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HIGHER ORDER MEAN FIELD EXPANSIONS
FOR LATTICE GAUGE THEORIES

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Abstract

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The leading contribution to the free energy of lattice gauge theories is evaluated in the mean field expansion to the two-loop level.

The methods are general but we only deal with the $U(1)$ case in this paper. The corrections improve the agreement with Monte Carlo calculations. We show that in order to obtain a satisfactory formalism it is necessary to include a new redundant parameter, γ , in the mean field expansion. For $\gamma \rightarrow 0$ we recover the usual mean field expansion whereas for $\gamma \rightarrow \infty$ we obtain the weak coupling expansion. Thus γ measures the amount of resummation that is done by the mean field formalism.

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1. Introduction

The mean field approach to lattice gauge theories can be formulated as a perturbative Euclidean quantum field theory [1,2,3]. The mean field configurations constitute a set of degenerate ground states which transform as an orbit of the gauge group. The quadratic form of fluctuations around any of these configurations is consequently degenerate.

As in perturbative quantum chromodynamics one can add a gauge fixing term to the action and apply the Faddeev-Popov technique to compensate for this gauge fixing term. The Faddeev-Popov determinant can be evaluated perturbatively after introducing ghost fields [4,5].

This straightforward scheme which renders mean field theory a formal structure analogous to perturbative QCD leads to certain technical problems if explicit calculations are performed. In order to describe them we have to scrutinize the formalism in some detail. According to the idea presented in Ref.[1] the mean field is defined as a stationary configuration in a set of field variables that are obtained by embedding the group manifold into a linear space and doubling this space introducing its Fourier dual. The gauge theory can then equivalently be formulated as a field theory with a complex action composed of the continued Wilson action, the logarithm of the Fourier transformed Haar measures and the imaginary quadratic form of the Fourier transformation connecting the pair of linear spaces. The exponent of this complex action is integrated with the Lebesgue measures of the linear spaces.

The ansatz for a mean field configuration leads to a pair of constraints each one guaranteeing stationarity with respect to one subspace. They involve the parameters ρ , the "mean field", α , its dual, and the inverse temperature β . One of these constraints fixes ρ as a function of α .

The second constraint expresses β as a function $\varphi(\rho, \alpha)$. In order to compute thermodynamic functions we ought to be able to eliminate the auxiliary parameter α in favour of β , i.e. to invert the function $\varphi(\rho(\alpha), \alpha)$. It turns out that this is not possible in general.

Namely in the weak coupling phase the inversion is possible in an interval

$$\beta_t \leq \beta < \infty$$

where the "turnover point" β_t is often larger than β_c . In mean field theory with axial gauge fixing accidentally $\beta_t < \beta_c$ and this problem does not arise. But axial gauge fixing is not complete and leads to infrared divergent propagators which hitherto could not consistently be regularized.

Therefore we have to use covariant gauge fixing and to face the problem just mentioned.

This problem has an easy solution. We add a term to the action which vanishes identically on the original gauge group manifold but is nonzero after continuation into the linear space. For simplicity we choose a quadratic expression which leaves one parameter γ free. The mean field expansion contains two ambiguous parameters then, γ and the gauge fixing parameter ξ . Their formal appearance is very similar. Whereas ξ enters the propagators of the massless fluctuations, γ enters the propagators of the massive fluctuations. A positive γ reduces β_t and β_t ceases to exist if γ exceeds a certain threshold value γ_0 .

Mean field perturbation theory is an expansion without an expansion parameter. Therefore it is not only impossible to estimate a priori how good any

approximation is but also to use the polynomial degree to order the expansion. However, we can use the number of loops to classify approximations.

As an example we calculate the free energy and the energy per plaquette for the U(1) gauge model in four dimensions including all two-loop diagrams whose number is twentyeight. The calculations are done in Feynman gauge and with the parameter γ fixed at γ_0 . Feynman gauge reduces the number of algebraic terms for many graphs considerably. The size of the two-loop terms compared with the largest one-loop term is surprisingly small in the transition region (smaller by a factor $\frac{1}{50}$ at least). Moreover, the twenty-eight terms cancel each other almost completely. The intersection of the mean field weak coupling curve with the strong coupling curve calculated with a strong coupling expansion to tenth order yields a phase transition of first order at $\beta_c = 1.014$, in very good agreement with the Monte Carlo calculations $\beta_c = 1.003$ [6] and $\beta_c = 1.008$ [7].

In the last section we discuss the limit $\gamma \rightarrow \infty$, in which the mean field loop expansion tends towards the $\frac{1}{3}$ loop expansion. This suggested limiting behaviour is verified in the case of the U(1) model up to and including the two-loop graphs.

2. The modified action

After the manipulations described in the Introduction, the complex action for any pure U(N) or SU(N) lattice gauge model assumes the form (notations as in Refs. [3,4])

$$\begin{aligned} Z_\ell &= Z_\ell^{(0)} = \rho \mathbf{1} \\ M_\ell &= M_\ell^{(0)} = -i \alpha \mathbf{1} \end{aligned} \quad (5)$$

$$\begin{aligned} S(\{Z_\ell\}, \{M_\ell\}) &= \frac{\beta}{N} \sum_x \frac{1}{2} \sum_{\mu, \nu=1}^d \text{Tr} (Z_{x\mu} Z_{x+\underline{\mu}\nu}^+, Z_{x+\underline{\nu}\mu}^+, Z_{x\nu}^+) \\ &+ \sum_{\ell} \phi \left(\frac{1}{N} \text{Tr} Z_\ell Z_\ell^+ - 1 \right) \\ &+ \sum_{\ell} \ell n \zeta (M_\ell) \\ &- i \sum_{\ell} \text{Re Tr} Z_\ell^+ M_\ell \end{aligned} \quad (1)$$

The subscript $\ell = (x, \mu)$ denotes the links of the lattice. To each link belongs a pair of field variables Z_ℓ, M_ℓ each of which takes values in complex $N \times N$ matrices. ζ is the Fourier transform of the Haar measure

$$\zeta = \int dU \exp(i \text{Re Tr} M^+ U) \quad (2)$$

and ϕ is an arbitrary function with

$$\phi(0) = 0 \quad (3)$$

We make the ansatz

$$\phi(x) = \frac{1}{2} \gamma x \quad (4)$$

For $N \rightarrow \infty$ we will scale this parameter γ as γ/N^2 .

The mean field ansatz

yields a stationary configuration for the action (1) if

$$\rho = \frac{\omega'(\alpha)}{N}, \quad \omega(\alpha) = \int_0^{\alpha} du \, e^{\alpha \operatorname{Re} \operatorname{Tr} U} \quad (6)$$

$$\left\{ \begin{array}{l} \bar{\beta} \rho^3 + \gamma \rho - \alpha N = 0 \\ \bar{\beta} = 2(d-1)\beta \end{array} \right. \quad (7)$$

At this configuration the action assumes the value

$$S_{\text{eff}} = V \left\{ \beta \left(\frac{d}{2} \right) \rho^4 + d \left[\frac{1}{2} \gamma (\rho^2 - 1) + \omega(\alpha) - \alpha \rho N \right] \right\} \quad (8)$$

which gives us the ground state contribution f_0 to the free energy density by

$$f_0 = \frac{S_{\text{eff}}}{V} \quad (9)$$

From (7) we obtain

$$\begin{aligned} \bar{\beta} &:= \varphi(\rho, \alpha) \\ &= \frac{1}{\rho^3} [\alpha N - \gamma \rho] \end{aligned} \quad (10)$$

For $\alpha \rightarrow \infty$ the function $\varphi(\rho(\alpha), \alpha)$ is monotonously increasing.

It ceases to be invertible at the stationary point

$$\frac{d}{d\alpha} \varphi(\rho(\alpha), \alpha) = 0 \quad (11)$$

whose solution α_t determines the turnover point β_t as a function of γ .
An implicit equation for α_t is easy to obtain from (11)

$$\gamma = N \frac{3 \alpha \rho'(\alpha) - \rho(\alpha)}{2 \rho'(\alpha) \rho(\alpha)} \quad (12)$$

The r.h.s. of (12) decreases monotonously from γ_0 at $\alpha = 0$ till $-\infty$ at $\alpha = \infty$. Using the asymptotic forms

$$\rho(\alpha) = \alpha \rho'(0) + \frac{\alpha^2}{2} \rho''(0) + o(\alpha^3) \quad \alpha \rightarrow 0 \quad (13)$$

$$\rho(\alpha) = 1 + o(\alpha^{-1}) \quad \alpha \rightarrow \infty \quad (14)$$

where $\rho''(0) \neq 0$ only for $SU(3)$, we obtain the leading singularity of $\varphi(\rho(\alpha), \alpha)$ at $\alpha = 0$ and for $\gamma \neq \gamma_0$ as

$$\varphi(\rho(\alpha), \alpha) = \frac{\gamma_0 - \gamma}{\rho'(0)^2} \cdot \frac{1}{\alpha^2} + o(\alpha^{-1}) \quad (15)$$

with

$$\gamma_0 = \frac{N}{\rho'(0)} = \frac{N^2}{\omega''(0)} \quad (16)$$

From (6) we get

$$\omega''(0) = \begin{cases} 1 & \text{for } SU(2) \\ \frac{1}{2} & \text{for } SU(N), N \geq 3 \text{ and } U(N), N \geq 1 \end{cases} \quad (17)$$

Again we conclude from (15) that a turnover point exists for $\gamma < \gamma_0$.

Its uniqueness for $\gamma < \gamma_0$ and nonexistence for $\gamma > \gamma_0$ follows from the monotony of the r.h.s. of (12).

If $\gamma = \gamma_0$ there is a finite turnover point

$$\bar{\beta}_{t,0} = -N \frac{\rho'''(0)}{6 \rho'(0)^4} \quad (18)$$

for all groups except SU(3). For U(1) we get

$$\bar{\beta}_{t,0} = 1, \quad \beta_{t,0} = \frac{1}{2(d-1)} \quad (19)$$

As two examples we plot $\bar{\beta}_t(\gamma)$ for U(1) and U(∞) in Fig.1. In order to combine both curves in one figure we use the scaled variables

$$\frac{\alpha}{N}, \frac{\bar{\beta}}{N^2}, \frac{\gamma}{N^2} \quad (20)$$

and restrict $\frac{\alpha}{N}$ to the weak coupling phase

$$1 \leq \frac{\alpha}{N} < \infty$$

of the Gross-Witten U(∞) model [8].

If β is kept fixed in the transition region, say

$$0.5 < \frac{\beta}{N^2} < 1.5 \quad (21)$$

$\frac{\alpha}{N}$ is a number proportional to $2(d-1)$ and therefore large compared to one. We can then to zeroth approximation replace ρ by one in (10) and see that a positive γ translates α by γ/N . This strong dependence of the

intermediary variable α on γ ought to cancel in the thermodynamic functions and indeed it does!

For fixed β and using (7) we get

$$\left(\frac{\partial f_0}{\partial \gamma}\right)_\beta = \frac{1}{2} d(\rho^2 - 1) \quad (22)$$

which by (14) is $O(\alpha^{-1})$ and therefore small compared with f_0 . If one takes into account one-loop diagrams (Section 3)

$$f' = f_0 + f_1 \quad (23)$$

one obtains after some algebra

$$\left(\frac{\partial f'}{\partial \gamma}\right)_\beta = \frac{N^2}{\alpha^2} + O(\alpha^{-3}) \quad (24)$$

and thus a term which is small compared to f_1 . We conjecture that the dependence of f on γ is always of higher order in this sense.

The phase transition point β_c for the U(1) model and $d=4$ is calculated using a zero loop and a one-loop approximation for the free energy in the weak coupling phase and plotted as a function of γ in Fig. 2.

The constancy of β_c for the one-loop approximation in the neighborhood of γ_0 is most remarkable.

3. One- and two-loop contributions

We introduce real fluctuating field variables by

$$\begin{aligned} Z_\ell - Z_\ell^{(0)} &= H_\ell + i A_\ell \\ M_\ell - M_\ell^{(0)} &= h_\ell + i a_\ell \end{aligned} \quad (25)$$

where all four matrices are hermitean. The quadratic form of the fluctuations for the term

$$\begin{aligned} &\sum_\ell \ln \zeta(M_\ell) \\ &\text{in (1) gives} \\ &-\frac{1}{2} \sum_\ell \{ \lambda_1(\alpha) \operatorname{Tr} h_\ell'^2 + \lambda_2(\alpha) \operatorname{Tr} a_\ell'^2 \\ &\quad + \lambda_3(\alpha) \frac{1}{N} (\operatorname{Tr} h_\ell) + \lambda_4(\alpha) \frac{1}{N} (\operatorname{Tr} a_\ell)^2 \} \end{aligned} \quad (26)$$

with

$$\begin{aligned} h_\ell' &= h_\ell - \frac{1}{N} \operatorname{Tr} h_\ell \cdot \mathbf{1} \\ a_\ell' &= a_\ell - \frac{1}{N} \operatorname{Tr} a_\ell \cdot \mathbf{1} \end{aligned} \quad (27)$$

The positive structure functions $\lambda_1(\alpha)$ satisfy the identities

$$\lambda_2(\alpha) = \lambda_4(\alpha) \quad \text{for } U(N) \quad (28)$$

$$\lambda_2(\alpha) = -\frac{\rho(\alpha)}{\alpha} \quad \text{for } U(N) \text{ and } SU(N) \quad (29)$$

The identity (28) renders the a_ℓ' and $\operatorname{Tr} a_\ell$ (analogously the A_ℓ' and $\operatorname{Tr} A_\ell$) fluctuations degenerate. The fluctuations A_ℓ for $U(N)$ and A_ℓ' for $SU(N)$ are massless and we add therefore a gauge fixing term to the quadratic form

(for $U(N)$, see [4])

$$S_{\text{g.f.}}(H, A) = -\frac{1}{2} \sum_x \operatorname{Tr} (\partial A_x)^2 - \ln N(\xi) \quad (30)$$

with

$$\partial A_x = \sum_\mu (A_{x\mu} - A_{x-\mu, \mu}) \quad (31)$$

$$\text{and} \quad \frac{1}{2} V \Delta = \begin{cases} N^2 - 1 & , \quad SU(N) \\ N^2 & , \quad U(N) \end{cases} \quad (32)$$

(A replaced by A' for $SU(N)$).

The Faddeev-Popov determinant can be evaluated with the help of ghost fields (for $U(N)$)

$$\begin{aligned} S_{\text{ghost}}(\bar{C}, C, H, A) &= \frac{1}{2} \sum_{x, \mu} \operatorname{Tr} [(\bar{C}_{x+\mu} - \bar{C}_x) \times \\ &\quad \times (C_{x+\mu} - C_x) + H_{x\mu}] - i [C_{x+\mu} + C_x, A_{x\mu}] \end{aligned} \quad (33)$$

C and $\bar{C}$ are $N \times N$ matrices with Grassmannian entries, the brackets [,] and { , } denote commutator and anticommutator in the matrix algebra. For $SU(N)$ replace $C, \bar{C}, H, A$ by the traceless matrices $C', \bar{C}'$; H', A' and the anticommutator by its traceless part.

In the abelian case $U(1)$ which we shall from now on consider only there is no coupling of the ghosts to the A -field. It is often advantageous not to use ghosts but to expand the logarithm of the Faddeev-Popov determinant in the fields A and H , e.g. for $U(1)$

$$\begin{aligned} \epsilon n \Delta(H, A) &= - \epsilon n \int \prod_X [dU_X \delta(\partial A_X^g - \partial A_X)] \\ &= V \int dk \epsilon n [2\pi \rho(k)] + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{Tr } T^n \end{aligned} \quad (34)$$

with

$$\begin{aligned} T_{xy} &= \frac{1}{\rho} \sum_{\mu} [H_{xu} (G_{xy} - G_{x+\mu, y}) \\ &\quad + H_{x-\mu, \mu} (G_{xy} - G_{x-\mu, y})] \end{aligned} \quad (35)$$

$$G_{xy} = \int dk \frac{e^{ik(x-y)}}{\lambda(k)} \quad (36)$$

$$\lambda(k) = \sum_{\mu=1}^d (2-2 \cos k_{\mu}) \quad (37)$$

$$\int dk = \prod_{\mu=1}^d \int_0^{2\pi} \frac{dk_{\mu}}{2\pi} \quad (38)$$

We define propagators by the expectation values of any pair of the field variables h_{ℓ} , a_{ℓ} , H_{ℓ} , A_{ℓ} with the exponent of the complex quadratic form of the fluctuations. This is well defined if we integrate first over a_{ℓ} , h_{ℓ} due to the positivity of the structure functions $\lambda_i(\alpha)$ (26), and over

A_{ℓ} , H_{ℓ} afterwards. For $U(1)$ only the terms with $i=3,4$ survive and h_{ℓ} , a_{ℓ} , H_{ℓ} , A_{ℓ} are simply real numbers. We obtain

$$\langle H_{\ell} H_{\ell'} \rangle = \lambda_3 D_{3, \ell\ell'} \quad (39)$$

$$\langle A_{\ell} A_{\ell'} \rangle = \lambda_4 D_{4, \ell\ell'} \quad (40)$$

For $\ell = (x, \mu)$, $\ell' = (y, \nu)$ we have

$$\begin{aligned} D_{3, \ell\ell'} &= D_{3, \mu\nu}(x, y) = \\ &= \int dk e^{ik(x-y)} [1 - \gamma \lambda_3 - \bar{\beta} \rho^2 \lambda_3 R(k)]_{\mu\nu}^{-1} \end{aligned} \quad (41)$$

The matrix $R(k)$ is defined by

$$R(k)_{\mu\nu} = \frac{1}{2(d-1)} [(|L|^2 - 2|L_{\mu}|^2) \delta_{\mu\nu} + L_{\mu} L_{\nu}] - \delta_{\mu\nu} \quad (42)$$

with

$$L_{\mu} = e^{ik_{\mu}} + 1, \quad |L|^2 = \sum_{\mu=1}^d |L_{\mu}|^2 \quad (43)$$

We shall calculate the inverse of the matrix in (41) (and correspondingly the logarithm of this matrix) by a power expansion in R . For large α this amounts to a power expansion in α^{-1} .

It is remarkable that the H-propagator depends on the parameter γ whereas the A-propagator depends on the parameter ξ

$$\begin{aligned} D_{4, \ell\ell'} &= D_{4, \mu\nu}(x, y) \\ &= \frac{1}{\lambda^4} \int dk \frac{e^{ik(x-y)}}{\lambda(k)} \left[\frac{1}{\xi_F} \left(\delta_{\mu\nu} - \frac{K_{\mu} \bar{K}_{\nu}}{\lambda(k)} \right) + \frac{1}{\xi} \frac{K_{\mu} \bar{K}_{\nu}}{\lambda(k)} \right] \end{aligned} \quad (44)$$

with

$$K_{\mu} = e^{ik_{\mu}} - 1 \quad (45)$$

$$\xi_F = \beta \rho^2 \quad (46)$$

We obtain Feynman gauge if $\xi = \xi_F$ in which case

$$D_{4,\mu\nu}(x,y) = \frac{1}{8 p^2 \lambda_4} \epsilon_{\mu\nu} g_{xy} \quad (47)$$

In the sequel all calculations will be done in Feynman gauge. Finally we complete the list of propagators

$$\langle H_{\ell} h_{\ell'} \rangle = -i D_{3,\ell\ell'} \quad (48)$$

$$\langle A_{\ell} a_{\ell'} \rangle = -i D_{4,\ell\ell'} \quad (49)$$

$$\langle h_{\ell} h_{\ell'} \rangle = \frac{1}{\lambda_3} (\delta_{\ell\ell'} - D_{3,\ell\ell'}) \quad (50)$$

$$\langle a_{\ell} a_{\ell'} \rangle = \frac{1}{\lambda_4} (\delta_{\ell\ell'} - D_{4,\ell\ell'}) \quad (51)$$

Define

$$\psi_0(\alpha^2) = \lambda n I_0(\alpha) \quad (52)$$

and cumulants $\psi_n(\alpha^2)$ of modified Bessel functions by

$$\psi_0(\alpha^2 + t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \psi_n(\alpha^2) \quad (53)$$

We can integrate (2) for $U(1)$ with

$$\text{Re } M = h - i\alpha \quad (54)$$

$$\text{Im } M = a$$

and obtain

$$\lambda n \zeta(M) = \sum_{n=0}^{\infty} \frac{1}{n!} (2i\alpha h - a^2 - h^2)^n \psi_n(\alpha^2) \quad (55)$$

Comparison with (26) yields

$$\begin{aligned} \lambda_3(\alpha) &= 2\psi_1 + 4\alpha^2 \psi_2 \\ &= \frac{1}{2\alpha^2} + \frac{1}{4\alpha^3} + \frac{3}{8\alpha^4} + o(\alpha^{-5}) \end{aligned} \quad (56)$$

and

$$\begin{aligned} \lambda_u(\alpha) &= \frac{\rho(\alpha)}{\alpha} = 2\psi_1 \\ &= \frac{1}{\alpha} \left[1 - \frac{1}{2\alpha} - \frac{1}{8\alpha^2} - \frac{1}{128\alpha^4} - \frac{25}{128\alpha^4} + o(\alpha^{-5}) \right] \end{aligned} \quad (57)$$

The one-loop contribution f_1 to the free energy density is composed of the part $f_1^{(0)}$ of the massless fluctuations and the ghosts and a part $f_1^{(m)}$ of the massive fluctuations

$$\begin{aligned} f_1^{(0)} &= -\frac{1}{2}(d-1) \ln \beta + (2 - \frac{3}{2}d) \ln \rho + \frac{1}{2} d \ln \alpha \\ &\quad - \frac{1}{2}(d-2) \int dk \ln \lambda(k) + \frac{1}{2} \ln 2\pi \end{aligned} \quad (58)$$

$$\begin{aligned} f_1^{(m)} &= -\frac{1}{2} \int dk \ln \det(1 - \gamma \lambda_3 - \bar{\beta} \rho^2 \lambda_3 R) \\ &= -\frac{1}{2} d \ln (1 - \gamma \lambda_3) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\bar{\beta} \rho^2 \lambda_3}{1 - \gamma \lambda_3} \right)^n \delta_n \end{aligned} \quad (59)$$

with

$$\begin{aligned} \delta_n &= \int dk \text{Tr } R^n, \\ \delta_1 &= 0, \quad \delta_2 = \frac{3d}{2(d-1)}, \quad \delta_3 = \frac{d(2d-1)}{2(d-1)^2} \end{aligned} \quad (60)$$

Terms with $n > 3$ in the series (59) will be neglected in our two-loop calculation. The integral in (58) has been evaluated [9]

$$\Lambda(d) = \int dk \ln \lambda(k) \quad (61)$$

$$\Lambda(4) = 1.99971 \quad (62)$$

For the two-loop calculations we need interaction terms that are homogeneous of degree three and four in the fluctuating fields. From the function $\Sigma \ln \zeta(M)$ (55) we obtain

$$\begin{aligned}
V_3(h,a) &= \sum_{\ell} \{ \frac{1}{3!} (2i\alpha h_{\ell})^3 \psi_3 - (2i\alpha h_{\ell}) (h_{\ell}^2 + a_{\ell}^2) \psi_2 \} \\
&= -i \sum_{\ell} \{ e_1 h_{\ell}^3 + e_2 h_{\ell} a_{\ell}^2 \} \quad (63) \\
V_4(h,a) &= \sum_{\ell} \{ \frac{1}{4!} (2i\alpha h_{\ell})^4 \psi_4 - \frac{1}{2!} (2i\alpha h_{\ell})^2 (h_{\ell}^2 + a_{\ell}^2) \psi_3 + \frac{1}{2!} (h^2 + a^2)^2 \psi_2 \} \\
&= \sum_{\ell} \{ g_1 h_{\ell}^4 + g_2 h_{\ell}^2 a_{\ell}^2 + g_3 a_{\ell}^4 \} \quad (64)
\end{aligned}$$

with

$$\begin{aligned}
e_1 &= -\frac{1}{6\alpha^3} + o(\alpha^{-4}) \quad (65a) \\
e_2 &= 4\alpha g_3 \quad (65b) \\
g_1 &= +\frac{1}{8\alpha^4} + o(\alpha^{-5}) \quad (66a) \\
g_2 &= +\frac{1}{2\alpha^3} - \frac{3}{4\alpha^4} - \frac{3}{8\alpha^5} + o(\alpha^{-6}) \quad (66b) \\
g_3 &= -\frac{1}{8\alpha^3} + \frac{1}{8\alpha^4} + \frac{3}{64\alpha^5} + o(\alpha^{-6}) \quad (66c)
\end{aligned}$$

Similarly we have from the Wilson action an interaction term $W_3(H,A)$ of degree three and a term $W_4(H,A)$ of degree four. They are elementary but too lengthy to be written down here. We have then to extract the two-loop terms of the expectation value of

$$\begin{aligned}
&V_4(h,a) + W_4(H,A) \\
&+ \frac{1}{2} V_3(h,a)^2 + \frac{1}{2} W_3(H,A)^2 \\
&+ V_3(h,a) W_3(H,A) \\
&+ (V_3(h,a) + W_3(H,A)) \cdot \text{Tr } T \\
&+ \frac{1}{2} [(\text{Tr } T)^2 - \text{Tr } T^2] \quad (67)
\end{aligned}$$

The complete list of twenty-eight two-loop graphs is depicted in Fig.3.

Evaluating these graphs we estimate their size by expanding in inverse powers of α and neglecting graphs of order α^{-3} or smaller. The propagators themselves have the size

$$\langle H_{\ell} H_{\ell'} \rangle = o(\alpha^{-2}) \quad (o(\alpha^{-3}) \text{ for } \ell \neq \ell') \quad (68a)$$

$$\langle A_{\ell} A_{\ell'} \rangle = o(\alpha^{-1}) \quad (68b)$$

$$\langle H_{\ell} h_{\ell'} \rangle = o(1) \quad (68c)$$

$$\langle A_{\ell} a_{\ell'} \rangle = o(1) \quad (68d)$$

$$\langle h_{\ell} h_{\ell'} \rangle = o(\alpha) \quad (o(1) \text{ for } \ell = \ell') \quad (68e)$$

$$\langle a_{\ell} a_{\ell'} \rangle = o(\alpha) \quad (68f)$$

We denote the contribution of each graph "n" to the free energy density by $f_2^{(n)}$ and obtain:

$$f_2^{(2)} = +d g_2 \cdot \frac{1-\Delta_3}{\lambda_3} \cdot \frac{1-\Delta_4}{\lambda_4} = o(\alpha^{-2}) \quad (69a)$$

$$f_2^{(3)} = +3d g_3 \cdot \left(\frac{1-\Delta_4}{\lambda_4} \right)^2 = o(\alpha^{-1}) \quad (69b)$$

$$f_2^{(6)} = +\frac{1}{\beta \rho^4} \left(\frac{d}{2} \right) c_1^2 = o(\alpha^{-1}) \quad (69c)$$

$$f_2^{(9)} = +\frac{1}{2} d e_2^2 \left(\frac{1-\Delta_4}{\lambda_4} \right)^2 \frac{\lambda + 3\beta \rho^2}{1 - \lambda_3(\gamma + 3\beta \rho^2)} = o(\alpha^{-1}) \quad (69d)$$

$$\begin{aligned}
 f_2^{(11)} &= + e_2^2 \left(\gamma + \gamma d \left(\frac{1 - \Delta_u}{\lambda_u} \right)^2 \right. \\
 &\quad + 4 \left(\frac{d}{2} \right) \cdot \frac{c_1^2}{\beta \rho^2 \lambda_u^4 (1 - \gamma \lambda_3)^2} \\
 &\quad + 12 \left(\frac{d}{2} \right) \cdot \frac{\beta^2 \rho^4 \lambda_3}{\lambda_u^2} (1 - 2 \Delta_u) \\
 &\quad + (d-1)^2 \frac{\lambda_3}{\lambda_u} c_2 \} + o(\alpha^{-3}) = o(\alpha^{-1}) \quad (69e) \\
 f_2^{(14)} &= + 2 d (d-1)^2 \cdot \frac{\lambda_3}{\rho^2} c_1^2 + o(\alpha^{-3}) = o(\alpha^{-2}) \quad (69f) \\
 f_2^{(16)} &= + \frac{\lambda_3}{\rho^2} c_3 + o(\alpha^{-3}) = o(\alpha^{-2}) \quad (69g) \\
 f_2^{(20)} &= - 4 \left(\frac{d}{2} \right) \cdot \frac{e_2}{\rho} \cdot c_1 \frac{1 - \Delta_u}{\lambda_u} \cdot \frac{1}{1 - \lambda_3 (\gamma + 3 \beta \rho^2)} \\
 &\quad = o(\alpha^{-1}) \quad (69h) \\
 f_2^{(22)} &= + \left(\frac{d}{2} \right) \frac{e_2 \lambda_3}{\rho \lambda_u^2} \cdot c_4 + o(\alpha^{-3}) = o(\alpha^{-2}) \quad (69i) \\
 f_2^{(24)} &= 2 (d-1) c_1 \cdot \frac{\lambda_3}{\rho^2} + o(\alpha^{-3}) = o(\alpha^{-2}) \quad (69k) \\
 f_2^{(26)} &= - \frac{e_2}{\rho} \cdot \frac{1 - \Delta_u}{\lambda_u} \cdot \frac{1}{1 - \lambda_3 (\gamma + 3 \beta \rho^2)} = o(\alpha^{-1}) \quad (69l)
 \end{aligned}$$

The contributions of the remaining graphs are of order α^{-3} or smaller,

namely

$$\begin{aligned}
 o(\alpha^{-3}): & f_2^{(5)}, f_2^{(8)}, f_2^{(10)}, f_2^{(18)}, f_2^{(19)}, f_2^{(25)}, f_2^{(27)} + f_2^{(28)}; \\
 o(\alpha^{-4}): & f_2^{(1)}, f_2^{(13)}, f_2^{(15)}, f_2^{(21)}, f_2^{(23)}; \\
 o(\alpha^{-5}): & f_2^{(4)}, f_2^{(7)}, f_2^{(17)}; \\
 o(\alpha^{-6}): & f_2^{(12)}.
 \end{aligned}$$

Δ_3 and Δ_u denote the diagonal elements of the matrices $D_{3,zz}$, (41) and $D_{4,zz}$, (47) respectively. In particular we find

$$\Delta_u = \frac{1}{\beta \rho^2 \lambda_u} F(d) \quad (70)$$

with the integral [9]

$$F(d) = \int dk \frac{1}{\lambda(k)} \quad (71)$$

$$F(4) = 0.15493 \quad (72)$$

In terms of this number we get

$$c_1 = \int dk \frac{\cos k}{\lambda(k)} = F(d) - \frac{1}{2d} \quad (73)$$

$$= 0.02993 \quad \text{for } d=4 \quad (74)$$

Other integrals used are

$$I_a = \int dk \frac{\cos^2 k}{\lambda(k)} = \frac{1}{4d} + \frac{1}{8d^2} + \frac{1}{4d^3} + \frac{9}{16d^4} + o(d^{-5}) \quad (75)$$

$$= 0.0764 \quad \text{for } d=4, \quad (76)$$

and

$$I_b = \int dk \frac{\cos k}{\lambda(k)} \frac{\cos k}{\lambda(k)} \frac{1}{\lambda(k)}, \quad \mu \neq \nu \quad (77)$$

which can be expressed by I_a and c_1

$$I_a(d) + (d-1) I_b(d) = d c_1(d) \quad (78)$$

implying

$$I_b(4) = 0.0144. \quad (79)$$

The remaining constants c_2, c_3, c_4 are expressed in terms of F, c_1, I_a, I_b by

$$c_2 = \frac{4d}{d-1} [2 F^2 + 2 c_1^2 + 2 I_a^2 - 2 F I_a + (d-1) I_b^2] \quad (80)$$

$$= 0.205 \quad \text{for } d = 4 \quad (81)$$

$$c_3 = 2 \binom{d}{2} [3 F^2 + 2 c_1^2 + I_b^2 + 4 I_a I_b - 2 F I_b - 4 c_1 I_a] \quad (82)$$

$$= 0.778 \quad \text{for } d = 4 \quad (83)$$

$$c_4 = 16 c_1 (F + I_b) \quad (84)$$

$$= 0.0811 \quad \text{for } d = 4 \quad (85)$$

The numerical evaluation is done with

$$\gamma = \gamma_0 = 2 \quad (86)$$

The free energy is plotted in Fig.4, the mean energy per plaquette in Fig. 5.

In order to illustrate the size of the different terms we give their values close to the phase transition point β_c (for the weak coupling phase of course) in the following table. Because of their equal functional dependence we have added up $f_2^{(14)}, f_2^{(16)}, f_2^{(24)}$ and $f_2^{(20)}, f_2^{(26)}$. Two things are very remarkable: the general smallness of all terms $f_2^{(n)}$ as compared to $f_1^{(0)}$ and the fact that the sum over all $f_2^{(n)}$ almost exactly cancels. This can probably be traced back to an optimal choice of the parameter γ , see Fig.2 and the comment added.

α	6.5000
β	0.9990
f_0	- 1.5757
$f_1^{(0)}$	+ 2.9997
$f_1^{(m)}$	+ 0.0547
$f_2^{(2)}$	+ 0.0249
$f_2^{(3)}$	- 0.0201
$f_2^{(6)}$	+ 0.0075
$f_2^{(8)}$	+ 0.0191
$f_2^{(11)}$	+ 0.0107
$f_2^{(14)} + f_2^{(16)} + f_2^{(24)}$	+ 0.0157
$f_2^{(20)} + f_2^{(26)}$	- 0.0499
$f_2^{(22)}$	- 0.0034
f	1.4832
E	0.6949

Table:

4. Large γ

The ansatz (1),(4) for the modified action suggests that in the limit $\gamma \rightarrow \infty$ the matrices Z_ℓ are forced to be unitary so that the procedure of embedding the group manifold into the linear matrix space is reversed. We investigate this limit now.

We consider the expectation value of any gauge invariant observable as obtained by the mean field loop expansion and cut this expansion at a fixed loop order n . Then we eliminate γ by (7)

$$\gamma = \frac{\alpha N}{\rho} - \bar{\beta} \rho^2 \quad (87)$$

and let α tend to infinity keeping $\bar{\beta}$ fixed. We propose that this limit exists and that it is equal to the standard $\frac{1}{\bar{\beta}}$ - expansion to the same loop order n .

We test this proposition for the case of the free energy and the $U(1)$ gauge group to loop order two. Using the well known asymptotic expansions of $\omega(\alpha)$ and $\rho(\alpha)$ we obtain

$$f_0 = \binom{d}{2} \bar{\beta} - \frac{1}{2} d \ln 2\pi \alpha + o(\alpha^{-1}) \quad (88)$$

$$f_1^{(0)} = -\frac{1}{2} (d-1) \ln \bar{\beta} + \frac{1}{2} d \ln \alpha - \frac{1}{2} (d-2) \int dk \ln \lambda(k) + \frac{1}{2} \ln 2\pi + o(\alpha^{-1}) \quad (89)$$

$$f_1^{(m)} = o(\alpha^{-1}) \quad (90)$$

Thus

$$\lim_{\gamma \rightarrow \infty} f_0 + f_1^{(0)} + f_1^{(m)}$$

exists and gives the correct free energy density of $\frac{1}{\bar{\beta}}$ perturbation theory including one loop corrections.

In the two-loop approximation we have finite contributions from the graphs 2,3,6,9,11,20,26, of Fig.3. The terms $f_2^{(3)}$, $f_2^{(9)}$, $f_2^{(11)}$ are of order α and do not possess a limit each. But their sum does

$$\lim_{\gamma \rightarrow \infty} (f_2^{(3)} + f_2^{(9)} + f_2^{(11)}) = \binom{d}{2} c_1^2 \frac{1}{\bar{\beta}} \quad (91)$$

The other four graphs possess limits and we obtain

$$\lim_{\gamma \rightarrow \infty} f_2^{(2)} = \frac{1}{2} d F \frac{1}{\bar{\beta}} \quad (92)$$

$$\lim_{\gamma \rightarrow \infty} f_2^{(6)} = \binom{d}{2} c_1^2 \frac{1}{\bar{\beta}} \quad (93)$$

$$\lim_{\gamma \rightarrow \infty} f_2^{(20)} = -2 \binom{d}{2} F c_1 \frac{1}{\bar{\beta}} \quad (94)$$

$$\lim_{\gamma \rightarrow \infty} f_2^{(26)} = -\frac{1}{2} F \frac{1}{\bar{\beta}} \quad (95)$$

(see (71),(73) for the notations c_1 and F).

The total sum is

$$\lim_{\gamma \rightarrow \infty} \sum_{n=1}^{28} f_2^{(n)} = \frac{d-1}{4d} \cdot \frac{1}{\bar{\beta}} \quad (96)$$

which is the well-known two-loop result [10].

The parameter γ interpolates between the mean field loop expansion and the $\frac{1}{\bar{\beta}}$ loop expansion. It is conceivable then that one should keep γ close to the value γ_0 to obtain good asymptotic behaviour in the transition domain where the $\frac{1}{\bar{\beta}}$ expansion converges rather badly. For comparison

we have plotted the energy per plaquette E obtained from (88), (89), (96),

$$E = 1 - \frac{1}{d\beta} - \frac{1}{2d^2\beta^2} \quad (97)$$

in Fig. 5.

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Figure Captions

1. The turnover point $\bar{\beta}_t$ as function of γ . The upper curve is for $U(1)$, the lower curve for $U(\infty)$. The $U(1)$ curve ends at $\bar{\beta}_{t,0} = 1$, $\gamma_0 = 2$.
2. The transition point β_c as a function of γ in tree approximation (curve 1) and one loop approximation (curve 2). Curve 3 is the turnover point β_t as a function of γ . The best value for β_c is obtained for γ between one and three. All curves are for $U(1)$ and $d = 4$.
3. The twenty-eight two-loop graphs for the free energy density of the $U(1)$ gauge model. Dotted lines denote ghost propagators, solid lines denote the propagators of the fluctuating fields as indicated. Numbering of the graphs corresponds to the order of the terms in (67).
4. The free energy density for the $U(1)$ model with $d = 4$ in the transition region. (A) denotes the mean field curve in two-loop approximation for the weak coupling phase; (B) is the curve for the strong coupling phase using the strong coupling series to tenth order. The curves intersect at $\beta_c = 1.014$.
5. The mean energy per plaquette for the $U(1)$ model with $d = 4$ in the transition region. (A) denotes the mean field curve for the weak coupling phase in two-loop approximation; (B) is the curve of perturbation theory for the weak coupling phase in two-loop approximation (97); (C) is the curve for the strong coupling phase using the strong coupling series to tenth order. Crosses are Monte-Carlo results taken from [6].

FIG. 1

FIG. 2

FIG. 3

FIG. 5

FIG. 4