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CHIRALITY INDEX AND DIMENSIONAL REDUCTION OF FERMIONS

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A B S T R A C T

The number of four-dimensional chiral fermions obtained from dimensional reduction of models with spinor matter fields coupled to pure gravity in $d > 4$ dimensions is linked to topological properties of the internal $d-4$ dimensional space. This gives important restrictions on possible ground states of such models consistent with a realistic four-dimensional unified theory. Connections with spontaneous symmetry breaking and Yukawa couplings of fermions in unified theories are discussed.

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In realistic four-dimensional unified gauge theories the unification group is spontaneously broken : non-singlet scalar fields acquire non-vanishing vev's and break G down to a subgroup H . The fermions which are chiral with respect to G may be vectorlike with respect to H . At the same time they may acquire masses through their Yukawa couplings to the vev's. Obviously, the number of chiral fermions at every step of spontaneous symmetry breaking and for every chain of spontaneous symmetry breaking is bounded from above by the number of chiral fermions with respect to the unification group $\bar{G}$. Again, a corresponding feature exists in Kaluza-Klein theories and we state the following theorem.

We define $\mathcal{M}^D$, the class of all metric spaces compatible with a given topology and differentiable structure on a compact D dimensional manifold and with negative (or positive) definite signature of the metric. Assume that $\mathcal{M}^D$ admits a maximal symmetry group $\bar{G}$ in the sense that every possible symmetry group G of a metric space in $\mathcal{M}^D$ is a subgroup of $\bar{G}$ and that there is only one $\bar{G}$ deformation class in $\mathcal{M}^D$. Then a chirality index $\bar{N}(\mathcal{M}^D)$ can be defined so that the number of chiral fermions is bounded from above by $\bar{N}$ for every metric space in $\mathcal{M}^D$.

In Kaluza-Klein theories, the unification group of the resulting four-dimensional theory can be identified with the maximal symmetry $\bar{G}$ of $\mathcal{M}^D$. Since the maximal number of chiral fermions is a property of $\mathcal{M}^D$, the existence of chiral fermions is intrinsically linked to the topological properties of the "internal" space ! As a consequence, for $\bar{N} > 0$, the existence of massless fermions is not accidental. No fine tuning of the parameters of the d dimensional theory is needed in order to obtain massless fermions. Given the topology and differentiable structure of the internal space, the existence of massless fermions is uniquely determined by the symmetry of the internal space, as it is the case for the massless spin-one bosons - the gauge bosons - or the massless spin-two boson - the graviton.

In Section 2, we give a detailed derivation of the chirality index $\bar{N}$. We discuss related numbers for the G deformation classes, compare $\bar{N}$ with the Atiyah-Singer index and give some further generalizations of the chirality index. In Section 3 the correspondence between deformations of the internal space and spontaneous symmetry breaking in the effective four-dimensional theory is established. We show how Yukawa couplings arise from dimensional reduction. In Section 4 we give sufficient conditions for a vanishing chirality index. Finally, in Section 5, we use these conditions to restrict possible choices of unified gauge theories which might be obtained from dimensional reduction of pure d dimensional gravity.

1. - INTRODUCTION

Chiral fermions are a crucial concept for unified models of weak, electromagnetic and strong interactions. The combination of Lorentz symmetry and gauge symmetry forbids a mass term for such fermions. (By definition, chiral fermions are Weyl spinors belonging to a complex representation of the gauge group.) This links the scale characterizing the masses of the observed fermions to the scale of spontaneous symmetry breakdown of the weak interaction group $SU(2)_L \times U(1)_Y$. Realistic unified models should have fermions in chiral representations with respect to $SU(3)_C \times SU(2)_L \times U(1)_Y$.

The requirement of chiral fermions restricts the possible choices of unified gauge theories. It makes it very hard to construct theories with generation unification in the context of only one simple gauge group. Chiral fermions are a major problem for all attempts of a unification of gauge interactions with gravity.

We discuss chiral four-dimensional fermions in the Kaluza-Klein type models ^{1),2)} of unification with gravity in the simple case of a d ($d > 4$) dimensional spinor coupled to pure d dimensional gravity. (No example of chiral fermions is known so far for this case. However, examples of chiral fermions from dimensional reduction exist if one introduces ^{3),4)} a supplementary gauge group in d dimensions.) For pure d dimensional gravity coupled to spinors, chiral fermions can only exist for $d = 2 \bmod 4$ and only a Majorana-Weyl spinor for $d = 2 \bmod 8$ can lead to a realistic mass spectrum of the four-dimensional fermions ⁵⁾. As a consequence, a minimal number of 14 supplementary "internal" dimensions is needed. In this paper we use topological considerations to give further restrictions on possible internal spaces admitting chiral fermions.

It is a well-known fact in four-dimensional unified gauge theories that the number of chiral fermions does not depend on vacuum expectation values (VEV's) of scalar fields which are singlets with respect to the unification group. In Kaluza-Klein theories the VEV's of singlets correspond to a change of the metric of the internal space K^D which leaves its symmetry group G unbroken (we call this G deformations). Indeed, we will find that the number of chiral fermions obtained from dimensional reduction on an internal space K^D is invariant with respect to G deformations of this space. [For similar ideas in a somewhat different context, see Ref. 3).]

2. - THE CHIRALITY INDEX

We consider pure d dimensional gravity coupled to a spinorial matter field. The action is invariant under general co-ordinate and Lorentz transformations in d dimensions (we denote this group by gen_d). It contains the usual covariant kinetic term for the spinor and gen_d invariant functions of the vielbein and its derivatives ^{*)}. No other fields are introduced. The spinor belongs to an irreducible representation of gen_d . We make a complete expansion of all fields on a space which is given by the direct product of four-dimensional Minkowski space and a compact D dimensional space K^D (without boundaries) admitting a symmetry group G of co-ordinate transformations. The integration over the co-ordinates y of K^D gives an effective four-dimensional action. The dimensional reduction of spinors has been discussed in detail in Ref. 5). Chiral four-dimensional fermions are forbidden for all gen_d spinors and spaces K^D with two possible exceptions : a Majorana-Weyl spinor in $d = 2 \bmod 8$ dimensions or a Weyl spinor in $d = 6 \bmod 8$ dimensions. For these cases the existence of chiral four-dimensional fermions depends explicitly on the space K^D . (The case $d = 6 \bmod 8$ is not consistent with phenomenology because of an unwanted doubling of all four-dimensional fermions.) Here we will show that if a space K^D admits chiral four-dimensional fermions, this is not an accident obtained by some special choice of the parameters of the d dimensional theory, but rather a feature intrinsically related to the topology of K^D .

We expand the d dimensional spinor [for details, including the notation, see Ref. 5)]

$$\psi(y, x) = \psi_i(y) \varphi_i(x) \quad (1)$$

The massless four-dimensional fermions $\phi_{0,j}(x)$ correspond to zero eigenmodes of the D dimensional Dirac operator

$$\Gamma^\alpha D_\alpha \psi_{0,j}(y) = 0 \quad (2)$$

For d even, the Weyl constraint links the zero eigenmodes contained in the functions

$$\begin{aligned} \psi^+(y) &= \frac{1}{2}(1 + \bar{\Gamma}_D) \psi(y) \\ \psi^-(y) &= \frac{1}{2}(1 - \bar{\Gamma}_D) \psi(y) \end{aligned} \quad (3)$$

^{*)} The d dimensional curvature scalar is such a function, but it may be useful to include other invariants ⁶⁾.

($\bar{1}_D$ is the D dimensional version of the γ^5 matrix) to left-handed and right-handed four-dimensional fermions : left-handed fermions are obtained from the expansion of $\psi^+(y)$, and right-handed fermions arise from $\psi^-(y)$.

Let n_C^+ be the number which denotes how many times a complex representation C of the group G appears in the expansion of $\psi^+(y)$, with corresponding numbers $n_C^-, n_C^+, n_{\bar{C}}^-, n_{\bar{C}}^+$, where $\bar{C}$ is the complex conjugate to C . The number N_C ,

$$N_C = f_d (n_C^+ - n_C^- - n_{\bar{C}}^+ + n_{\bar{C}}^-) \quad (4)$$

denotes the total number ^{*)} of unpaired left-handed fermions transforming as a representation C under the gauge group G . The corresponding right-handed fermions belonging to $\bar{C}$ are obtained by the usual charge conjugation operation. With "unpaired left-handed fermions", we mean the following : whenever we have two fermions corresponding either to ψ_C^+ and $\psi_{\bar{C}}^-$ or to ψ_C^+ and $\psi_{\bar{C}}^+$, we pair them to a vectorlike representation and discard them from the counting of chiral representations. For the massive four-dimensional fermions, the pairing is obvious : every mass eigenstate involves a left-handed and a right-handed fermion. (This also holds for Majorana spinors, where the left-handed and the right-handed fermions are charge conjugate to each other.) For massless spinors the pairing is arbitrary. However, since we expect only a finite number of massless fermions, the number N_C is well-defined ^{**)} . For $d = 2 \bmod 4$, D dimensional charge conjugation implies ⁵⁾

$$n_C^+ - n_{\bar{C}}^- = 0, \quad n_{\bar{C}}^+ - n_C^- = 0 \quad (5)$$

For a Majorana-Weyl spinor in $d = 2 \bmod 8$, the Majorana constraint identifies the spinors ψ_C^+ and $\psi_{\bar{C}}^-$ and we account for this by the factor f_d which is $\frac{1}{2}$ for a Majorana-Weyl spinor ^{***)} and 1 for a Weyl spinor. Gauge symmetry combined with Lorentz symmetry forbids a mass term for N_C generations of left-handed fermions belonging to C . Denoting by d_C the dimension of the representation C

^{*)} For $N_C < 0$ we take the positive number $N_{\bar{C}}$.

^{**)} This prescription defines N_C even for the case where n_C^+, n_C^- , etc., are not finite.

^{***)} N_C remains integer due to Eq. (5).

one obtains by summing ^{*)} over all complex representations contained in $\psi(y)$

$$N = \sum_c d_c |N_c| \quad (6)$$

It follows that the total number of massless fermions $n_0(K^D)$ obtained from a dimensional reduction on K^D is bounded from below by N

$$n_0(K^D) \geq N \quad (7)$$

Consider now a different space $\tilde{K}^D$ which can be obtained from K^D by a continuous change of the vielbein and the metric, where the same co-ordinates are used ^{**)} to parametrize K^D and $\tilde{K}^D$. We call such a transformation a deformation from K^D to $\tilde{K}^D$. Such a deformation can always be written as a differentiable function of some deformation parameters s so that $s = 0$ corresponds to the identity transformation. This allows us to write finite deformations as integrals of infinitesimal deformations. It is trivial to define an inverse deformation from $\tilde{K}^D$ to K^D . K^D and $\tilde{K}^D$ have the same topology and we call the equivalence class of all spaces which can be obtained by deformations from K^D the deformation class $\mathcal{M}^D$ ^{***)}. A G (invariant) deformation is defined as a deformation which can be written as an integral of infinitesimal deformations leaving the symmetry group G unbroken. The spaces which are obtained by G deformations from K^D form the G deformation class $\mathcal{M}_G^D$. Note that $\mathcal{M}_D$ may contain several distinct G deformation classes with the same gauge group G .

G deformations leave the numbers N_c unchanged and every space $\tilde{K}^D$ in the G deformation class of K^D has the same number of chiral fermions N !

^{*)} We do not sum over C and $\bar{C}$ separately. Otherwise we would have to include a factor $\frac{1}{2}$ in the definition of N .

^{**)} This is for convenience. The situation remains of course unchanged if we perform co-ordinate transformations on $\tilde{K}^D$ or K^D .

^{***)} We use the name "deformation class" because of its intuitive physical meaning. In a more mathematical language $\mathcal{M}^D$ is the class of all metric spaces consistent with a given topology and a given differentiable structure on a D dimensional compact manifold with negative (or positive) definite signature of the metric. Indeed, if $g_{1\alpha\beta}$ and $g_{2\alpha\beta}$ are two metrics consistent with a given topology and differentiable structure, $g_{3\alpha\beta} = (1-s)g_{1\alpha\beta} + s g_{2\alpha\beta}$ is also a metric of this type for all s , $0 \leq s \leq 1$, and the spaces with metric $g_{1\alpha\beta}$ and $g_{2\alpha\beta}$ belong to the same deformation class.

Indeed, we may expand an arbitrary spinor function $\psi(y)$ in eigenmodes of the Dirac operator on K^D and representations H of the gauge group G :

$$\psi(y) = \psi_{nHmk}(y) \varphi_{nHmk} \quad (8)$$

with

$$\Gamma^\alpha D_\alpha \psi_{nHmk}(y) = m \psi_{nHmk}(y) \quad (9)$$

and

$$\delta_G \psi_{nHmk}(y) = -\theta^z S_z \psi_{nHmk}(y) = -\theta^z \psi'_{n'H'm'k'}(y) (T_z)_{n'n} \quad (10)$$

Here δ_G is an infinitesimal gauge transformation with parameters θ^z [the operators S_z are given in Ref. 5]. The index n denotes the components of the representation H and k counts how many times a representation H with mass m appears in the expansion of $\psi(y)$. A similar expansion can be done on $\tilde{K}^D$:

$$(\Gamma^\alpha D_\alpha)' \psi'_{n'H'm'k'}(y) = m' \psi'_{n'H'm'k'}(y) \quad (11)$$

$$S_z' \psi'_{n'H'm'k'}(y) = \psi'_{n''H'm'k'}(y) (T_z)_{n''n'} \quad (12)$$

Since the ψ_{nHmk} are a complete system of functions, we can expand every function $\psi'_{n'H'm'k'}(y)$ as

$$\psi'_{n'H'm'k'}(y) = A_{m'm'H'H'm'm'k'k'} \psi_{nHmk}(y) \quad (13)$$

Inversely, we can expand every ψ_{nHmk} in terms of the complete system of functions $\psi'_{n'H'm'k'}$:

$$\psi_{nHmk}(y) = B_{m'm'H'H'm'm'k'k'} \psi'_{n'H'm'k'}(y) \quad (14)$$

Thus A is regular, $BA = 1$, and we have a one-to-one correspondence between linearly independent ψ_{nHmk} and $\psi'_{n'H'm'k'}$. No degrees of freedom are lost or added by a deformation $K^D \rightarrow \tilde{K}^D$. If a deformation is G invariant, we will show that A can be cast in a block diagonal form

$$A_{m'm'H'H'm'm'k'k'} = A_{m'm'k'k'} \delta_{m'm} \delta_{H'H} \quad (15)$$

The representation content of $\psi(y)$ is not changed by a deformation. We may use alternatively a basis of eigenstates of $\bar{T}_D$ (which are not eigenstates of the Dirac operator except for zero eigenmodes)

$$\psi(y) = \psi_{nH\ell i}(y) \varphi_{nH\ell i} \quad (16)$$

with $\psi_{nH\ell+}$ contained in $\psi^+(y)$ and $\psi_{nH\ell-}$ contained in $\psi^-(y)$, ℓ counting how many times H appears in the expansion of ψ_+ or ψ_- . Again, one has, for the G deformations :

$$A_{mm'HH'\ell\ell'ii'} = A_{\ell\ell'} \delta_{nn'} \delta_{HH'} \delta_{ii'} \quad (17)$$

The numbers N_G remain unchanged, since there is a one-to-one correspondence between every representation H contained in $\psi^{+(-)}$ on K^D and $\tilde{K}^D$. It is sufficient to prove Eqs. (15) and (17) for arbitrary infinitesimal G deformations. The operators S_z depend continuously on the deformation parameter, and so do all Casimir operators constructed from S_z and classifying the representations H . Since the Casimir operators have discrete values for a compact group G , they cannot change by an infinitesimal deformation and we obtain the factor $\delta_{HH'}$ in Eqs. (15) and (17). The factor $\delta_{ii'}$ in Eq. (17) is trivial since $\bar{T}_D$ does not depend on the deformation and the factor $\delta_{nn'}$ can always be obtained by an appropriate constant unitary transformation on the ψ'_{nH} .

The eigenvalues m of the Dirac operator also depend continuously on the deformation parameters. In most cases we expect a finite mass gap between the zero mass fermions and the fermions with lowest positive $m^2 = m_1^2$. For sufficiently small deformations we have a correspondence between massless spinors on K^D and "light" spinors on $\tilde{K}^D$: let us denote by δm , $(\delta m)^2 \ll m_1^2$, the maximal change of the mass for ψ_{nHok} or ψ_{nHm_1k} induced by a sufficiently small (but not necessarily infinitesimal) deformation. The "light" fermions are obtained from ψ_{nHok} . They decouple effectively from all "heavy" fermions with $m^2 \gtrsim m_1^2$, since all contributions to their mass (and interactions) from mixing effects with heavy fermions are suppressed by a factor $\delta m/m_1$ at least

$$\psi'_{nHm'k}(\ell)(y) = A_{\ell\ell'} \psi_{nHok}(y) + O\left(\frac{\delta m}{m_1}\right) \quad (18)$$

Their masses m' are small : $m'^2 \leq (\delta m)^2$. Of course, for N non-vanishing, we remain at least with N massless eigenmodes on $\tilde{K}^D$. The "paired" *) fermions of complex representations can be combined to vectorlike fermions and acquire a small mass term.

Massless fermions which are not chiral are often accidental **). A small change in the parameters of the d dimensional theory may result in a small G deformation giving all fermions a mass. Since the typical mass scale of the internal space is the Planck mass, a very accurate fine tuning of parameters would be needed in order to obtain fermions with masses below the scale of weak interactions. On the contrary, the existence of chiral fermions in general should not depend on small parameter changes. If a d dimensional theory admits chiral fermions, this should be the case for a finite range of all the parameters and no fine tuning is needed to obtain massless fermions.

It is easy to generalize for deformations which break the gauge group G to a subgroup $\tilde{G}$: after the deformation, every representation of G is reduced to irreducible $\tilde{G}$ representations which may have different masses. However, for sufficiently small deformations and discrete values of m the mass splitting between $\tilde{G}$ representations obtained from a given G representation H is small compared with a typical separation of mass eigenvalues Δm of the Dirac operator on K^D . We again can establish a correspondence between fermions with mass m on K^D and a fermion multiplet with fine splitting of the masses on $\tilde{K}^D$. Especially, we still have the effective decoupling of the "light" fermions.

However, we now have to define new numbers $\tilde{N}_{\tilde{C}}$ and $\tilde{N} = \sum_{\tilde{C}} d_{\tilde{C}} |\tilde{N}_{\tilde{C}}|$ with respect to the remaining gauge group $\tilde{G}$. Some of the $\tilde{G}$ representations contained in C may be real or pseudoreal with respect to $\tilde{G}$. Another possibility is that new "pairings" are possible between complex fermions $\tilde{C}$. We conclude

*) A mass term is always allowed for real representations. Pseudoreal representations require a similar "pairing" to allow a non-vanishing mass and for an odd number of massless pseudoreal ψ_{noPk} at least one massless representation remains after the deformation.

**) Hitchin has given examples 7) of massless fermions obtained from dimensional reduction on a three-dimensional internal space with the topology of S^3 and a symmetry group $SU(2) \times U(1)$. However, these fermions are not chiral. Massless fermions correspond to particular choices of a parameter and a small G deformation gives small masses of the order δm to all of these fermions.

$$\tilde{N} \leq N \quad (19)$$

and there may be less chiral fermions on $\tilde{K}^D$ than on K^D . This is very similar to what we observe in nature : as long as the group $SU(3)_C \times SU(2)_L \times U(1)_Y$ is unbroken, we have several generations of chiral massless fermions. After spontaneous breakdown of $SU(2)_L \times U(1)_Y$ to $U(1)_{em}$ all complex representations with respect to $SU(3)_C \times U(1)_{em}$ can be paired and the fermions become massive. Another example is the breakdown of the unifying group $SO(10)$ to $SU(3)_C \times SU(2)_L \times U(1)_Y$: the right-handed neutrinos are a real representation with respect to the latter group and the number of chiral fermions is reduced. In the next section, we will give a more detailed description of the relation between chirality numbers like N_C and spontaneous symmetry breakdown.

For the moment, we go one step further and assume that $\mathcal{M}^D$ admits a maximal symmetry group $\bar{G}$ in the sense that for every metric space K^D belonging to $\mathcal{M}^D$ and admitting a symmetry G , this symmetry group G is either $\bar{G}$ or a subgroup of $\bar{G}$, $G \subseteq \bar{G}$. We further assume that all spaces $\tilde{K}^D$ in $\mathcal{M}^D$ admitting the symmetry $\bar{G}$ can be transformed into each other by $\bar{G}$ deformations. (There is only one $\bar{G}$ deformation class in $\mathcal{M}^D$.) Then the numbers N_C , $\bar{N} = \sum_C d_C |\bar{N}_C|$ are the same for all spaces $\tilde{K}^D$ with symmetry $\bar{G}$. With our assumptions, the group $\bar{G}$ is uniquely defined for the deformation class $\mathcal{M}^D$ and the numbers $\bar{N}_C$ and $\bar{N}$ can be seen as properties of $\mathcal{M}^D$ rather than of a special space $\tilde{K}^D$. We call $\bar{N}_C$ the C chirality index and $\bar{N}$ the total chirality index of $\mathcal{M}^D$ *). $\bar{N}$ gives the maximal number of chiral fermions consistent with $\mathcal{M}^D$ and for every space K^D in this class we have

$$N \leq \bar{N} \quad (20)$$

For a given subgroup G of $\bar{G}$, it is possible to use the C chirality indices $\bar{N}_C$ to determine the numbers N_C .

This has important consequences for the question of naturalness of obtaining massless or light fermions from dimensional reduction. If a given sub-space of the parameter-space of the d dimensional theory gives rise to ground

*) We do not know if there are conditions on $\mathcal{M}^D$ for the assumption of a unique maximal symmetry $\bar{G}$ with a unique $\bar{G}$ deformation class to be valid. Is it fulfilled for arbitrary $\mathcal{M}^D$? If not, the definition of the chirality index has to be generalized since one has to consider different G deformation classes separately.

states belonging to a deformation class $\mathcal{M}^D$ with $\bar{N}(\mathcal{M}^D) > 0$, the existence of massless fermions is uniquely determined by the symmetry of the ground state. (Additional "accidental" massless fermions are still possible, but we will not consider this case here.) If the symmetry of K^D is $\bar{G}$ we will find $\bar{N}$ massless fermions. If for some range of parameters within this parameter-subspace the unification group is spontaneously broken to a subgroup G , the numbers $N_{\bar{C}}$ of the corresponding ground state are uniquely determined by the C chirality indices $\bar{N}_C$ of $\mathcal{M}^D$ via reducing the representations C of $\bar{G}$ with respect to G . The number of massless fermions therefore depends only on G and its embedding in $\bar{G}$! Thus chiral massless fermions in Kaluza-Klein theories have the same status of naturalness as the massless gauge bosons. Massless fermions and gauge bosons correspond to some unbroken gauge group G . If this gauge group G is spontaneously broken, a certain well-defined number of fermions and gauge bosons acquire masses. These masses are determined by the scale of spontaneous symmetry breakdown.

The total chirality index $\bar{N}$ can be used as a criterion for excluding a large variety of ground states for a possible construction of a realistic low energy theory out of higher dimensional gravity : all spaces belonging to a deformation class $\mathcal{M}^D$ with $\bar{N}(\mathcal{M}^D) = 0$ do not admit chiral fermions! In Section 4 we discuss some necessary conditions for K^D to admit chiral fermions.

Let us compare the chirality index with the Atiyah-Singer⁸⁾ index I . I denotes the number of massless eigenmodes in $\psi^+(y)$ minus the massless eigenmodes in $\psi^-(y)$. Enlarging our language to real representations R and pseudoreal representations P with dimensions d_R and d_P , one has

$$\begin{aligned} I_C &= n_C^+ - n_C^- \\ I_R &= n_R^+ - n_R^- \\ I_P &= n_P^+ - n_P^- \\ I &= \sum_C d_C (I_C + I_{\bar{C}}) + \sum_R d_R I_R + \sum_P d_P I_P \\ N_C &= f_d (I_C - I_{\bar{C}}) \end{aligned} \tag{21}$$

Massless fermions exist for every non-vanishing I_C , I_R or I_P . Clearly, it is possible that the Atiyah-Singer index of $\mathcal{M}^D$ vanishes whereas the chirality index $\bar{N}$ is different from zero. For the interesting cases $D = 2 \bmod 4$ the existence of a charge conjugation operation for $\psi(y)$ implies⁵⁾ $I = 0$ [compare with Eq. (5)] whereas these are the only cases where $\bar{N}$ may be different from zero!

Let us finally discuss some generalizations of the chirality index. The numbers I_C , I_R , I_P and the eigenvalues of $\Gamma_{\alpha}^{D_D}$ depend only on K^D . The above discussion is therefore valid for dimensional reduction on every space which is a direct product of K^D and an arbitrary E dimensional space. For E even, the notion of left-handed and right-handed fermions can be generalized and mass terms for N_C generations of fermions belonging to the representation C are again forbidden by combined gen_E and gauge symmetry. For E odd, we still have N_C generations of massless fermions, even if a mass term is now allowed by the symmetries of the E dimensional action. For I_R (I_P) odd, the mass term of one generation belonging to the representation $R(P)$ may be forbidden by $\text{gen}_E \times G$ invariance for $E = 2, 6, 8, 9 \bmod 8$ ($E = 2, 4, 5, 6 \bmod 8$). [For details, see Ref. 5) ; especially, for $D = 2 \bmod 4$ the numbers I_R and I_P vanish due to D dimensional charge conjugation.]

So far, the numbers I_H have been defined, Eq. (21), only for D even and are closely related to the existence of Weyl spinors. There is a possible generalization for D odd if the irreducible spinor representations of gen_D are Majorana spinors and the corresponding charge conjugation operator C_D anti-commutes with the Dirac operator. This is the case for $D = 9 \bmod 8$. We now can define ψ^+ and ψ^- as eigenstates of C_D ($C_D \psi^{\pm} = \pm \psi^{\pm}$). The numbers I_H are defined as in Eq. (21) and are again invariant with respect to G deformations. In general we have at least $f_d |I_H|$ generations of massless fermions belonging to a representation H . Here a generation refers to an irreducible $\text{gen}_E \times G$ representation and f_d is given for arbitrary d and E by

$$\begin{aligned}
 f_d &= \tilde{f}_d \times f_E \times f_{ED} \\
 \tilde{f}_d &= \begin{cases} \frac{1}{2} & \text{for Majorana-Weyl spinors } (d = 2 \bmod 8) \\ 1 & \text{for Weyl or Majorana spinors } (d = 3, 4, 6, 8, 9 \bmod 8) \\ 2 & \text{for Dirac spinors } (d = 5, 7 \bmod 8) \end{cases} \\
 f_E &= \begin{cases} 2 & \text{for Majorana-Weyl spinors (with respect to } \text{gen}_E) \\ 1 & \text{for Weyl or Majorana spinors} & " \\ \frac{1}{2} & \text{for Dirac spinors} & " \end{cases} \\
 f_{ED} &= \begin{cases} 2 & \text{for both } E \text{ and } D \text{ odd} \\ 1 & \text{otherwise} \end{cases} \quad (22)
 \end{aligned}$$

A special case exists for pseudoreal representations and $D = 5 \bmod 8$, where I_p may be defined as the number of unpaired pseudoreal representations contained in ψ . For $I_p = 1$ we again have f_d generations of massless fermions [compare Ref. 5)].

A further generalization includes supplementary fields in the d dimensional action. One interesting case is for non-vanishing torsion. Here the gauge group G is still a subgroup of gen_D . We also may start with a supplementary gauge symmetry F in d dimensions where G is now a subgroup of $gen_D \times F$. All our arguments are only based on the G transformation properties of the functions in the expansion of $\psi(y)$. The numbers I_H and N_C can be defined and are invariant under G deformations. For both torsion and supplementary gauge interactions the modified Dirac operator still anticommutes with $\bar{F}_D$ and we have massless fermions for every I_H different from zero.

3. - SPONTANEOUS SYMMETRY BREAKDOWN AND YUKAWA COUPLINGS

The deformations described in the last section can be understood as vacuum expectation values (vev) of four-dimensional scalar fields. Indeed, the space K^D on which we expanded all d dimensional fields was not necessarily the ground state. If the ground state belongs to the same deformation class as K^D it can be interpreted as vev's of scalar fields determined by minimizing the four-dimensional scalar potential. We denote the vielbein of K^D by $e_{i\alpha}^a(y)$. We now make an expansion for the field $e_{i\alpha}^a(y, x)$:

$$e_{i\alpha}^a(y, x) = e_{i\alpha}^a(y) + F_{i\alpha}^a(y) S_i(x) \quad (23)$$

The $F_{i\alpha}^a(y)$ form a complete set of functions and may again be taken as representations of G and eigenfunctions of an appropriate mass operator. The $S_i(x)$ are four-dimensional scalars transforming as a corresponding linear representation of G . The $F_{i\alpha}^a$ have to be properly normalized to obtain the correct normalization for the four-dimensional kinetic term of the scalars ^{*)}. The four-dimensional scalar potential $V(S_i)$ is then obtained from the y integration over all pieces of the action containing only the field $e_{i\alpha}^a(y, x)$ and its derivatives with respect to y ⁶⁾.

^{*)} If $e_{i\alpha}^a(y, x)$ is dimensionless, the dimension of $F_{i\alpha}^a$ is $(\text{mass})^{-1}$.

Consider now a different space $\tilde{K}^D$ in the deformation class of K^D . Its vielbein $e_{2\alpha}^a(y)$ can be expanded according to (23)

$$e_{2\alpha}^a(y) = e_{1\alpha}^a(y) + F_{i\alpha}^a(y) S_i^{(1)} \quad (24)$$

Thus every deformed space $\tilde{K}^D$ corresponds to constant values $S_i^{(1)}$ of scalar fields in the formulation of the model on K^D . Especially, this holds for the ground state K_O^D if it belongs to the same class $\mathcal{M}^D$ as K^D . The choice of K^D within $\mathcal{M}^D$ is completely arbitrary. Every choice gives an equivalent effective four-dimensional theory and corresponds to different possible choices of a point in scalar field space around which we may make a polynomial expansion of the scalar potential. If we start with K_O^D , no spontaneous symmetry breakdown occurs by definition. But it is more convenient to start with a space $\tilde{K}^D$ in $\mathcal{M}^D$ which has the maximal symmetry $\bar{G}$. If the ground state symmetry G_O is smaller than $\bar{G}$, some vev's of G_O singlets transforming non-trivially with respect to $\bar{G}$ must acquire a non-zero vev - this is called spontaneous symmetry breakdown. In a formulation on $\tilde{K}^D$ spontaneous symmetry breakdown always occurs if the ground state symmetry is lower than the maximal symmetry on $\mathcal{M}^D$. In view of the flexibility of the theory due to the free parameters in the action ⁶⁾ we expect this to be a very common feature of higher dimensional gravity. We finally note that the existence of a unique maximal symmetry on $\mathcal{M}^D$ with unique $\bar{G}$ deformation class gives a well-defined meaning to the concept of a unifying gauge group for all gauge interactions : this is just $\bar{G}$, and it is the maximal symmetry so that all symmetry breaking effects can be understood in terms of the Higgs-Kibble mechanism.

In the following, we will always expand on a space $\tilde{K}^D$ with maximal symmetry $\bar{G}$. The masses of the fermions in the ground state can then be understood from their couplings to scalars which acquire non-zero vev's. The fermion masses and non-derivative couplings to scalars are all contained in the contribution to the action from the term

$$\begin{aligned} L_{\psi}^{(m)} &= i \bar{\psi}(y, x) \gamma^\alpha \partial_\alpha \psi(y, x) \\ &= i \bar{\psi}(y, x) \gamma^5 \left[\Gamma^a e_a^\alpha(y, x) \partial_\alpha + \frac{1}{4} \omega_{abc}(y, x) \Gamma^a \Gamma^b \Gamma^c \right] \psi(y, x) \end{aligned} \quad (25)$$

(with $\gamma^a = \gamma^5 \Gamma^a$). We split the inverse vielbein e_a^α and the spin connection ω_{abc} in pieces $\bar{e}_a^\alpha(y)$, $\bar{\omega}_{abc}(y)$ corresponding to the vielbein of $\tilde{K}^D$ and giving $\bar{G}$ invariant masses to the fermions and functions depending on the scalar fields $S_i(x)$ [see Eq. (23)]

$$e_a^\mu(y, x) = \bar{e}_a^\mu(y) + f_a^\mu(y, S_i(x))$$

$$\omega_{abc}(y, x) = \bar{\omega}_{abc}(y) + g_{abc}(y, S_i(x)) \quad (26)$$

The x dependence of f_a^α and g_{abc} is implicit through the fields $S_i(x)$. The functional dependence of f_a^α and ω_{abc} on $S_i(x)$ may be very complicated, in general it is non-polynomial. Expanding the spinors $\psi(y, x)$ and integrating out the y co-ordinates leaves us with effective four-dimensional fermion scalar couplings of the form

$$L_S = \bar{\varphi}_{nn'HH'mm'kk'}(x) \gamma^5 h_{nn'HH'mm'kk'}(S_i(x)) \varphi_{n'H'm'k'}(x) \quad (27)$$

The $h_{nn'HH'mm'kk'}$ are again complicated (non-polynomial) functions of the $S_i(x)$, constrained of course by the requirement that the term (27) is a $\bar{G}$ singlet. The ground state K_0^D corresponds to vev's S_i minimizing the scalar potential and the fermion masses corresponding to the ground state get contributions from the couplings (27) to these vev's. Fermion masses forbidden by $\bar{G}$ invariance (for $N \neq 0$) may be induced by spontaneous symmetry breaking and fermion couplings to S_i transforming non-trivially with respect to $\bar{G}$.

Let us look at the interesting case of a gauge group G (not necessarily the unification group $\bar{G}$) which is only broken at a scale much below the Planck mass M_P . This implies that there is a space K^D [with vielbein $e_{i\alpha}^a(y)$] so that all vev's S_i are much smaller than M_P . For scalar excitations with a characteristic mass scale $M \ll M_P$ we can expand f_a^α and g_{abc} in powers of $S_i(x)$:

$$f_a^\mu(y, S_i(x)) = -S_i(x) F_{ia}^\mu(y) + O\left(\frac{S_i S_j}{M_P}\right)$$

$$g_{abc}(y, S_i(x)) = -S_i(x) F_{ia}^d(y) \omega_{adb c}(y)$$

$$+ S_i(x) (D_c F_{iba}(y) - D_b F_{ica}(y)) + O\left(\frac{S_i S_j}{M_P}\right) \quad (28)$$

with

$$F_{ia}^\mu(y) = e_{ia}^\beta(y) F_{i\beta}^\mu(y) e_{ib}^\alpha(y)$$

$$F_{iab}(y) = F_{ia}^{\alpha}(y) e_{\alpha b}(y)$$

$$D_c F_{iba} = e_{ic}^{\alpha} \partial_{\alpha} F_{iba} + \omega_{acb}^d F_{ida} + \omega_{aca}^d F_{ibd} \quad (29)$$

The full expansion of (25) up to terms of the order M/M_p reads :

$$\begin{aligned} L_{\psi}^{(m)} = & i \bar{\varphi}_{nHm\mathbb{E}}(x) \gamma^5 \varphi_{n'H'm'\mathbb{E}'}(x) \cdot \\ & \left\{ m \psi_{nHm\mathbb{E}}^{\dagger}(y) \varphi_{n'H'm'\mathbb{E}'}(y) - S_i(x) \left[\frac{1}{2} \psi_{nHm\mathbb{E}}^{\dagger}(y) \Gamma_1^{\beta} \right. \right. \\ & \psi_{n'H'm'\mathbb{E}'}(y) (D_{\alpha} F_{i\beta}^{\alpha}(y) - D_{\beta} F_{i\alpha}^{\alpha}(y) + \\ & \left. \left. + \psi_{nHm\mathbb{E}}^{\dagger}(y) \Gamma_1^{\beta} D_{1\alpha} \psi_{n'H'm'\mathbb{E}'}(y) F_{i\beta}^{\alpha}(y) \right] \right\} \end{aligned} \quad (30)$$

with

$$\begin{aligned} D_{1\alpha} \psi &= (\partial_{\alpha} - \frac{1}{2} \omega_{1\alpha bc} \Sigma^{bc}) \psi \\ D_{\gamma} F_{i\alpha}^{\beta} &= \partial_{\gamma} F_{i\alpha}^{\beta} - \Gamma_{1\gamma\alpha}^{\delta} F_{i\delta}^{\beta} + \Gamma_{1\gamma\delta}^{\beta} F_{i\alpha}^{\delta} \\ \Gamma_1^{\beta} &= \Gamma^{\alpha} e_{1\alpha}^{\beta} \end{aligned} \quad (31)$$

where $\Gamma_{1\gamma\alpha}^{\delta}$ is the Christoffel-symbol corresponding to the vielbein $e_{1\alpha}^a$. The integration over the y co-ordinates leaves us with Yukawa-couplings between fermions and scalars :

$$L_S = i h_{i, mm'HH'mm'\mathbb{E}\mathbb{E}'} \bar{\varphi}_{nHm\mathbb{E}}(x) \gamma^5 \varphi_{n'H'm'\mathbb{E}'}(x) S_i(x) \quad (32)$$

with

$$\begin{aligned} h_{i, mm'HH'mm'\mathbb{E}\mathbb{E}'} = & - \int d^D y e_D \left\{ \frac{1}{2} \psi_{nHm\mathbb{E}}^{\dagger} \Gamma_1^{\beta} \psi_{n'H'm'\mathbb{E}'} \cdot \right. \\ & \left. (D_{\alpha} F_{i\beta}^{\alpha} - D_{\beta} F_{i\alpha}^{\alpha}) + \psi_{nHm\mathbb{E}}^{\dagger} \Gamma_1^{\beta} D_{1\alpha} \psi_{n'H'm'\mathbb{E}'} F_{i\beta}^{\alpha} \right\} \end{aligned} \quad (33)$$

We note at this place that the next-to-leading order gives terms of the form

$$i \frac{\alpha_{ij}}{M_P} \bar{\psi} \gamma^5 \psi S_i S_j \quad (34)$$

which could play a rôle for the fermion masses of the lowest generation ⁹⁾ or for right-handed neutrinos ¹⁰⁾. The case $S_i \ll M_P$ corresponds to the above-discussed case of "small deformations". If the typical mass gap between zero mass fermions and the lowest massive fermions on K^D is of the order M_P , we can consider an effective theory of massless fermions with Yukawa couplings ^{*}). In this scenario, spontaneous symmetry breakdown may even occur in steps at different mass scales below M_P . If we can find spaces with non-vanishing $\bar{N}$, it may well be possible to construct a theory resembling very much what we observe in Nature !

4. - CASES OF VANISHING CHIRALITY INDEX

In the light of our above discussion we can restrict our problem to the question whether some spaces K^D with maximal symmetry $\bar{G}$ admit chiral fermions. The interesting cases are for a non-vanishing total chirality index $\bar{N}$ even if the number of chiral fermions in the ground state may finally vanish due to spontaneous symmetry breakdown. However, there are some constraints limiting the possibilities of having $\bar{N} \neq 0$. In this short section we give several criteria implying $\bar{N} = 0$. Such cases have to be avoided for the construction of a realistic theory.

A. - The index $\bar{N}$ depends on the d dimensional spinor we start with. As a result of Ref. 5), we find $\bar{N} = 0$ for all d dimensional spinors except for Weyl or Majorana-Weyl spinors in $d = 2 \bmod 4$ dimensions, where $\bar{N}$ depends on the specific $\mathcal{M}^D$.

B. - The chirality index vanishes for all $\mathcal{M}^D$ where the maximal symmetry group $\bar{G}$ has no complex representations. For example, this is the case for $\bar{G} = SO(N)$ except for $N = 2 \bmod 4$. Thus the maximal symmetric spaces S^D with symmetry $SO(D+1)$ do not admit chiral fermions.

^{*}) If we want to consider effective terms of the order $1/M_P$ as in Eq. (34), we have also to include mixing effects between the light and heavy fermions which are of the same order.

C. - There is a powerful theorem due to Lichnerowicz ¹¹⁾ that no massless spinors exist for compact spaces (without boundaries) and with D dimensional curvature scalar ^{*)} $R \leq 0$ but not identically zero. Such spaces do not admit chiral fermions and $\bar{N}$ vanishes if $\bar{K}^D$ can be $\bar{G}$ deformed into a space of this kind. Furthermore, we may apply Lichnerowicz's theorem to spaces which are direct products of subspaces where one subspace has $R \leq 0, R \neq 0$. The chirality index vanishes if the $\bar{G}$ deformation class of $\bar{K}^D$ contains a space which is a direct product of a D' dimensional compact space with $R \leq 0, R \neq 0$ and an arbitrary $D-D'$ dimensional space.

5. - RESTRICTIONS ON UNIFIED GAUGE THEORIES OBTAINED FROM DIMENSIONAL REDUCTION

Unfortunately, no example of a non-vanishing number of chiral fermions is known so far in models of a d dimensional spinor coupled to pure d dimensional gravity. The few spaces for which a systematic expansion of D dimensional spinors has been discussed all fulfil one of the no-go criteria of Section 4. However, these no-go criteria are not exhaustive and we see so far no reason why chiral fermions should not exist for relatively simple spaces with non-negative curvature. Waiting for a solution of this problem, we may use the results of Section 4 for restrictions on possible unification groups (depending on the number of dimensions).

The most popular unification groups without family unification are $SU(5)$ ¹²⁾, $SO(10)$ ¹³⁾ and E_6 ¹⁴⁾. The minimal dimension for $SU(5)$ [$SO(10)$] symmetric spaces is 8 (9). A large variety of topologically distinct fourteen-dimensional $SU(5)$ or $SO(10)$ symmetric spaces exists and only a few special cases have been classified so far. Are there $SU(5)$ or $SO(10)$ symmetric spaces admitting chiral fermions for $D = 14$? E_6 symmetric spaces have $D \geq 22$. The space with minimal dimension is the homogeneous space $E_6/SO(10) \times U(1)$ which has a negative scalar curvature ¹⁵⁾. E_6 unification therefore needs $D \geq 30$. $SU(8)$ has been proposed as a unification group including the unification of the different fermion generations. The only $D = 14$ $SU(8)$ symmetric space, $SU(8)/SU(7) \times U(1)$, has a negative scalar curvature and this type of unification needs $D \geq 22$.

*) We use conventions where R is negative for spheres.

For realistic theories with $SO(10)$ unification, the $SO(10)$ group should not be embedded in a higher orthogonal group $SO(N)$, $N > 10$. Otherwise, the fermions cannot be chiral with respect to $SO(10)$ ¹⁶⁾. For $D = 14$, a gauge group with generation unification must be of the form $SO(10) \times G_F$ where G_F is a family group. The maximal group G_F consistent with $D = 14$ is $G_F = SO(6)$. This is only realized for $K^D = S^9 \times S^5$. Chiral fermions cannot exist in this case and $G_F = SO(6)$ is excluded.

A more interesting family group is $G_F = SO(5)$. There are several topologically different fourteen-dimensional spaces with $SO(10) \times SO(5)$ symmetry. It is intriguing to note that the number of components of a Majorana-Weyl spinor in $d = 18$ corresponds to the degrees of freedom needed for four generations of fermions within $SO(10)$ unification. Is it possible to find a fourteen-dimensional space with $SO(10) \times SO(5)$ symmetry and admitting chiral fermions with ψ_+ transforming as $(16, 4)$? The flat $D = 15$ space has massless spinors transforming as $(16, 4) + (\overline{16}, 4)$ with respect to a $SO(10) \times SO(5)$ symmetry. Is it possible to project this spinor on a suitable $D = 14$ space so that a massless $(16, 4)$ is contained in ψ_+ and a massless $(\overline{16}, 4)$ is contained in ψ_- ? This would lead to four generations of chiral fermions transforming as the four-dimensional spinor representation of $SO(5)$. Needless to say that this would be an interesting attempt to unification, but so far this is only speculation. A more detailed study of fourteen dimensional $SO(10)$ or $SU(5)$ symmetric spaces is needed.

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