\beta} \omega^{\mu\nu}. \]

One then obtains

\[ U(\Lambda) M_{\mu\nu} U^{-1}(\Lambda) = \Lambda^{\alpha\mu} \Lambda^{\beta\nu} M_{\alpha\beta}, \]

which are the transformation properties of a tensor of rank two.

Taking also for \( U(\Lambda) \) a generic infinitesimal transformation, one gets

\[ [M_{\mu\nu}, M_{\rho\sigma}] = -(M_{\rho\sigma})^{\alpha\mu} M_{\alpha\nu} - (M_{\rho\sigma})^{\beta\nu} M_{\mu\beta}, \]

which corresponds to the commutation relation (3.60), as can be seen by use of the explicit expression of \( (M_{\mu\nu})^{\rho\sigma} \) derived in Problem 3.7.

4.5 The operators \( W_{\mu} \) transform as \( P_{\mu} \), i.e. according to (4.35):

\[ W'_{\mu} = U^{-1}(\Lambda) W_{\mu} U(\Lambda) = \Lambda^{\mu\nu} W_{\nu}. \]

We use for \( \Lambda \) the pure Lorentz transformation \( L_{p}^{-1} \) which brings a state \( |p, \zeta> \) at rest; its explicit expression is obtained immediately from (3.27) by changing the sign of the space-like components:

\[ L_{p}^{-1} = \begin{pmatrix} \frac{p^0}{m} & \frac{p^i}{m} \\ -\frac{p_j}{m} & \delta_j^i - \frac{p^i p_j}{m(p^0 + m)} \end{pmatrix}. \]

One then obtains

\[ W'_0 = \frac{1}{m} \left( p^0 W_0 + p^i W_i \right) = 0 \quad \text{(see Eq. (4.20))}, \]

\[ W'_i = \frac{1}{m} \left( -p_i W_0 - \frac{p_i p^j W_j}{p^0 + m} \right) + W_i = W_i - \frac{p_i}{p^0 + m} W_0. \]

Let us write the following commutator

\[ [W'_i, W'_j] = [W_i, W_j] - \frac{p_i}{p^0 + m} [W_0, W_j] - \frac{p_j}{p^0 + m} [W_i, W_0]. \]
With the definition \( J_k = \frac{1}{m} W'_k \), making use of (4.21) and having in mind that the operators are applied to momentum eigenstates \( |p, \zeta> \), one gets
\[
[J_i, J_j] = i \epsilon_{ijk} J_k .
\]
The calculation is straightforward but lengthy: for simplicity we fix \( i = 1, j = 2 \):
\[
[W'_1, W'_2] =
\]
\[
= -i(W_0 p_3 - p_0 W_3) + \frac{ip_1}{p^0 + m} (W_1 p_3 - p_1 W_3) + \frac{ip_2}{p^0 + m} (W_2 p_3 - p_2 W_3) =
\]
\[
= -i W_0 p_3 + i W_3 \left( p^0 - \frac{p_1^2 + p_2^2}{p^0 + m} \right) + i \frac{p_3}{p^0 + m} (W_1 p_1 + W_2 p_2) =
\]
\[
= -i W_0 p_3 + i m W_3 - i \frac{p_3}{p^0 + m} (W^i p_i) =
\]
\[
= im \left( W_3 - \frac{p_3}{p^0 + m} W_0 \right) = im W'_3 .
\]
q. d. e.

4.6 From the identity (see Eqs. (4.19), (3.61)),
\[
U(\Lambda) = e^{-i n^\sigma W_\sigma} = e^{-\frac{i}{2} \epsilon_{\sigma \mu \nu} p^\mu M^{\mu \nu}} = e^{-\frac{i}{2} \omega_{\mu \nu} M^{\mu \nu}},
\]
it follows
\[
\omega_{\mu \nu} = \epsilon_{\sigma \rho \mu \nu} n^\sigma p^\rho ,
\]
which shows that the given transformation belongs to \( \mathcal{L}^1_+ \) (the operators are considered acting on a one-particle state \( |p, \zeta> \)). The matrix elements of \( \Lambda \) are then
\[
\Lambda^{\mu \nu} = g^{\mu \nu} + \omega^{\mu \nu} = g^{\mu \nu} + \epsilon^{\sigma \rho \mu \nu} n_\sigma p_\rho ,
\]
so that
\[
p'^\mu = \Lambda^{\mu}_{\nu} p^\nu = p^\mu + \epsilon^{\sigma \rho \mu \nu} n_\sigma p_\rho p_\nu = p^\mu ,
\]
as follows from the antisymmetry of the Levi-Civita tensor.

Problems of Chapter 5

5.1 We note that the generic element of the little group of a given four-vector \( \mathbf{p} \) depends on \( \Lambda \) and \( p \) \((p^2 = \mathbf{p}^2)\), arbitrarily chosen in \( \mathcal{L}^1_+ \) and \( H_{p^2} \), respectively. It is then easy to show that the set of elements \( \mathcal{R} \) satisfies the group properties.
a) Product:
\[ \mathcal{R}_a = \mathcal{R}(\Lambda_a, p_a) = L_{p_a^{-1}p_a}^{-1} \Lambda_a L_{p_a^{-1}p_a} \]
with \( p'_a = \Lambda_a p_a \),
\[ \mathcal{R}_b = \mathcal{R}(\Lambda_b, p_b) = L_{p_b^{-1}p_b}^{-1} \Lambda_b L_{p_b^{-1}p_b} \]
with \( p'_b = \Lambda_b p_b \),
\[ \mathcal{R}_b \mathcal{R}_a = L_{p_b^{-1}p_b}^{-1} \Lambda_b L_{p_b^{-1}p_b} L_{p_a^{-1}p_a}^{-1} \Lambda_a L_{p_a^{-1}p_a} = \]
\[ = L_{p_b^{-1}p_b}^{-1} \Lambda_b L_{p_a^{-1}p_a} = \mathcal{R}(\Lambda_{ba}, p_a) \]
with \( p'_b = \Lambda_{ba} p_a \).

It is easy to check that the product is associative.

b) Identity: it corresponds to the choice \( \Lambda = I \); in fact
\[ \mathcal{R}(I, p) = L_{p^{-1}p}^{-1} L_{p^{-1}p} = I \, . \]

c) Inverse: it is given by:
\[ \mathcal{R}^{-1} = \left( L_{p^{-1}p}^{-1} \Lambda L_{p^{-1}p} \right)^{-1} = L_{p^{-1}p}^{-1} A^{-1} L_{p^{-1}p} = \]
\[ = \mathcal{R}(A^{-1}, p') \]
with \( p' = A p \).

5.2 We suppose that a different standard vector \( \overline{p}' \) has been fixed, instead of \( \overline{p} \), belonging to the little Hilbert space \( H_{p^2} \) (i.e. \( \overline{p}'^2 = \overline{p}^2 = p^2 \)). Then for any given element \( \mathcal{R} = \mathcal{R}(A, p) \) of the little group of \( \overline{p} \) we can build
\[ \mathcal{R}' = L_{\overline{p}'^{-1}p}^{-1} \mathcal{R} L_{\overline{p}'^{-1}p} \, , \]
which is the element of the little group of \( \overline{p}' \) corresponding to \( \Lambda \) and \( p \). In fact, we can invert Eq. (5.6) as follows
\[ A = L_{p'\overline{p}^{-1}} \mathcal{R}(A, p) L_{p'\overline{p}^{-1}}^{-1} = L_{p'\overline{p}^{-1}} \mathcal{R}' L_{p'\overline{p}^{-1}}^{-1} \, , \]
so that
\[ \mathcal{R}' = L_{p'\overline{p}^{-1}} A L_{p'\overline{p}^{-1}} = \mathcal{R}'(A, p) \, . \]
There is a one-to-one correspondence between \( \mathcal{R}(A, p) \) and \( \mathcal{R}'(A, p) \), which is preserved under multiplication:
\[ \mathcal{R}_2' \mathcal{R}_1' = L^{-1} \mathcal{R}_2 L L^{-1} \mathcal{R}_1 L \, . \]
Therefore there is an isomorphism between the two little groups relative to the standard vectors \( \overline{p} \) and \( \overline{p}' \). One can remove the restrictions that \( \overline{p} \) and \( \overline{p}' \) are in the same little Hilbert space, keeping in mind that the little group of a standard vector \( \overline{p} \) is the little group also of any standard vector \( c\overline{p} \) \( (c \neq 0) \), being
\[ \mathcal{R}(c\overline{p}) = c\overline{p} \quad \text{if} \quad \mathcal{R}(\overline{p}) = \overline{p} \, . \]
Combing the results, we notice that the isomorphism is extended to all little groups of vectors in the same class, i.e. time-like, space-like and light-like.

5.3 From what proved in the previous problem, the little group is defined by any standard vector $\bar{p}$ in the class. Let us choose the vector $\bar{p} = (0, 0, 0, p_3)$: it is clear that any rotation about the third axis (i.e. in the $(1, 2)$ plane), any pure Lorentz transformation in the $(1, 2)$ plane, and any combination of such rotations and pure Lorentz transformations leave the chosen standard vector unchanged. In fact, the required transformation matrices must be a subset of $L^1_+$ (i.e. $\tilde{A}gA = g$ and $\det A = +1$) satisfying the condition $A\bar{p} = \bar{p}$. Then such matrices have the form

$$A = \begin{pmatrix} \lambda & 0 \\ -\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\lambda$ is a $3 \times 3$ matrix which satisfies the condition

$$\tilde{\lambda}g\lambda = g, \quad \det \lambda = 1 \quad \text{with} \quad g = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

It is easy to check that the set of the matrices $\lambda$ can be identified with the group $SO(1, 2)$ (for the definition, see Section 1.2). We note that the transformations of this group leave invariant the quantity $(x^0)^2 - (x^1)^2 - (x^2)^2$.

5.4 We start from the explicit expression for a generic pure Lorentz transformation (see Eq. (3.27))

$$L = \begin{pmatrix} \gamma & \gamma\beta_j \\ -\gamma\beta_j & \delta_j - \frac{\beta^i\beta_j}{\beta^2}(\gamma - 1) \end{pmatrix},$$

and of the particular ones

$$L_p = \begin{pmatrix} \frac{p_0}{m} & -\frac{p_j}{m} \\ -\frac{p^i}{m} & \delta^i_j - \frac{p^ip_j}{m(p^0 + m)} \end{pmatrix}$$

and

$$L_p^{-1} = \begin{pmatrix} \frac{p_0'}{m} & -\frac{p_j'}{m} \\ -\frac{p^i'}{m} & \delta^i_j - \frac{p^ip_j'}{m(p^0 + m)} \end{pmatrix}.$$
After some algebra, making use of the relation \( p' = Lp \), one obtains

\[
R_{p'p} = L_p^{-1} L L_p =
\begin{pmatrix}
1 & -\frac{\beta^i \beta_j}{\beta^2} (\gamma - 1) + \frac{\gamma \beta^j p_j}{p^0 + m} - \frac{\gamma p'^i \beta_j}{p'^0 + m} + \frac{(\gamma - 1)p'^i p_j}{(p^0 + m)(p'^0 + m)} \\
0 & \delta^i_j
\end{pmatrix}
\]

(a).

When applied to \( p \), \( R_{p'p} \) gives

\[
(R_{p'p})^i_j p^j = \frac{p^0 + \gamma m}{p'^0 + m} p'^i + \gamma m \beta^i.
\]

In the ultrarelativistic limit \((p^0 \gg m, p'^0 \gg m)\) this corresponds to

\[
(R_{p'p})^i_j p^j = \frac{p'^i}{p'^0},
\]

so that \( R_{p'p} \) represents the rotation of the velocity vector (note that \(|p|/p^0 \simeq |p'|/p'^0 \simeq 1|\).

In the non-relativistic limit one gets simply \((R_{p'p})^i_j \simeq \delta^i_j\) and there is no rotation at all.

It is instructive to express \( R_{p'p} \) in terms of the 2 \( \times \) 2 representation of \( L, L_p, L_p^{-1} \). In this case one has

\[
L = e^{-\frac{1}{2} \alpha \sigma \cdot \nu}, \quad L_p = e^{-\frac{1}{2} \psi \sigma \cdot \nu}, \quad L_{p'} = e^{-\frac{1}{2} \psi' \sigma \cdot \nu'},
\]

where

\[
cosh \alpha = \gamma, \quad \cosh \psi = \frac{p^0}{m}, \quad \cosh \psi' = \frac{p'^0}{m},
\]

and

\[
e = \frac{\beta}{|\beta|}, \quad n = \frac{p}{|p|}, \quad n' = \frac{p'}{|p'|}.
\]

Then we can write

\[
R_{p'p} = e^{\frac{i}{2} \psi' \sigma \cdot n'} e^{-\frac{i}{2} \alpha \sigma \cdot e} e^{-\frac{i}{2} \psi \sigma \cdot n},
\]

which, with some algebra, can be expressed in the form

\[
R_{p'p} = \frac{1}{\cosh \psi'} \left\{ \cosh \frac{\alpha}{2} \cosh \frac{\psi}{2} + \sinh \frac{\alpha}{2} \sinh \frac{\psi}{2} \left[ (e \cdot n) + i \sigma \cdot (e \times n) \right] \right\}.
\]

If one write

\[
R_{p'p} = e^{-\frac{i}{2} \theta \sigma \cdot \nu},
\]

one can determine the angle \( \theta \) and the direction \( \nu \) through the relations
\[ \cos \frac{\theta}{2} = \frac{1}{\cosh \frac{\psi'}{2}} \left\{ \cosh \frac{\alpha}{2} \cosh \frac{\psi}{2} + \sinh \frac{\alpha}{2} \sinh \frac{\psi}{2} (e \cdot n) \right\} , \]

\[ \sin \frac{\theta}{2} \nu = -\frac{1}{\cosh \frac{\psi'}{2}} \sinh \frac{\alpha}{2} \sinh \frac{\psi}{2} (e \times n). \]

5.5 In the present case Eq. (5.13) becomes

\[ R_{p'p} = L_{p'}^{-1} R L_p \quad \text{with} \quad p' = Rp \quad \text{i.e.} \quad p'^0 = p^0, \quad |p'| = |p|. \]

One can write (see Eq. (5.21)):

\[ L_p = R_p L_3(p) R_p^{-1}, \]

where \( R_p^{-1} \) is a rotation which transforms \( p \) along the direction of the \( x^3 \)-axis, and \( L_3(p) \) is the boost along the \( x^3 \)-direction. Analogously

\[ L_{p'} = R_{p'} L_3(p') R_{p'}^{-1}, \]

with \( L_3(p') = L_3(p) \) since \( |p| = |p'| \). Making use of the above relations, one gets

\[ R_{p'p} = R_{p'} L_3^{-1}(p') R_{p'}^{-1} R R_p L_3(p) R_p^{-1}, \]

and, since \( R_{p'}^{-1} R R_p \) is a rotation about the \( x^3 \)-axis (a vector along \( x^3 \) is not transformed) and therefore it commutes with the boost \( L_3(p) \), one has the desired results

\[ R = R_{p'} R_{p'}^{-1} R R_p R_p^{-1} = R. \]

5.6 From Eqs. (5.22) and (5.14), one gets

\[ U(a, \Lambda) |p, \lambda> = U(a, \Lambda) \sum_{\sigma} D^{(s)}_{\sigma \lambda}(R_p) |p, \sigma> = \]

\[ = e^{-ip'a} \sum_{\sigma} D^{(s)}_{\sigma \lambda}(R_p) \sum_{\sigma'} D^{(s)}_{\sigma' \sigma} \left( L_{p'}^{-1} \Lambda L_p \right) |p', \sigma'> = \]

\[ = e^{-ip'a} \sum_{\sigma'} D^{(s)}_{\sigma' \lambda} \left( L_{p'}^{-1} \Lambda L_p \right) |p', \sigma'> = \]

\[ = e^{-ip'a} \sum_{\sigma'} D^{(s)}_{\sigma' \lambda} \left( R_{p'} R_{p'}^{-1} L_{p'}^{-1} \Lambda L_p R_p \right) |p', \sigma'>. \]

Making use of (5.21) and again of (5.22), one finally gets
\[ U(a, A)|p, \lambda> = e^{-ip'a} \sum_{\sigma' \lambda'} D^{(s)}_{\sigma' \lambda'}(R_p) D^{(s)}_{\lambda' \lambda} \left( L_3^{-1}(p')R_p^{-1}A R_pL_3(p) \right) |p', \sigma'> = \]
\[ = e^{-ip'a} \sum_{\lambda'} D^{(s)}_{\lambda' \lambda} \left( L_3^{-1}(p')R_p^{-1}A R_pL_3(p) \right) |p', \lambda'> , \]
which concides with Eq. (5.20).

5.7 The transformation properties of \(|p, \lambda>\) under a rotation are given by Eq. (5.20), where the little group matrices are of the type
\[ \mathcal{R} = L_3^{-1}(p')R_p^{-1}RR_pL_3(p) , \]
with \(p' = Rp\). We observe that \(R_p^{-1}RR_p\) is a rotation about the \(x^3\)-axis (in fact, it leaves unchanged the vectors along this axis), which commutes with a boost along the same axis (see (3.58)). Then, since \(L_3(p') = L_3(p)\) (being \(|p| = |p'|\)), the little group matrix reduces just to that rotation
\[ \mathcal{R} = R_p^{-1}RR_p . \]

Now we use for \(R\) the explicit expression (2.9) of a generic rotation in terms of an angle \(\phi\) and a unit vector \(n\). An explicit expression for \(R_p\) can be obtained in the form (it corresponds to \(n_3 = 0\))
\[ R_p = \begin{pmatrix} (1 - \cos \theta) \sin^2 \beta + \cos \theta & -(1 - \cos \theta) \sin \beta \cos \beta & \sin \theta \cos \beta \\ -(1 - \cos \theta) \sin \beta \cos \beta & (1 - \cos \theta) \cos^2 \beta + \cos \theta & \sin \theta \sin \beta \\ -\sin \theta \cos \beta & -\sin \theta \sin \beta & \cos \theta \end{pmatrix} , \]
and one has
\[ R_p \begin{pmatrix} 0 \\ 0 \\ |p| \end{pmatrix} = \begin{pmatrix} |p| \sin \theta \cos \beta \\ |p| \sin \theta \sin \beta \\ |p| \cos \theta \end{pmatrix} . \]

One can express \(R_p\) in the same form, introducing two angles \(\theta'\) and \(\beta'\). The rotation \(R_p\) can also be expressed in terms of \(\theta, \beta\) and \(n, \phi\) making use of the condition
\[ R_p' \begin{pmatrix} 0 \\ 0 \\ |p| \end{pmatrix} = RR_p \begin{pmatrix} 0 \\ 0 \\ |p| \end{pmatrix} . \]
In fact, with the notation \(S = RR_p\) the above relation gives \(\sin \theta' \cos \beta' = S_{13}, \sin \theta' \sin \beta' = S_{23}, \cos \theta' = S_{33}\), so that one can write (compare with the expression given in Problem 3.1)
\( R_{p'} = \begin{pmatrix}
\frac{S_{23}^2}{1 + S_{33}} + S_{33} & -S_{13}S_{23} & S_{13} \\
\frac{-S_{13}S_{23}}{1 + S_{33}} & \frac{S_{13}^2}{1 + S_{33}} + S_{33} & S_{23} \\
-S_{13} & -S_{23} & S_{33}
\end{pmatrix} \).

Performing the product \( \mathcal{R} = R_{p}^{-1}S \) one then obtains
\[
\mathcal{R} = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
where
\[
\cos \alpha = \frac{S_{11} + S_{22}}{1 + S_{33}}, \quad \sin \alpha = \frac{S_{21} - S_{12}}{1 + S_{33}}.
\]

Finally, expressing the elements of the matrix \( S \) in terms of the parameters \( \theta \), \( \phi \) and \( \mathbf{n} \), one finds\(^2\), after a lengthy calculation, the following expression for the rotation \( \mathcal{R} \):
\[
e^{-ia} = \frac{(1 + \cos \theta) \sin \phi + \hat{p}^0 \cdot (\hat{p} \times \mathbf{n}) - i(\hat{p}_0 + \hat{p}) \cdot \mathbf{n}(1 - \cos \phi)}{(1 + \cos \theta) \sin \phi + \hat{p}^0 \cdot (\hat{p} \times \mathbf{n}) + i(\hat{p}_0 + \hat{p}) \cdot \mathbf{n}(1 - \cos \phi)},
\]
where
\[
\hat{p}_0 = (0, 0, 1) \quad , \quad \hat{p} = (\sin \theta \cos \beta, \sin \theta \sin \beta, \cos \theta).
\]
It is interesting to note that, in the specific case \( \mathbf{n} = (0, 0, 1) \), the above relation, as expected, reduces to the identity \( \alpha = \phi \).

5.8 We can consider the little group matrix given by Eq. (5.30) taking for \( \Lambda \) a rotation \( R \):
\[
\mathcal{R} = R_{p}^{-1}L_{p'}^{-1}RL_{p}R_{p} = L_{\hat{p}'}^{-1}R_{p}^{-1}RR_{p}L_{p} \quad , \quad (p' = Rp).
\]
Since \( R_{p}^{-1}RR_{p} \) is a rotation about the \( x^3 \)-axis, it commutes with any boost along the same axis; moreover, since \( p' = Rp \), one has \( \hat{p}' = \hat{p} \). The two boosts cancel and one gets
\[
\mathcal{R} = R_{p}^{-1}RR_{p},
\]
as in the case of a massive particle (see Problem 5.5).

5.9 We can take \( \mathbf{p} = (\rho, 0, 0, \rho) \) as standard vector. The element \( E \) of the little group has to satisfy the conditions
By applying the above conditions to a generic $4 \times 4$ real matrix, $E$ can be written in terms of 3 independent real parameters, which are conveniently denoted by $x$, $y$ and $\cos \alpha$. A sign ambiguity is eliminated by the condition $\det E = +1$. The final expression of the matrix $E$ is the following:

$$E = \begin{pmatrix}
1 + \frac{1}{2}(x^2 + y^2) & -(x \cos \alpha + y \sin \alpha) & x \sin \alpha - y \cos \alpha & -\frac{1}{2}(x^2 + y^2) \\
-x & \cos \alpha & -\sin \alpha & x \\
-y & \sin \alpha & \cos \alpha & y \\
\frac{1}{2}(x^2 + y^2) & -(x \cos \alpha + y \sin \alpha) & x \sin \alpha - y \cos \alpha & 1 - \frac{1}{2}(x^2 + y^2)
\end{pmatrix}.$$  

If we introduce a two-dimensional vector $r$ and a rotation $R$ of the form

$$r = \begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

the matrix $E$ can be written in the form

$$E = \begin{pmatrix}
1 + \frac{1}{2}r^2 & -\bar{r}R & -\frac{1}{2}r^2 \\
-r & R & r \\
\frac{1}{2}r^2 & -\bar{r}R & 1 - \frac{1}{2}r^2
\end{pmatrix}.$$  

The elements $(r, R)$ form a group with the composition law

$$(r_2, R_2)(r_1, R_1) = (r_2 + R_2r_1, R_2R_1),$$

which is analogous to Eq. (4.3), except that now $r$ is two-dimensional and $R$ is a $2 \times 2$ matrix with the condition $RR^\dagger = I$. The group is then the group of translations and rotations in a plane, which is called the Euclidean group in two dimensions.

It is easy to find the generators from three independent infinitesimal transformations, i.e. translation along $x$, $y$ and rotation in the $(x, y)$ plane:

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

Their commutation relations are given by

$$[\Pi_1, \Pi_2] = 0, \quad [J_3, \Pi_1] = i\Pi_2, \quad [J_3, \Pi_2] = -i\Pi_1,$$

in agreement with Eq. (5.25).
5.10 The generic element of the little group, which is the two-dimensional Euclidean group, is given by Eq. (5.30), with $\Lambda$ taken as a pure Lorentz transformation $L$. Accordingly, it can be written as

$$E = L^{-1}_{p\bar{p}} R^{-1}_{p'} L R_{p} L_{p\bar{p}},$$  \hspace{1cm} (a)$$

where the general form for $E$ has been obtained in Problem 5.9 in terms of a rotation and a two-dimensional translation. In principle, since we are interested only in the rotation (see Section 5.3), we could neglect the translation from the beginning. Moreover, one can realize that the two boosts are irrelevant for the calculation of the rotation angle. However, for the sake of completeness, we perform the full calculation. Assuming

$$\bar{p} = (\rho, 0, 0, \rho) \ , \ \bar{p} = (|p|, 0, 0, |p|) \ , \ \bar{p}' = (|p'|, 0, 0, |p'|),$$

from the condition

$$L_{p\bar{p}} \bar{p} = \bar{p},$$

one gets

$$L_{p\bar{p}} = \begin{pmatrix}
\frac{1 + \delta^2}{2\delta} & 0 & 0 & \frac{1 - \delta^2}{2\delta} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1 - \delta^2}{2\delta} & 0 & 0 & \frac{1 + \delta^2}{2\delta}
\end{pmatrix}$$

with $\delta = \frac{\rho}{|p|}$, $\delta'$ of the same form in terms of $\delta' = \frac{\rho}{|p'|}$. The rotation $R_{p}$ can be written in the form (compare with Problem 3.1)

$$R_{p} = \begin{pmatrix}
1 - \frac{(n_1)^2}{1 + n_1^3} & -\frac{n_1 n_2}{1 + n_1^3} & n_1 \\
-\frac{n_1 n_2}{1 + n_1^3} & 1 - \frac{(n_2)^2}{1 + n_3^3} & n_2 \\
-n_1 & -n_2 & n_3
\end{pmatrix}$$

in terms of the components of $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and similarly for $R_{p'}$ expressed in terms of $\mathbf{n}' = \mathbf{p'}/|\mathbf{p'}|$. Performing the matrix product in (a), taking for $L$ a generic Lorentz transformation of the type (3.27), one gets the matrix $E$, in the form reported in Problem 5.9. In particular, the parameters $x$ and $y$ of the translation are given by

$$x = E^1_{3} = \frac{\gamma \delta}{1 + n_1^3} \left[ \beta_1 - (\beta \cdot \mathbf{n} \,) n_1' - k_2' \right]$$
$$y = E^2_{3} = \frac{\gamma \delta}{1 + n_3'} \left[ \beta_2 - (\beta \cdot \mathbf{n} \,) n_2' - k_1' \right]$$
and the rotation angle $\alpha$ by
\[
\cos \alpha = E_1^1 = 1 - \frac{1}{(1 + n_3)(1 + n'_3)} \frac{\gamma - 1}{\beta^2} k_3 k'_3,
\]
\[
\sin \alpha = E_2^1 = \frac{1}{(1 + n_3)(1 + n'_3)} \left[ \frac{1}{\gamma} \frac{\gamma - 1}{\beta^2} (k_3 + k'_3 + \gamma \beta_3 k_3) + \gamma n'_3 k_3 \right],
\]
where
\[ k = \beta \times n \quad , \quad k' = \beta \times n' = \frac{\delta'}{\delta} k. \]

5.11 Instead of performing the products in Eq. (5.35), we make use of Eqs. (5.21) and (5.13) that allow to rewrite $R_{1}^{-1}(\epsilon)$ in the form
\[
R_{1}^{-1}(\epsilon) = R_{1}^{-1}(\theta)L_{p}^{-1}L_{2}(-v')L_{3}(-v) = R_{1}^{-1}(\theta)R_{p'p},
\]
where we have
\[ p = (p^0, 0, 0, |p|) \quad \text{with} \quad (p^0 = \gamma m, |p| = p^0 v = \beta \gamma m) \]
and
\[ p' = L_{2}(-v')p = (\gamma p^0, 0, -\gamma p^0 v', |p|) \quad \text{i.e.} \quad p' = (\gamma' m, 0, \beta' \gamma' m, \beta' \gamma m). \]

In the above relations the usual definitions of the relativistic parameters $\gamma, \gamma'$ and $\beta, \beta'$ in terms of $v$ and $v'$ are adopted.

Then we evaluate $R_{p'p}$ according to Eq. (a) of Problem 5.4; in the present case $R_{p'p}$ reduces to a rotation about the $x^1$-axis through the angle $\delta$. One finds
\[
\cos \delta = \frac{\gamma + \gamma'}{1 + \gamma \gamma'} = \frac{p^0 + \gamma m}{p^0 + m},
\]
\[
\sin \delta = \frac{\beta \gamma \beta' \gamma'}{1 + \gamma \gamma'} = \frac{\gamma |p| v'}{p^0 + m}.
\]
It follows
\[
\tan \delta = \frac{\gamma |p| v'}{p^0 + \gamma m} = \frac{\beta' \gamma' \gamma'}{\gamma + \gamma'} = \frac{\beta \beta'}{\sqrt{1 - \beta^2} + \sqrt{1 - \beta'^2}},
\]
and from $R_{1}^{-1}(\delta)R_{1}(\theta)$, making use of Eq. (5.34), one gets
\[
\tan \epsilon = \tan(\theta - \delta) = \frac{v'}{v} \sqrt{1 - v^2}.
\]
We note that the same result for $\tan \delta$ can be obtained directly from the formulae (b) of Problem 5.4, that we re-write here as
Solutions

$$\frac{\cos \delta}{2} = \frac{1}{\cosh \frac{\psi'}{2}} \left[ \cosh \frac{\alpha}{2} \cosh \frac{\psi}{2} - \sinh \frac{\alpha}{2} \sinh \frac{\psi}{2} (e \cdot n) \right]$$

$$\frac{\sin \delta}{2} \nu = \frac{1}{\cosh \frac{\psi'}{2}} \sinh \frac{\alpha}{2} \sinh \frac{\psi}{2} (e \times n)$$

by identifying the Wigner rotation as

$$R_{\rho' p} = e^{\frac{i}{2} \psi' \sigma \cdot n'} e^{-\frac{1}{2} \alpha \sigma \cdot e} e^{-\frac{i}{2} \psi \sigma \cdot n} = L_{\rho'}^{-1} L_2(-v') L_3(-v')$$

where now $n = (1, 0, 0)$, $e = (0, 1, 0)$ and

$$\cosh \psi = \gamma, \quad \cosh \alpha = \gamma', \quad \cosh \psi' = \gamma \gamma'.$$

By applying the previous relations one easily obtains $\nu = (1, 0, 0)$ and the expression of $\tan \delta$ reported above.

Problems of Chapter 6

6.1 Knowing that the spin of the $\pi^0$ is zero, the quantities which enter the matrix element of the decay $\pi^0 \rightarrow \gamma \gamma$ are the relative momentum $k$ of the two photons in the c.m. system and their polarization (three-component) vectors $\epsilon_1$ and $\epsilon_2$. The matrix element is linear in the vectors $\epsilon_1$ and $\epsilon_2$, and it must be a scalar quantity under rotations.

One can build the two combinations, which are scalar and pseudoscalar under parity, respectively, and therefore correspond to even and odd parity for the $\pi^0$:

$$\epsilon_1 \cdot \epsilon_2, \quad k \cdot \epsilon_1 \times \epsilon_2.$$

In the first case, the polarization vectors of the two photon tend to be parallel, in the second case, perpendicular to each other. These correlations can be measured in terms of the planes of the electron-positron pairs in which the photons are converted, and then one can discriminate between the two cases. The parity of the $\pi^0$ was determined in this way to be odd (see R. Plano et al., Phys. Rev. Lett. 3, 525 (1959)).

The positronium is an $e^+e^-$ system in the state $^1S_0$ (spin zero and $S$-wave): since $\ell = 0$, the parity of this state is determined by the intrinsic parity. The situation is similar to the case of the $\pi^0$; a measurement would detect that the polarization vectors of the two photons produced in the annihilation tend to be perpendicular, corresponding to the case of odd relative parity for electron and positron.
6.2 Knowing that all the particles in the reaction \( K^- + He^4 \rightarrow \Lambda H^4 + \pi^0 \) have spin zero, angular momentum conservation implies that the orbital momentum is the same in the initial and final states. Then, the parity of the \( He^4 \) and \( \Lambda H^4 \) being the same, the occurrence of the above reaction is in itself a proof that the parity of \( K^- \) is the same of that of \( \pi^0 \), i.e. odd.

6.3 The system of particles 1 and 2 can be considered a single particle of spin \( \ell \) and intrinsic parity \( \eta_{12} = \eta_1 \eta_2 (-1)^\ell \). The parity of the total system is simply given by the product:

\[
\eta = \eta_3 \eta_{12} (-1)^L = \eta_1 \eta_2 \eta_3 (-1)^{\ell+L}.
\]

6.4 In the decay \( \rho^0 \rightarrow \pi^0 \pi^0 \), the two spinless pions would be in a state of orbital momentum \( \ell = 1 \), and therefore antisymmetric under the exchange of the two \( \pi^0 \). This state is not allowed by Bose statistics, which require that the two identical pions can be only in symmetrical state.

In a similar way, in the decay \( \rho^0 \rightarrow \gamma \gamma \) the final state must be symmetric under the interchange of the two photons. Then we have to build a matrix element in terms of the two polarization vectors \( \epsilon_1, \epsilon_2 \) and of the relative momentum \( k \) of the two \( \gamma \)'s in the c.m. frame. The matrix element is linear in \( \epsilon_1, \epsilon_2 \), symmetric under the two \( \gamma \) exchange (i.e. under \( \epsilon_1 \leftrightarrow \epsilon_2 \) and \( k \leftrightarrow -k \)) and it transforms as a vector (since the spin of the \( \rho^0 \) is 1). Out of the three independent combinations of \( \epsilon_1, \epsilon_2, k \) which transform as vectors, i.e.

\[
\epsilon_1 \times \epsilon_2, \quad (\epsilon_1 \cdot \epsilon_2) k \quad \text{and} \quad (k \cdot \epsilon_2) \epsilon_1 - (k \cdot \epsilon_1) \epsilon_2,
\]

only the last one is symmetric; however it is excluded by the transversality condition \( k \cdot \epsilon_1 = k \cdot \epsilon_2 = 0 \) required by electromagnetic gauge invariance. Therefore the decay is forbidden.

6.5 We notice that the electric dipole moment of a particle is proportional to its spin. For a spin \( \frac{1}{2} \) particle (the electron \( e \), for instance) one has \( \mu_e = \mu_e \sigma \) which is an axial vector (it does not change under parity, see Eq. (6.2)). The Hamiltonian of the particle in an external electric field \( E \) contains the interaction term \( H_I = -\mu_e \sigma \cdot E \), which is pseudoscalar, since \( E \) is a polar vector. The existence of an electric dipole moment would then indicate violation of parity.

6.6 From Problem 6.3 and the assignment \( J^P = 0^- \) for the \( \pi^- \)-meson, we can determine the parity of the \( J = 0 \) \( \pi^+ \pi^+ \pi^- \) state; in fact, if \( \ell \) is the relative momentum of the two identical \( \pi^+ \), \( \ell \) must be even and \( J = \ell + L \) implies
that also $L$ is even. The parity is then given by $-(-1)^{\ell+L} = -1$. On the other hand, the system $\pi^+\pi^0$ must have parity $+1$, being $J = \ell = 0$. Then the $K^+$ would decay into two systems of opposite parity. The occurrence of these decays, since the $K^+$ has spin zero and a definite parity ($JP = 0^-$), shows that parity is violated.

6.7 In general, in order to detect parity violation, one has to look for some quantity which is odd under parity (e.g., pseudoscalar); a non-vanishing expectation value of this quantity indicates that parity is violated. Suppose that the $\Lambda^0$ hyperon has been produced in the reaction $\pi^- + p = \Lambda^0 + K^0$; one can determine the normal versor to the production plane

$$n = \frac{p_\pi \times p_\Lambda}{|p_\pi \times p_\Lambda|},$$

where $p_\pi$ and $p_\Lambda$ are the momenta of the incident $\pi^-$ and of the outgoing $\Lambda^0$, respectively. The quantity

$$\frac{n \cdot k}{|k|} = \cos \theta,$$

where $k$ is the momentum of the $\pi^-$ in the decay $\Lambda^0 \to \pi^- + p$, is clearly pseudoscalar; it was found that the average value of $\cos \theta$ is different from zero (which means asymmetry with respect to the production plane), and therefore that parity is violated in the decay.

The appearance of the $\cos \theta$ can be understood by noting that, if parity is not conserved, the $\Lambda^0$ ($JP = \frac{1}{2}^+$) can decay both in a $JP = \frac{1}{2}^- (S$-wave) and in a $JP = \frac{1}{2}^+ (P$-wave) $\pi^-p$ state; the interference between the $S$- and $P$-waves gives rise to the $\cos \theta$ term.

6.8 The Maxwell equations, in natural units ($\hbar = c = 1$), are:

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0,$$

$$\nabla \times B = j + \frac{\partial E}{\partial t}, \quad \nabla \times E = -\frac{\partial B}{\partial t}.$$

The last equation shows that the vectors $E$ and $B$ behave in opposite ways under time reversal. From the usual definitions of the charge density $\rho$ and the current density $j$, one realizes that $\rho$ is unchanged under time reversal, while the current density changes direction: $j \to -j$. Therefore, assuming that the Maxwell equations are invariant under time reversal, one has $E \to E$ and $B \to -B$.

6.9 As discussed in Problem 6.5, the interaction Hamiltonian is given by $H_I = -\mu_e \sigma \cdot E$. Since under time reversal $\sigma \to -\sigma$, $E \to E$, this interaction
term changes its sign under time reversal. The presence of an electric dipole moment would then indicate, besides parity violation, non-invariance under time reversal.

Problems of Chapter 7

7.1 The IR’s $D^{(1,0)}$ and $D^{(0,1)}$ of $\mathcal{L}_+^\dagger$ are irreducible also with respect to the rotation group $SO(3)$: within this subgroup they are both equivalent to $D^{(1)}$, so that they describe a vector particle. In a covariant description, a three-vector is then replaced by a selfdual (or an anti-selfdual) antisymmetric tensor (see also Problems 3.10, 3.11).

7.2 By differentiation of the antisymmetric tensor $f^{\mu\nu}$ one gets

$$\partial_\mu f^{\mu\nu} = \partial_\mu \partial^\mu \Phi^\nu - \partial_\nu \partial^\nu \Phi^\mu = \square \Phi^\nu - \partial^\nu \partial_\mu \Phi^\mu = -m^2 \Phi^\nu ,$$

which are the Proca equations. Conversely, by differentiation of the Proca equations

$$\partial_\nu (\partial_\mu f^{\mu\nu} + m^2 \Phi^\nu) = \partial_\nu \partial_\mu f^{\mu\nu} + m^2 \partial_\nu \Phi^\nu = 0$$

the subsidiary condition $\partial_\nu \Phi^\nu = 0$ follows, since $f^{\mu\nu}$ is antisymmetric. If the subsidiary condition is now used in Proca equations, one gets

$$\partial_\mu (\partial^\nu \Phi^\nu - \partial^\nu \Phi^\mu) + m^2 \Phi^\nu = \square \Phi^\nu - \partial^\nu \partial_\mu \Phi^\mu + m^2 \Phi^\nu = 0 ,$$

which is the Klein-Gordon equation (7.17).

We observe that the above equivalence no longer holds if $m = 0$, since the condition $m \neq 0$ is used in the derivation.

7.3 Under a Lorentz transformation $x'^\mu = A_\mu^\nu x^\nu$, the Dirac equation (7.79)

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad (a)$$

will be transformed into

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0 , \quad (b)$$

where $\psi'(x') = S(A)\psi(x)$ (see Eq. (7.52)), and

$$\partial_\mu = A_\mu^\nu \partial'_\nu \quad \left( \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ , \ \partial'_\nu \equiv \frac{\partial}{\partial x'^\nu} \right) .$$
The matrices $\gamma'^\mu$ in (b) are equivalent to the matrices $\gamma^\mu$ in (a) up to a unitary transformation, so that we can simply replace $\gamma'$ by $\gamma$ (see J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, p.18). Making use of the above relations, Eq. (a) becomes

$$iA'^\mu S\gamma^\mu S^{-1}\partial'_\nu\psi'(x') - m\psi'(x') = 0,$$

which coincides with Eq. (b), if one makes use of

$$A'^\mu S\gamma^\mu S^{-1} = \gamma^\nu \quad \text{i.e.} \quad S^{-1}\gamma^\mu S = A'^\mu \gamma^\mu.$$

### 7.4
Let us start from the infinitesimal Lorentz transformations written as (compare with Problem 3.7)

$$A^\rho_\sigma = g^\rho_\sigma + \delta\omega^\rho_\sigma \quad \text{with} \quad \delta\omega^\rho_\sigma = -\delta\omega^\sigma_\rho,$$

and assume for $S(A)$ the infinitesimal form

$$S = I - \frac{i}{4} \sigma_{\mu\nu} \delta\omega^{\mu\nu},$$

where $\sigma_{\mu\nu}$ are six $4 \times 4$ antisymmetric matrices. Inserting both the above infinitesimal transformations into Eq. (7.76), at the first order one finds

$$\frac{i}{4} \delta\omega^{\mu\nu}(\sigma_{\mu\nu}\gamma^\alpha - \gamma^\alpha\sigma_{\mu\nu}) = \delta\omega^\alpha_\beta\gamma^\beta,$$

that, taking into account the antisymmetry of the $\delta\omega^{\mu\nu}$, can be rewritten in the form

$$[\gamma^\alpha, \sigma_{\mu\nu}] = 2i(g^\alpha_\mu \gamma^\nu - g^\alpha_\nu \gamma^\mu),$$

satisfied by

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu].$$

### 7.5
Under a Lorentz transformation (neglecting for simplicity the argument $x$) the spinor $\psi$ transforms as

$$\psi \rightarrow \psi' = S\psi,$$

so that for $\overline{\psi}'$ one gets

$$\overline{\psi}' = \psi^\dagger \gamma^0 S^{-1} = \overline{\psi} S^{-1}.$$

Then, for the scalar quantity, it follows immediately:

$$\overline{\psi}'\psi' = \overline{\psi} S^{-1} S\psi = \overline{\psi}\psi \quad (a).$$
For the pseudoscalar quantity, one obtains:
\[
\bar{\psi}' \gamma_5 \psi' = \bar{\psi} S^{-1} \gamma_5 S \psi ,
\]
and, since
\[
S^{-1} \gamma_5 S = iS^{-1} \gamma^0 \gamma^1 \gamma^2 \gamma^3 S = iA^0_\mu A^1_\nu A^2_\sigma A^3_\tau \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\tau = \epsilon^{\mu \nu \sigma \tau} A^0_\mu A^1_\nu A^2_\sigma A^3_\tau \gamma^0 \gamma^1 \gamma^2 \gamma^3 = (\det A) \gamma_5 ,
\]
it follows
\[
\bar{\psi}' \gamma_5 \psi' = (\det A) \bar{\psi} \gamma_5 \psi . \tag{b}
\]
Space inversion corresponds to \( \det A = -1 \), so that \( \bar{\psi} \gamma_5 \psi \) behaves as a pseudoscalar under \( L^\dagger \).

A quantity that transforms as a four-vector satisfies the relation
\[
\bar{\psi}' \gamma^\mu \psi' = \bar{\psi} S^{-1} \gamma^\mu S \psi = \Lambda^\mu_\nu \bar{\psi} \gamma^\nu \psi . \tag{c}
\]
Similarly, one finds that \( \bar{\psi} \gamma_5 \gamma^\mu \psi \) transforms as a pseudo-vector and
\[
\bar{\psi} [\gamma_\mu, \gamma_\nu] \psi \tag{d}
\]
as an antisymmetric tensor.

7.6 From Eq. (7.85) one gets
\[
\bar{\psi} \xrightarrow{t} \bar{\psi}' = \bar{\psi} \gamma_5 \gamma^0 .
\]
Then the transformation properties under time reversal of the quantities (7.87) can be easily derived as follows:
\[
\begin{align*}
\bar{\psi} \psi & \quad \longrightarrow \quad -\bar{\psi} \gamma_5 \gamma^0 \gamma^0 \gamma_5 \psi = -\bar{\psi} \psi , \quad (a) \\
\bar{\psi} \gamma_5 \psi & \quad \longrightarrow \quad -\bar{\psi} \gamma_5 \gamma^0 \gamma_5 \gamma^0 \gamma_5 \psi = \bar{\psi} \gamma_5 \psi , \quad (b) \\
\bar{\psi} \gamma^\mu \psi & \quad \longrightarrow \quad -\bar{\psi} \gamma_5 \gamma^0 \gamma^\mu \gamma^0 \gamma_5 \psi = \begin{cases} 
\bar{\psi} \gamma^0 \psi & (\mu = 0) , \\
-\bar{\psi} \gamma^k \psi & (\mu = k = 1, 2, 3) , 
\end{cases} \quad (c) \\
\bar{\psi} \gamma_5 \gamma^\mu \psi & \quad \longrightarrow \quad -\bar{\psi} \gamma_5 \gamma^0 \gamma_5 \gamma^\mu \gamma^0 \gamma_5 \psi = \begin{cases} 
-\bar{\psi} \gamma_5 \gamma^0 \psi & (\mu = 0) , \\
\bar{\psi} \gamma_5 \gamma^k \psi & (\mu = k = 1, 2, 3) , 
\end{cases} \quad (d) \\
\bar{\psi} \gamma^\mu \gamma^\nu \psi & \quad \longrightarrow \quad -\bar{\psi} \gamma^\mu \gamma^\nu \psi . \quad (e)
\end{align*}
\]
Problems of Chapter 8

8.1 The state $|pd\rangle$ is a pure $|I = \frac{1}{2}, I_3 = \frac{1}{2}\rangle$ state. Since the pion is an isotriplet and $^3He$ and $^3H$ form an isodoublet with $I_3 = \frac{1}{2}$ and $I_3 = -\frac{1}{2}$, respectively, the final states can have either $I = \frac{1}{2}$ or $I = \frac{3}{2}$. Making use of the relevant Clebsh-Gordan coefficients (see Table A.3), one gets

\[ |\pi^+ \, 3H\rangle = |1, 1\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle = \sqrt{\frac{1}{3}} \, |\frac{3}{2}, \frac{1}{2}\rangle + \sqrt{\frac{2}{3}} \, |\frac{1}{2}, \frac{1}{2}\rangle, \]
\[ |\pi^0 \, 3He\rangle = |1, 0\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle = \sqrt{\frac{2}{3}} \, |\frac{3}{2}, \frac{1}{2}\rangle - \sqrt{\frac{1}{3}} \, |\frac{1}{2}, \frac{1}{2}\rangle. \]

From isospin invariance, one obtains for the S-matrix elements:

\[ <\pi^+ \, 3H | S | pd\rangle = \sqrt{\frac{2}{3}} \, A_{\frac{1}{2}}, \]
\[ <\pi^0 \, 3He | S | pd\rangle = -\sqrt{\frac{1}{3}} \, A_{\frac{1}{2}}, \]

where $A_{\frac{1}{2}}$ is the amplitude for pure $I = \frac{1}{2}$ state. Then the ratio of the relative cross-sections is:

\[ R = \frac{\sigma(pd \rightarrow \pi^+ \, 3H)}{\sigma(pd \rightarrow \pi^0 \, 3He)} = 2. \]

8.2 Since the pion is an isotriplet, we have three independent isospin amplitudes: $A_0, A_1, A_2$, which refer to $I = 0, 1, 2$, respectively. The isospin analysis of the various states, making use of the Clebsh-Gordan coefficients (see Section A.2 of Appendix A), gives

\[ |\pi^+ \pi^+\rangle = |2, 2\rangle, \]
\[ |\pi^+ \pi^-\rangle = \sqrt{\frac{1}{6}} \, |2, 0\rangle + \sqrt{\frac{1}{2}} \, |1, 0\rangle + \sqrt{\frac{1}{3}} \, |0, 0\rangle, \]
\[ |\pi^0 \pi^0\rangle = \sqrt{\frac{2}{3}} \, |2, 0\rangle - \sqrt{\frac{1}{3}} \, |0, 0\rangle. \]

Then one obtains

\[ <\pi^+ \pi^+ | S | \pi^+ \pi^+\rangle = A_2, \]
\[ <\pi^+ \pi^- | S | \pi^+ \pi^-\rangle = \frac{1}{6} A_2 + \frac{1}{2} A_1 + \frac{1}{3} A_0, \]
\[ <\pi^0 \pi^0 | S | \pi^+ \pi^-\rangle = \frac{1}{3} A_2 - \frac{1}{3} A_0, \]
\[ <\pi^0 \pi^0 | S | \pi^0 \pi^0\rangle = \frac{2}{3} A_2 + \frac{1}{3} A_0. \]

8.3 By identifying the basis of the IR $D^{(1)}$ with
\[ x^1 = \xi_1^1 \xi_1^1, \]
\[ x^2 = \frac{1}{\sqrt{2}} (\xi_1^1 \xi_2^2 + \xi_2^2 \xi_1^1), \]
\[ x^3 = \xi_2^2 \xi_2^2, \]

and taking into account the transformation properties of the contravariant vector \( \xi \) under \( I_i = \frac{1}{2} \sigma_i \), i.e.

\[ I_1 \left( \frac{\xi_1^1}{\xi_2^2} \right) = \frac{1}{2} \left( \frac{\xi_2^2}{\xi_1^1} \right), \quad I_2 \left( \frac{\xi_1^1}{\xi_2^2} \right) = i \frac{1}{2} \left( -\frac{\xi_2^2}{\xi_1^1} \right), \quad I_3 \left( \frac{\xi_1^1}{\xi_2^2} \right) = \frac{1}{2} \left( \frac{\xi_1^1}{-\xi_2^2} \right), \]

one can apply them independently to the factors of the products which appear in \( x^1, x^2, x^3 \). For example, one obtains

\[ I_1 x^1 = I_1 (\xi_1^1 \xi_1^1) = (I_1 \xi_1^1) \xi_1^1 + \xi_1^1 (I_1 \xi_1^1) = \frac{1}{2} (\xi_2^2 \xi_1^1 + \xi_1^1 \xi_2^2) = \frac{1}{\sqrt{2}} x^2; \]

similarly

\[ I_1 x^2 = \frac{1}{\sqrt{2}} (x^1 + x^3), \]
\[ I_1 x^3 = \frac{1}{\sqrt{2}} x^2, \]

and analogously for \( I_2 \) and \( I_3 \). From these relations the following matrix structure of the generators in the three-dimensional IR can be easily derived:

\[ I_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_2 = i \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

**8.4** From Eqs. (8.54) it is easy to find the matrix \( S \) which transforms the vector \( \pi = (\pi_1, \pi_2, \pi_3) \) into the vector of components \( \pi^+, \pi^0, \pi^- \):

\[ S = \sqrt{\frac{1}{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}. \]

By transforming, by a similarity transformation, the usual form of the \( SU(2) \) generators

\[ (I_i)_{jk} = -i \epsilon_{ijk}, \]

one finds the expressions for \( I_1, I_2, I_3 \) given in the solution of the previous problem, so that \( \pi^+, \pi^0, \pi^- \) are eigenstates of \( I_3 \) with eigenvalues \( +1, 0, -1 \), respectively. By introducing the raising and lowering operators \( I_{\pm} = I_1 \pm iI_2 \), given explicitly by

\[ I_+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_- = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]
it is easy to verify:
\[ I_{\pm} \pi^0 = \sqrt{2} \pi^\pm. \]

8.5 Following the recipe given in Appendix C, we obtain the following decompositions, besides the $8 \otimes 8$, already reported in Section C.3:

\[
\begin{array}{ccc}
8 & \otimes & 10 \\
\hline
35 & \oplus & 27 & \oplus & 10 & \oplus & 8
\end{array}
\]

\[
\begin{array}{ccc}
10 & \otimes & 10 \\
\hline
64 & \oplus & 27 & \oplus & 10 & \oplus & 1
\end{array}
\]

Let us now examine the $SU(2)_I \otimes U(1)_Y$ content of some of the above IR’s. We start with the IR’s 8 and 10, obtained from the decomposition of the product $3 \otimes 3 \otimes 3$. It is convenient to make use of the strangeness $S$ since the $U(1)_Y$ counts the non-strange $S = 0$ and the strange ($S = -1$) quarks. We obtain:

\[
\begin{array}{c}
8 \\
\hline
2, S = 0 & \oplus & 3, -1 & \oplus & 1, -1 & \oplus & 1, -2
\end{array}
\]

\[
\begin{array}{c}
10 \\
\hline
4, S = 0 & \oplus & 3, -1 & \oplus & 2, -2 & \oplus & 1, -3
\end{array}
\]

Since each quark has $B = \frac{1}{3}$, in the present case we have $Y = S + B = S + 1$, and the above relations, in terms of $Y$, read

\[
\begin{array}{c}
8 \\
\hline
(2, Y = 1) & \oplus & (3, 0) & \oplus & (1, 0) & \oplus & (2, -1)
\end{array}
\]

\[
\begin{array}{c}
10 \\
\hline
(4, Y = 1) & \oplus & (3, 0) & \oplus & (2, -1) & \oplus & (1, -2)
\end{array}
\]

Next we consider the representation 27:
27 = (3, S = 0) ⊕ (2, -1) ⊕ (4, -1) ⊕ (1, -2) ⊕ (3, -2) ⊕ (5, -2) ⊕ (2, -3) ⊕ (4, -3) ⊕ (3, -4) .

Taking into account that the initial tableau contains 6 boxes, we can go from $S$ to $Y$ by means of $Y = S + B = S + 2$. Then we get:

$$27 = (3, Y = 2) ⊕ (2, 1) ⊕ (4, 1) ⊕ (1, 0) ⊕ (3, 0) ⊕ (5, 0) ⊕ (2, -1) ⊕ (4, -1) ⊕ (3, -2) .$$

Similarly:

$$35 = (5, Y = 2) ⊕ (6, 1) ⊕ (4, 1) ⊕ (5, 0) ⊕ (3, 0) ⊕ (4, -1) ⊕ (2, -1) ⊕ (3, -2) ⊕ (1, -2) ⊕ (2, -3)$$

and

$$64 = 27 ⊕ (4, 3) ⊕ (5, 2) ⊕ (6, 1) ⊕ (7, 0) ⊕ (6, -1) ⊕ (5, -2) ⊕ (4, -3) .$$

8.6 According to their definition, it is easy to find the following commutation relations among the shift operators:

$$[I_3, I_\pm] = ±I_\pm , \quad [Y, I_\pm] = 0 ,$$

$$[I_3, U_\pm] = ±\frac{1}{2}U_\pm , \quad [Y, U_\pm] = ±U_\pm ,$$

$$[I_3, V_\pm] = ±\frac{1}{2}V_\pm , \quad [Y, V_\pm] = ±V_\pm ,$$

and

$$[I_+, I_-] = 2I_3 , \quad [U_+, U_-] = 2U_3 = \frac{3}{2}Y - I_3 , \quad [V_+, V_-] = 2V_3 = \frac{3}{2}Y + I_3 ,$$

$$[I_\pm, U_\pm] = ±V_\pm , \quad [I_\pm, V_\pm] = ±U_\pm , \quad [U_\pm, V_\pm] = ±I_\pm ,$$

$$[I_+, U_-] = [I_+, V_-] = [U_+, V_+] = 0 .$$

It follows that $I_\pm, U_\pm$ and $V_\pm$ act as raising and lowering operators, specifically:

$I_\pm$ connects states with $\Delta I_3 = ±1, \Delta Y = 0$ ;

$U_\pm$ connects states with $\Delta I_3 = ±\frac{1}{2}, \Delta Y = ±1 (\Delta Q = 0)$ ;

$V_\pm$ connects states with $\Delta I_3 = ±\frac{1}{2}, \Delta Y = ±1 (\Delta S = \Delta Q)$ .

This action on the states in the $(I_3, Y)$ plane has been shown in Fig. 8.8. For each IR, all states can be generated, starting from whatever of them, by a repeated application of the shift operators.

In order to obtain the matrix elements between two given states, it is usual to fix the relative phase according to
\[ I_{\pm}|I, I_3, Y> = [I \mp I_3](I \pm I_3 + 1)]^{\frac{1}{2}}|I, I_3 \pm 1, Y>, \quad (a) \]
\[ V_{\pm}|V, V_3, Q> = \sum_{I'} a^{\pm}(I, I', I_3, Y)|I', I_3 \pm \frac{1}{2}, Y \pm 1>, \quad (b) \]
by requiring the coefficients \( a^{\pm} \) always real non-negative numbers\(^3\). Let us note that at this point the action of \( U^{\pm} \) is uniquely fixed by the commutation relations. The following convention is also adopted, which connects the eigenstates of two conjugate IR’s \( D(p_1, p_2), D(p_2, p_1) \):
\[ |(p_1, p_2); I, I_3, Y>^* = (-1)^{I_3+\frac{1}{2}}|p_1, p_2); I, -I_3, -Y>. \]

Let us now consider the 8 IR, whose isospin eigenstates are represented in Fig. 8.11 in terms of the \( \frac{1}{2}^+ \) baryon octet. Making use of the previous convention, it is easy to find the matrix elements of the shift operators reported in the figure.

The transition operated by \( I_{\pm} \) are expressed by:
\[ I_\mp p = n, \quad I_\mp A^0 = 0, \quad I_\mp \Sigma^0 = \sqrt{2} \Sigma^- . \]
which can be easily derived from Eq. (a). The action of the \( V_{\pm} \) operators can be obtained making use of the commutation relations and of Eq. (b), e.g.
\[ V_\mp p = a_0^- A^0 + a_1^- \Sigma^0 \]
with \( a_0^-, a_1^- \) both positive and normalized according to
\[ (a_0^-)^2 + (a_1^-)^2 = 1 \]
then from
\[ [I_-, V_-] = 0, \]
being
\[ I_- V_- p = a_1^- \sqrt{2} \Sigma^-, \quad V_- I_- p = \Sigma^-, \]
one finds
\[ V_- p = \sqrt{\frac{3}{2}} A^0 + \frac{1}{\sqrt{2}} \Sigma^0 . \]
Let us now consider \( U_{\pm} \) and the \( \Sigma_u^0, A_u^0 \) combinations of the \( \Sigma^0, A^0 \) states. Assuming, in analogy with the isospin eigenstates,
\[ \sqrt{2} \Sigma_u^0 = U_- n, \]
from
\[ U_- = [V_-, I_+] \]
one finds
\[ U_- p = V_- p - I_+ \Sigma^- = \left[ \sqrt{\frac{3}{2}} A^0 + \frac{1}{\sqrt{2}} \Sigma^0 \right] - \sqrt{2} \Sigma^0 = \sqrt{\frac{3}{2}} A^0 - \frac{1}{\sqrt{2}} \Sigma^0 , \]
\(^3\) J.J. de Swart, Rev. of Mod. Phys. 35, 916 (1963).
so that, accordingly with Eq. (8.124),

\[ \Sigma_u^0 = \sqrt{\frac{3}{2}} \Lambda^0 - \frac{1}{\sqrt{2}} \Sigma^0 , \]

\[ \Lambda^0_u \] being the orthogonal combination.

In a similar way one can derive the matrix elements of the shift operators for the decuplet.

8.7 We recall that the photon is U-spin singlet (\( U = 0 \)). On the other hand, \( \pi^0 \) and \( \eta^0 \) are superpositions of \( U = 0 \) (\( \eta^0_U \)) and \( U = 1, U_3 = 0 \) (\( \pi^0_U \)) states; in analogy with Eq. (8.124) one has

\[ \pi^0_U = -\frac{1}{2} \pi^0 + \frac{\sqrt{3}}{2} \eta^0 , \]
\[ \eta^0_U = \frac{\sqrt{3}}{2} \pi^0 + \frac{1}{2} \eta^0 , \]
i.e.
\[ \pi^0 = -\frac{1}{2} \pi^0_U + \frac{\sqrt{3}}{2} \eta^0_U , \]
\[ \eta^0 = \frac{\sqrt{3}}{2} \pi^0_U + \frac{1}{2} \eta^0_U . \]

Only the \( \eta^0_U \) term can contribute to the decays \( \pi^0 \to 2\gamma \) and \( \eta^0 \to 2\gamma \); therefore, the corresponding amplitudes satisfy the relation

\[ A(\pi^0 \to 2\gamma) = \sqrt{3} A(\eta^0 \to 2\gamma) \]

Taking into account the phase space corrections, the ratio of the decay widths is given by

\[ \frac{\Gamma(\pi^0 \to 2\gamma)}{\Gamma(\eta^0 \to 2\gamma)} = \frac{1}{3} \left( \frac{m_\eta}{m_\pi} \right)^3 . \]

8.8 We recall Eq. (8.154):

\[ \omega = \cos \theta \omega_1 + \sin \theta \omega_8 , \]
\[ \phi = -\sin \theta \omega_1 + \cos \theta \omega_8 , \]
i.e.
\[ \begin{pmatrix} \omega \\ \phi \end{pmatrix} = R(\theta) \begin{pmatrix} \omega_1 \\ \omega_8 \end{pmatrix} \]
with
\[ R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} . \]
The two mass matrices

\[ M_{\omega_1,\omega_8}^2 = \begin{pmatrix} m_{1^2} & m_{18^2} \\ m_{18^2} & m_8^2 \end{pmatrix}, \quad M_{\omega,\phi}^2 = \begin{pmatrix} m_\omega^2 & 0 \\ 0 & m_\phi^2 \end{pmatrix} \]

are connected by

\[ R(\theta)M_{\omega_1,\omega_8}^2 R^{-1}(\theta) = M_{\omega,\phi}^2 \]

from which Eqs. (8.156) and (8.157) follow.

The two reactions \( \omega \rightarrow e^+e^- \), \( \phi \rightarrow e^+e^- \) occur through an intermediate photon, and \( \omega_1 \not\rightarrow \gamma \), \( \omega_8 \rightarrow \gamma \), since \( \gamma \) transforms as the \( U=Q=0 \) component of an octet. Then, neglecting the phase space correction, one obtains for the ratio of the decay amplitudes:

\[ \frac{A(\omega \rightarrow e^+e^-)}{A(\phi \rightarrow e^+e^-)} = \tan \theta. \]

8.9 Taking into account Eq. (8.124) one gets, from U-spin invariance:

\[ \mu_{\Sigma_0} = \langle \Sigma_0^0 | \mu | \Sigma_0^0 \rangle = -\frac{1}{2} \Sigma_0^0 + \frac{\sqrt{3}}{2} A_0^0 | \mu | - \frac{1}{2} \Sigma_0^0 + \frac{\sqrt{3}}{2} A_0^0 = \]

\[ = \frac{1}{4} \mu_\Sigma^0 + 3 \mu_{\lambda_0} - \frac{\sqrt{3}}{2} \mu_{A_0^0 \Sigma_0^0} , \]

\[ < \Sigma_0^0 | \mu | A_0^0 > = -\frac{1}{2} \Sigma_0^0 + \frac{\sqrt{3}}{2} A_0^0 | \mu | \frac{\sqrt{3}}{2} \Sigma_0^0 + \frac{1}{2} A_0^0 = \]

\[ = \frac{\sqrt{3}}{4} \mu_\Sigma_0 + \frac{\sqrt{3}}{4} \mu_{\lambda_0} + \frac{3}{2} \mu_{A_0^0 \Sigma_0^0} . \]

The required relations follow immediately from the above equations.

8.10 We can write the Casimir operator in terms of the shift operators (see Problem 8.6) and of \( I_3, Y \), in the form

\[ F^2 = F_i F_i = \frac{1}{2} \{ I_+ , I_- \} + \frac{1}{2} \{ U_+ , U_- \} + \frac{1}{2} \{ V_+ , V_- \} + I_3^2 + \frac{3}{4} Y^2 . \]

Since \( F^2 \) is an invariant operator, its eigenvalue in a given IR can be obtained by applying it to a generic state. It is convenient to take into account the so-called maximum state, \( \psi_{\text{max}} \), defined as the state with the maximum eigenvalue of \( I_3 \). It is easy to verify that for each IR there is only one such a state, with a specific eigenvalue of \( Y \). Moreover, because of its properties

\[ I_+ \psi = V_+ \psi = U_- \psi = 0 . \]

Looking at the eight dimensional IR, \( \psi_{\text{max}} \) corresponds to the eigenvalues 1 and 0 for \( I_3 \) and \( Y \), respectively. Since
\[ I_+ I_- \psi_{\text{max}} = 2 \psi_{\text{max}} , \quad U_+ U_- \psi_{\text{max}} = \psi_{\text{max}} , \quad V_+ V_- \psi_{\text{max}} = \psi_{\text{max}} , \]
\[ I_3^2 \psi_{\text{max}} = \psi_{\text{max}} , \quad Y^2 \psi_{\text{max}} = 0 , \]
the eigenvalue of \( F^2 \) in the adjoint IR is 3. Making use of the definition of \( F^2 \) and of its eigenvalue, Eq. (8.145) is then checked, once Eqs. (8.140) and (8.94) are taken into account.

More generally, it is possible to obtain the expression of the eigenvalues of \( F^2 \) in terms of the two integers \( p_1 \) and \( p_2 \) which characterize a given IR. Since \( p_1 \) and \( p_2 \) represent the number of times that the representations 3 and \( \bar{3} \) are present in the direct product from which \( D(p_1, p_2) \) is obtained, then the maximum eigenvalue of \( I_3 \) is given by
\[ I_3 \psi_{\text{max}} = \frac{1}{2} (p_1 + p_2) \psi_{\text{max}} , \]
while the eigenvalue of \( Y \) is
\[ Y \psi_{\text{max}} = \frac{1}{3} (p_1 - p_2) \psi_{\text{max}} . \]

From the commutation relations of the shifting operators, one obtains:
\[ \{ I_+ , I_- \} \psi_{\text{max}} = I_+ I_- \psi_{\text{max}} = [I_+ , I_-] \psi_{\text{max}} = 2I_3 \psi_{\text{max}} = (p_1 + p_2) \psi_{\text{max}} , \]
\[ \{ U_+ , U_- \} \psi_{\text{max}} = U_+ U_- \psi_{\text{max}} = [U_+ , U_-] \psi_{\text{max}} = -2U_3 \psi_{\text{max}} = p_2 \psi_{\text{max}} , \]
\[ \{ V_+ , V_- \} \psi_{\text{max}} = V_+ V_- \psi_{\text{max}} = [V_+ , V_-] \psi_{\text{max}} = 2V_3 \psi_{\text{max}} = p_1 \psi_{\text{max}} . \]

By inserting these relations in the expression of \( F^2 \), one finds the general expression for its eigenvalues:
\[ \frac{1}{3} (p_1^2 + p_2^2 + p_1 p_2) + p_1 + p_2 . \]

8.11 By inserting the \( \lambda \) matrices in the Jacobi identity
\[ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \]
and making use of the commutation relations (8.86), one finds:
\[ f_{j \ell k} [\lambda_i , \lambda_k] + f_{\ell ik} [\lambda_j , \lambda_k] + f_{ijk} [\lambda_\ell , \lambda_k] = 0 . \]

By multiplying by \( \lambda_m \) and taking the traces according to the relation (8.89), one obtains the identity
\[ f_{j \ell k} f_{ikm} + f_{\ell ik} f_{jkm} + f_{ijk} f_{\ell km} = 0 , \quad (a) \]
and, taking into account the definition (8.137), one gets the commutation relations (8.92)
\[ [F_i , F_j] = if_{ijk} F_k . \]
In analogous way, making use of the identity

$$[A, \{B, C\}] = \{[A, B], C\} + \{[A, C], B\} ,$$

and of the relations (8.89) and (8.90), one finds

$$d_{jk\ell} f_{i\ell m} = f_{ij\ell} d_{\ell km} + f_{ik\ell} d_{\ell jm} , \quad (b)$$

which, taking into account the definitions (8.137) and (8.138), corresponds to Eq. (8.139):

$$[F_i, D_j] = i f_{ijk} D_k .$$

By multiplying the previous relation by $-i f_{ij\ell}$ and summing over $(i, j)$ one finds

$$-i f_{ij\ell} [F_i, D_j] = F^2 D_\ell , \quad (c)$$

where $F^2 = F_i F_i$ is the quadratic Casimir operator in the adjoint $D(1, 1)$ representation; it is equal to 3 times the unit matrix (make use of the expression obtained for $F^2$ in Problem 8.10 with $p_1 = p_2 = 1$). On the other hand, the l.h.s., making use of the relation (b) and of the symmetry properties of the $f$ and $d$ coefficients, can be written in the form:

$$-i f_{ij\ell} [F_i, D_j] = 2 F^2 D_k - d_{\ell mn} \{F_m, F_n\} = 2 F^2 D_\ell - 2 d_{\ell mn} F_m F_n . \quad (d)$$

By comparison of (c) and (d) one gets

$$F^2 D_\ell = 2 d_{\ell mn} F_m F_n ,$$

which coincides with Eq. (8.140).

8.12 In $SU(6)$ the diquark states $d$ correspond to

$$6 \otimes 6 = 21 \oplus 15 ,$$

so that the S-wave symmetric states belong to the IR 21. Its content in terms of the subgroup $SU(3) \otimes SU(2)_S$ is given by

$$21 = (6, 3) \oplus (\overline{3}, 1) .$$

The baryon states will be described as bound $dq$ systems:
21 \otimes 6 = 56 \oplus 70 .

Including a relative \( d - q \) orbital momentum \( L \), one can easily verify that the symmetric states belong to the multiplets \((56, L_{\text{even}}^+)\) and \((70, L_{\text{odd}}^-)\), which are the only ones definitively established.

On the other hand, if one build the \( \bar{d} \bar{d} \) mesons, one obtains lots of exotic states. In fact, according to

\[
21 \otimes 21 = 1 \oplus 35 \oplus 405 ,
\]

one gets, besides the IR 1 and 35, the \( SU(6) \) multiplet 405 whose \( SU(3) \otimes SU(2)_S \) content is:

\[
405 = (1 + 8 + 27, 1) \oplus (8 + 8 + 10 + \bar{10} + 27, 3) \oplus (1 + 8 + 27, 5) .
\]

8.13 Taking into account the spin, the four quarks \((u, d, s, c)\) belong to the IR 8 of \( SU(8) \), which corresponds, in term of the subgroup \( SU(4) \otimes SU(2)_S \), to

\[
8 = (4, 2) .
\]

The meson states are assigned to the representation

\[
8 \otimes \bar{8} = 1 \oplus 83 ,
\]

and the 83-multiplet has the following content in terms of the above subgroups:

\[
83 = (15, 1) \oplus (15 + 1, 3) .
\]

We see that one can fit into the same multiplet both the 15 pseudoscalar mesons \((K, \bar{K}, \pi, \eta, F, D, F^\ast, D^\ast)\) and the 16 vector mesons \((K^\ast, \bar{K}^\ast, \rho, \omega, \phi, F^*, D^*, F^{\ast*}, D^{\ast*})\), where \( F^* \) and \( D^* \) are the vector counterparts of the scalar mesons \( F \) and \( D \).

The baryon states are classified according to the IR’s:

\[
8 \otimes 8 \otimes 8 = 120 \oplus 168 \oplus 168 \oplus 56 .
\]

The lowest S-wave states can be fitted into the completely symmetric IR 120:
The content of the other two IR’s is given by:

\[168 = (20, 2) \oplus (20', 2) \oplus (20', 4) \oplus (1, 2)\]

\[56 = (20', 2) \oplus (1, 4)\]

8.14 In analogy with the strangeness \( S \) (\( S = -1 \) for the \( s \)-quark), a quantum number \( b \) is introduced for beauty (\( b = -1 \) for the \( b \)-quark). Then the Gell-Mann Nishijima formula (8.45) is replaced by (compare with Eqs. (8.180), (8.181))

\[Q = I_3 + \frac{1}{2}(B + S + b) \quad \text{and} \quad Y = B + S + b\]

The \( 0^- \) \( b \)-mesons can be assigned to the 15-multiplet of \( SU(4) \) and the situation is analogous to that represented in Fig. 8.15 (replacing \( C \) by \( -b \)): there are an iso-doublet \( B^+(u\overline{b}), B^0(\overline{u}\overline{b}) \) and an iso-singlet \( B_s^0(s\overline{b}) \), all with \( b = 1 \), and the corresponding antiparticles with \( b = -1 \): \( B^-(\overline{u}b), B^0(\overline{d}b) \) and \( B_s^0(\overline{s}b) \).

The situation for the \( 1^- \) \( b \)-mesons is similar to that of the charmed ones: they should be assigned to a 15 + 1 multiplet with a mixing giving rise to a pure \( Y(b\overline{b}) \) state, which is the analogue of the \( J/\psi(c\overline{c}) \) state.

The \( \frac{1}{2}^+ \) \( b \)-baryons can be assigned to the 20' multiplet (we limit ourselves to this case, since many of these states have been observed experimentally) and the situation is the following (where \( q \) stands for \( u \) or \( d \)):

- Baryons with \( b = -1 \): a triplet \( \Sigma_b(qq\overline{b}) \), a doublet \( \Xi_b(qs\overline{b}) \) and two singlets \( \Lambda_b(ud\overline{b}) \) and \( \Omega_b(ss\overline{b}) \).
- Baryons with \( b = -2 \): a doublet \( \Xi_{bb}(qbb) \) and a singlet \( \Omega_{bb}(sbb) \).

Problems of Chapter 9

9.1 We rewrite here the Lagrangian (9.46)

\[\mathcal{L}(x) = \sum_j \bar{q}^j(x)(i\gamma^\mu D_\mu - m_j)q^j(x) - \frac{1}{2}\text{Tr}(G_{\mu\nu}G^{\mu\nu}),\]
where $D_\mu$ is the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + ig_s G_\mu(x),$$

and $G_{\mu\nu}$ the field strength

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + ig_s [G_\mu, G_\nu],$$

and require its invariance under the transformation

$$q(x) \to U(x)q(x).$$

In order to obtain the invariance of the first term of $L$, we write the transformation properties to the quantity $D_\mu q(x)$:

$$(D_\mu q(x))' = U(D_\mu q(x)) = U\partial_\mu q + ig_s UG_\mu q.$$}

and require that they correspond with what we expect from its invariance

$$(D_\mu q(x))' = (\partial_\mu + ig_s G'_\mu)Uq = (\partial_\mu U)q + U\partial_\mu q + ig_s G'_\mu Uq.$$}

A comparison of these two relations gives the required transformation properties of $G_\mu$

$$G'_\mu = UG_\mu U^\dagger + \frac{i}{g_s}(\partial_\mu U)U^\dagger,$$

which coincide with Eq. (9.50).

We can prove the invariance of the second term in $L$ by considering the following commutator:

$$[D_\mu, D_\nu]q(x) = [\partial_\mu + ig_s G_\mu, \partial_\nu + ig_s G_\nu]q(x) =$$

$$= ig_s (\partial_\mu G_\nu - \partial_\nu G_\mu)q(x) - g_s^2[G_\mu, G_\nu]q(x).$$

Comparing with the expression of the field strength $G_{\mu\nu}$, we obtain the relation

$$G_{\mu\nu} = -\frac{i}{g_s} [D_\mu, D_\nu].$$

From this relation we can derive the transformation properties of the field strength $G_{\mu\nu}$

$$ig_s G'_{\mu\nu} = [D'_\mu, D'_\nu] = [UD_\mu U^\dagger, UD_\nu U^\dagger] = U[D_\mu, D_\nu]U^\dagger = ig_s UG_{\mu\nu} U^\dagger.$$}

It follows

$$Tr(G_{\mu\nu} G^{\mu\nu}) = Tr(U G_{\mu\nu} U^\dagger U G^{\mu\nu} U^\dagger) = Tr(G_{\mu\nu} G^{\mu\nu}),$$

which concludes the proof of the gauge invariance of the Lagrangian.
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9.2 If we adopt the compact notation

\[
\Phi = \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_N
\end{pmatrix}
\]

\(\mathcal{L}\) can be written in the form

\[
\mathcal{L} = \frac{i}{2} \partial_\mu \Phi \partial^{\mu} \Phi - V(\Phi^2),
\]

where

\[
V(\Phi^2) = \frac{1}{2} \mu^2 \Phi^2 + \frac{i}{4} \lambda \Phi^4
\]

is the "potential" and

\[
\Phi^2 = \bar{\Phi} \Phi = \sum_{i=1}^{N} \phi_i^2.
\]

The minimum of the potential corresponds to the v.e.v.

\[
(\Phi^2)_0 = -\frac{\mu^2}{\lambda} = v^2,
\]

and one can choose a coordinate system such that

\[
(\Phi)_0 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
v
\end{pmatrix}.
\]

With the definition

\[
\Phi(x) = \begin{pmatrix}
\xi_1(x) \\
\vdots \\
\xi_{N-1}(x) \\
\eta(x) + v
\end{pmatrix},
\]

\(\mathcal{L}\) can be rewritten in the form

\[
\mathcal{L} = \frac{i}{2} \partial_\mu \eta \partial^{\mu} \eta - \frac{i}{2} \sum_{i=1}^{N-1} \partial_\mu \xi_i \partial^{\mu} \xi_i + \mu^2 \eta^2 - \frac{i}{4} \lambda \eta^4 - \frac{i}{4} \sum_{i=1}^{N-1} \xi_i^4 - \frac{i}{4} \lambda \eta^2 \sum_{i=1}^{N-1} \xi_i^2.
\]

Only the field \(\eta(x)\) has a mass different from zero, with the value \(m_\eta^2 = -2\mu^2\), while the \(N-1\) fields \(\xi_i(x)\) are massless: they are the Goldstone bosons. According to the Goldstone theorem, their number is equal to \(n - n'\), where \(n = N(N-1)/2\) and \(n' = (N-1)(N-2)/2\) are the numbers of generators of the group \(O(N)\) and of the subgroup \(O(N-1)\), respectively; in fact, \(n - n' = N-1\).
9.3 The leading $W^\pm$-exchange contribution to the $\nu_e e^-$ invariant scattering amplitude is given by:

$$\mathcal{M}(W^\pm) = \left(\frac{g}{2\sqrt{2}}\right)^2 \left[\bar{\nu}_e \gamma^\mu (1 - \gamma_5) e \frac{-ig_{\mu\nu}}{q^2 - M_W^2} \bar{e} \gamma_\nu (1 - \gamma_5) \nu_e\right]$$

and, since the momentum transfer satisfies the condition $q^2 \ll M_W^2$, one can write:

$$\mathcal{M}(W^\pm) \approx -i \frac{g^2}{8M_W^2} \left[\bar{\nu}_e \gamma^\mu (1 - \gamma_5) e \right] \left[\bar{e} \gamma_\mu (1 - \gamma_5) \nu_e\right].$$

Starting from the Fermi Lagrangian (9.105), one obtains a similar expression: the only difference is that the factor $\frac{g^2}{8M_W^2}$ is replaced by $\frac{G_F}{\sqrt{2}}$, so that the requested relation is the following:

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2} = \frac{1}{\sqrt{2}v^2}.$$

The analogous contribution due to the $Z^0$-exchange is given by

$$\mathcal{M}(Z^0) \approx -i \frac{g^2}{8M_Z^2 \cos^2 \theta_w} \left[\bar{\nu}_e \gamma^\mu (1 - \gamma_5) \nu_e\right] \left[\bar{e} \gamma_\mu (1 - \gamma_5) e + 4 \sin^2 \theta_w \bar{e} \gamma_\mu e\right].$$

The ratio between the couplings of the neutral and the charged current is then given by (comparing with Eq. (9.135))

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_w} = 1.$$

9.4 The real scalar triplet and its v.e.v. can be written as follows

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad <\Psi>_0 = \begin{pmatrix} 0 \\ V \\ 0 \end{pmatrix}$$

and we have to include in the Lagrangian (9.118) the additional terms

$$\frac{1}{2} (D_\mu \tilde{\Psi})(D^\mu \Psi) - V(\Psi^2),$$

where in particular, being $Y_\psi = 0$,

$$D_\mu \Psi = \partial_\mu \Psi - igI_i A_i^\mu \Psi .$$

The contribution to the vector boson masses comes from the covariant derivative when the fields are shifted and are substituted by their v.e.v.’s. Concerning the field $\Psi$, its contribution is then given by

$$\frac{1}{2} [(D_\mu \tilde{\Psi})(D^\mu \Psi)]_0 .$$
However, this term contributes only to the mass of the charged bosons: in fact, since the $\Psi$ component which develops the v.e.v. different from zero must be a neutral component, from the assumption $Y_\Psi = 0$ we get also $I_{3,\Psi} = 0$. Accordingly

$$\frac{1}{2}[(D_\mu \tilde{\Psi})(D^\mu \Psi)]_0 = \frac{1}{2}g^2 V^2 (I_{1,\Psi} A^{(1)}_{\mu} + I_{2,\Psi} A^{(2)}_{\mu})^\dagger (I_{1,\Psi} A^{(1)\mu} + I_{2,\Psi} A^{(2)\mu}) =$$

$$= \frac{1}{2}g^2 V^2 [I^- W^{(-)}_{\mu} + I^+ W^{(+)}_{\mu}] [I^+ W^{(+)}_{\mu} + I^- W^{(-)}_{\mu}] =$$

$$= \frac{1}{2}g^2 V^2 [I^+ I^- + I^- I^+] W^{(+)}_{\mu} W^{(-)}_{\mu} =$$

$$= \frac{1}{2}g^2 V^2 2 \{I_1^2 - I_3^2\} = \frac{1}{2}g^2 V^2 I_\Psi (I_\Psi + 1) = g^2 V^2.$$

since $I_{3,\Psi} = 0$ and $\Psi$ is a triplet ($I_\Psi = 1$) of weak isospin. This contribution must be added to the term $\frac{1}{4}g^2 v^2$ coming from the isospin doublet $\Phi$ of Eq. (9.125): we see that the value of $M_W$ of the standard model is modified into

$$M_W = \frac{1}{2}g\sqrt{v^2 + 4V^2}.$$

The mass $M_Z$ is not changed, so that the ratio $M_W/M_Z$ is replaced by

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2 + g'^2}} \sqrt{1 + \frac{4V^2}{v^2}}.$$

If only the triplet $\Psi$ were present in the scalar sector, only $SU(2)_L$ would be broken and the vacuum symmetry would be $U(1)_{I_1} \otimes U(1)_Y$. As a consequence, only the charged vector fields $W_{\mu}^{\pm}$ acquire mass different from zero, while the two fields $A^{(3)}_{\mu}$ and $B_{\mu}$ remain massless.

9.5 Let us generalize the Lagrangian density $\mathcal{L}_{\text{gauge}}$ of Eq. (9.118) by introducing several Higgs fields $\phi_\ell$. The terms contributing to the masses of the vector bosons and to their couplings with the Higgs fields are given by

$$\sum_\ell \left\{ \left[ gI_{i,\ell} A_i^\mu + g'\frac{1}{2} Y_\ell B_\mu \right] \phi_\ell \right\}^\dagger \left\{ \left[ gI_{i,\ell} A_i^{\mu} + g'\frac{1}{2} Y_\ell B_\mu \right] \phi_\ell \right\}.$$

This expression can be rewritten in terms of the physical fields, introducing the charged vector fields $W_{\mu}^{(\pm)}$ of Eq. (9.128), the neutral fields $A_{\mu}$ and $Z_\mu$ through the relations (9.132), and the operators $I_{\pm} = I_1 \pm iI_2$. For the terms in the second bracket, we obtain

$$\left\{ \left[ gI_{i,\ell} A_i^{\mu} + g'\frac{1}{2} Y_\ell B_\mu \right] \phi_\ell \right\} = \left\{ g \left[ I_\ell^+ W_{\mu}^{(+)} + I_\ell^- W_{\mu}^{(-)} \right] \phi_\ell +$$

$$+ \frac{1}{\sqrt{g^2 + g'^2}} \left[ gg' (I_{3,\ell} + \frac{1}{2} Y_\ell) A_{\mu} + (g^2 I_{3,\ell} - g'^2 \frac{1}{2} Y_\ell) Z_{\mu} \right] \phi_\ell \right\}.$$

We suppose that each Higgs field $\phi_\ell$ develops a vacuum expectation value different from zero, defined by $\frac{1}{\sqrt{2}} v_\ell$, where $v_\ell$ must correspond to a neutral
component of $\phi_\ell$, otherwise also the electromagnetic gauge invariance would be broken. This means, on the basis of the relation (9.112), that for each v.e.v. $v_\ell$ one gets $I_{3,\ell} + \frac{1}{2} Y_\ell = 0$, independently of the specific representation of $\phi_\ell$. It follows that the coefficient of the e.m. field $A_\mu$ goes to zero ($A_\mu$ is massless, as required) and the previous relation becomes

$$\frac{1}{\sqrt{2}} g \left[ I_\ell^+ W^{(+)}_\mu + I_\ell^- W^{(-)}_\mu \right] - \sqrt{g^2 + g'^2} \frac{1}{2} Y_\ell Z_\mu Z^\mu \phi_\ell .$$

Finally, introducing the v.e.v. $v_\ell$, we get

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} \sum_\ell v_\ell^2 \left\{ \frac{1}{\sqrt{2}} g^2 \left[ I_\ell^+ I_\ell^- + I_\ell^+ I_\ell^- \right] W^{(+)}_\mu W^{(-)}_\mu + (g^2 + g'^2) \frac{1}{4} Y_\ell^2 Z_\mu Z^\mu \right\} =$$

$$= \frac{1}{2} \sum_\ell v_\ell^2 \left\{ g^2 \left[ I_\ell(I_\ell + 1) - \frac{1}{4} Y_\ell^2 \right] W^{(+)}_\mu W^{(-)}_\mu + (g^2 + g'^2) \frac{1}{4} Y_\ell^2 Z_\mu Z^\mu \right\} ,$$

and, by taking the ratio of the two squared masses, we obtain the required expression

$$\rho = \frac{\sum_\ell v_\ell^2 (I_\ell(I_\ell + 1) - \frac{1}{4} Y_\ell^2)}{\frac{1}{2} \sum_\ell v_\ell^2 Y_\ell^2} .$$

9.6 The Lagrangian contains two terms

$$(D_{L,R}^\mu \Phi_L)^\dagger (D_{L,R}^\mu \Phi_L) + (D_{L,R}^\mu \Phi_R)^\dagger (D_{L,R}^\mu \Phi_R) ,$$

where

$$(D_{L,R})_\mu = \partial_\mu - i g \frac{1}{2} \tau_i (A_{L,R}^i_\mu) - \frac{1}{2} g' B_\mu .$$

Introducing the charged vector fields

$$(W_{L,R}^\pm)_\mu = \frac{1}{\sqrt{2}} \left[ (A_{L,R}^3)_\mu \mp i (A_{L,R}^1)_\mu \right] ,$$

one obtains

$$(D_{L,R})_\mu \Phi_{L,R} = - \frac{i}{2} \left\{ g (W_{L,R}^+)_\mu \begin{pmatrix} v_{L,R} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \left[ g (A_{L,R}^3)_\mu + g' B_\mu \right] \begin{pmatrix} 0 \\ v_{L,R} \end{pmatrix} \right\} .$$

It follows that all the charged vector bosons become massive with masses given by

$$M_{W_L} = \frac{1}{2} g v_L , \quad M_{W_R} = \frac{1}{2} g v_R .$$

Since there is no experimental evidence of other vector bosons besides those of the Standard Model, one must assume : $v_R \gg v_L$.

For the squared masses of the neutral gauge bosons we obtain the matrix

$$M^2 = \frac{1}{4} g^2 v_R^2 \begin{pmatrix} y & 0 & -\kappa y \\ 0 & 1 & -\kappa \\ -\kappa y & -\kappa & (1 + y) \kappa^2 \end{pmatrix} ,$$
where

\[ \kappa = \frac{g'}{g} \quad \text{and} \quad y = \frac{v_R^2}{v_R^2}. \]

Since \( \det M^2 = 0 \), one eigenvalue is equal to zero. For the other two eigenvalues, in the case \( v_R \gg v_L \), one gets the approximate values:

\[ M_1^2 \approx \frac{1}{4} g^2 v_L^2 \left( \frac{1 + 2\kappa}{\kappa^2 + 1} \right), \quad M_2^2 \approx \frac{1}{4} g^2 v_R^2 (\kappa^2 + 1). \]

9.7 Let us consider first the IR 5 of \( SU(5) \); its content in terms of the subgroup \( SU(3) \otimes SU(2) \), according to Eq. (9.169), is given by

\[ 5 = (1, 2)_{-1} + (3, 1)_{\pm \frac{1}{3}}, \]

where the subscripts stand for the values of \( Y \) which can be read from Tables 9.2 and 9.3.

The IR 24 can be obtained by the direct product

\[ 5 \otimes 5 = 1 \oplus 24 \]

and its content in terms of the subgroup \( G_{SM} \) follows from the above relation:

\[ 24 = (8, 1)_{0} \oplus (1, 1)_{0} \oplus (1, 3)_{0} \oplus (3, 2)_{-\frac{5}{3}} \oplus (\bar{3}, 2)_{+\frac{5}{3}}. \]

Let us define

\[ (\bar{3}, 2)_{+\frac{5}{3}} \rightarrow \left( \begin{array}{ccc} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{array} \right). \]

From the usual relation \( Q = I_3 + \frac{1}{2} Y \) we obtain the electric charges of the \( X \) and \( Y \) particles:

\[ Q(X_i) = \frac{4}{3}, \quad Q(Y_i) = \frac{1}{3}. \]

The multiplet \( (3, 2)_{-\frac{5}{3}} \) contains the antiparticles \( \bar{X} \) and \( \bar{Y} \) which have opposite values of \( Q \). These particles are called lepto-quarks because they have the same quantum numbers of the lepton-quark pairs. They mediate new interactions which violate baryon and lepton numbers.

9.8 We recall the decomposition of the 24-multiplet \( \Phi \) in terms of the subgroup \( G_{SM} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \):

\[ 24 = (1, 1)_{0} \oplus (1, 3)_{0} \oplus (8, 1)_{0} \oplus (3, 2)_{-5/3} \oplus (\bar{3}, 2)_{+5/3}, \]

and write \( \Phi \) as a \( 5 \times 5 \) traceless tensor. It is than clear that, in order to preserve the \( G_{SM} \) symmetry, the v.e.v. \( < \Phi >_0 \) must behave as \( (1, 1)_{0} \) and then it must have the following form
\[
\langle \Phi \rangle_0 = \begin{pmatrix} a & a & a \\ b & a & b \\ b & b & b \end{pmatrix}
\]

and, since the matrix is traceless, it can be written as

\[
\langle \Phi \rangle_0 = V \begin{pmatrix} 1 & 1 \\ 1 & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}
\]

The vacuum symmetry is \( G_{SM} \); therefore the 12 gauge vectors fields of \( G_{SM} \) remain massless, and the other twelve acquire mass: they are the multiplets \((3,2)\) and \((\overline{3},2)\) considered in Problem 9.7.

In order to break also

\[
G_{SM} \rightarrow SU(3)_c \otimes U(1)_Q
\]

one needs a scalar multiplet containing the doublet \( \phi \) of the Standard Model. The minimal choice is then a field

\[
\phi_5 \sim (3,1) \oplus (1,2)
\]

with v.e.v. which transforms as the neutral component of \((1,2)\).
We list here some of the books which were used while preparing the manuscript, and other books which can be consulted to broaden and deepen the different topics. The titles are collected according to the main subjects, that often concern more than one chapter. We are sure that there are many other good books on the same topics, and the list is limited to those we know better.

**Group and Representation Theory**


Herman, R., *Lie Groups for Physicists*, Benjamin, New York, 1966


**Rotation, Lorentz and Poincaré Groups**


**Quantum Mechanics and Relativity**


**Quantum Field Theory**


**Particle Physics**


**Internal Symmetries**


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