A STUDY OF THE NON-COMPACT NON-LINEAR $\sigma$ MODEL:
A SEARCH FOR DYNAMICAL REALIZATIONS OF NON-COMPACT SYMMETRIES

Y. Cohen $^{*+}$ and E. Rabinovici $^{*+}$
CERN -- Geneva

ABSTRACT
The issue of dynamical linear realizations of non-compact symmetries is discussed. The two-dimensional non-compact O(N,1)/O(N) is studied using various approximations. Monte Carlo simulations of the O(2,1)/O(2) non-linear $\sigma$ model are performed. No significant deviations from the perturbative region are detected.

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In this paper we report on Monte Carlo simulations performed on the O(2,1) non-linear sigma model in two dimensions. This study is done in an attempt to understand the structure of field theories with an internal non-compact symmetry.

The idea of classifying the fundamental particles in unitary representations of a non-compact group has been tried in the past\(^1\). However, this approach was abandoned in favour of the so-called standard SU(3) × SU(2) × U(1) model, in which the fundamental particles fall into unitary representations of compact groups.

In addition to the phenomenological success of compact models, it was also not clear how to formulate a sensible field theory symmetric under a global non-compact internal symmetry G. The most naive objection to such theories was that their action was not classically positive definite.

Cremmer and Julia\(^2\), in their attempts to make N = 8 supergravity accessible to research, have solved this problem en passant. They have shown that by gauging the maximal compact subgroup H of G, a field theoretical non-linear realization, in terms of G/H, may be possible. In particular, they have shown that N = 8 supergravity can be cast in the form G/H, reminiscent of the CP\(^{N-1}\) model\(^3\), H being a local SU(8) symmetry and G a global non-compact E\(_7\) symmetry. The theory possesses a formal local SU(8) symmetry, but no perturbative SU(8) gauge fields. This work encouraged Ellis, Gaillard, Maini and Zumino\(^4\) to try and derive the low-energy effective Lagrangian of N = 8 supergravity. Stimulated by that work, we have re-examined the analogy with CP\(^{N-1}\) models. This has led one of us\(^5\) to propose that under the assumption that N = 8 supergravity exhibits dynamical phenomena similar to CP\(^{N-1}\), emphasis should be put not only on the formation of SU(8) bound states, but also on the possibility of the restoration of the global symmetry G. It has been shown\(^6\) that one could analyze the CP\(^{N-1}\) model from that point of view, the U(1) gauge fields serving to restore the global SU(N)/Z(N) symmetry. [In the supersymmetric O(3) model, the global symmetry is not restored.] Thus, in the N = 8 case the symmetry to be restored is non-compact, and its unitary linear representations would be infinite dimensional. The insufficiency of finite dimensional representations became apparent to Ellis, Gaillard and Zumino\(^7\) from other points of view. One is thus faced with two questions, the first being whether it will still be possible to relate N = 8 supergravity to a "conventional" low-energy particle spectrum as envisaged by EGZ, and the second question being whether there exists a meaningful field theoretical linear realization of a non-compact symmetry.

In this note, we discuss the second question. As an attempt will be made actually to derive results, we will abandon four dimensions in favour of the
neighbourhood of two dimensions, where an arsenal of techniques has been devised to study the linearization of compact systems. The results of our searches for linear realizations are at best inconclusive; in some cases they are negative. We hope that this reflects our inability and we present our results in order that they may stimulate more research into the problem.

In studying non-perturbative properties of four-dimensional gauge theories, it was found very useful to analyze their two-dimensional counterparts, namely theories with a global $O(N)$ symmetry. It was shown, for example, that in two-dimensions, the $O(N+1)/O(N)$ non-linear $\sigma$ model parametrized perturbatively by $N$ Goldstone bosons is actually realized linearly, the low-energy excitations forming a degenerate massive $(N+1)$ multiplet. This was shown to be true for large $N$, and is expected to be true down to $O(3)/O(2)$ (for $\theta = 0$). One should note that in the supersymmetric $\text{CP}^1$ this is not true. In $2\varepsilon$ dimensions both phases are possible, the symmetric one manifesting itself in the strong coupling region. Using a lattice regularization, these results are reconfirmed with additional structures appearing for $N = 2$. In that case, a Kosterlitz-Thouless transition separates the two phases.

A general characteristic of two-dimensional non-linear $\sigma$ models was discovered by D. Frieden. He has shown that the coefficient of the leading term in the $\beta$ function in these models is equal to the curvature of the field manifold, thus compact theories are asymptotically free, while the non-compact theories are infra-red free. Infra-red free field theories do not carry clout. It is claimed that they are trivial in the dimensions in which they are non-normalizable, and maybe even in the dimension in which they are renormalizable. These claims were worked out for $\varphi^4$ field theory. Nevertheless, one can show that the $O(N,1)/O(N)$ propagator is given, in the one-loop approximation, by $(1-c \log p^2)^N / (N-1) / p^2$ ($c > 0$). Thus the propagator has a zero instead of a Landau pole. In any case, a non-compact model in two dimensions with a zero $\beta$ function could have been a better study case. One could also study the theory in $2\varepsilon$ dimensions, where it would be asymptotically free; however, we have not yet done that.

We also note that for large $N$ and in perturbation theory, the two-loop $\beta$ function has a non-trivial ultra-violet fixed point. As the value of the fixed point is large in both cases, we cannot rely on its existence, but one could enquire if it persists using a lattice regularization. With these reservations in mind, we proceed.

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*) Asymptotically free non-compact theories appear in the study of the localization problem in solid state physics. See Ref. 13 for some references.
The Lagrangian of the $O(N,1)/O(N)$ non-linear $\sigma$ model is given by

$$L = \partial_\mu \vec{\pi} \cdot \partial_\mu \vec{\pi} - \vec{\sigma} \cdot \vec{\sigma}$$  \hspace{1cm} (1)

where the $N+1$ component vector $(\vec{\pi},\sigma)$ is constrained to one of the sheets of the two-sheeted hyperboloid

$$\vec{c}^2 - \vec{\pi} \cdot \vec{\pi} = \frac{1}{g}$$  \hspace{1cm} (2)

The Lagrangian (1) constrained by (2), as well as the corresponding Hamiltonian, is classically semi-positive definite.

In the compact case, a large $N$ Hamiltonian analysis demonstrates that the $O(N+1)$ symmetry is restored linearly\(^6,9\). Using a similar analysis for (1) and assuming that the excitations of the system are of the single particle type, one obtains once again the gap equation; however, it has only the trivial massless solution. The $N = \infty$ limit, if not singular, shows that no symmetry restoration occurs. This result does not rule out the possibility that the system has no particle interpretation\(^15\), but has instead a very complicated analytic structure. For example, $(\log p^2)^\alpha$ terms in the inverse propagator for $p^2 = 0$ would be a way to circumvent Coleman's theorem.

Another method used to extract non-perturbative results in the compact case was the lattice regularization. In particular, we have studied the $O(2,1)/O(2)$ case. The lattice formulation of the model is given by

$$L = \frac{1}{g} \sum_{i,r} \left\{ \sigma(i) \sigma(i + e_r) - \vec{\pi}(i) \cdot \vec{\pi}(i + e_r) \right\} \equiv \frac{1}{g} \sum_{i} \vec{s}(i) \cdot \vec{s}(i + e_r) \hspace{1cm} (3)$$

where

$$1 = \sigma(i) - \vec{\pi}(i) \cdot \vec{\pi}(i) = \vec{s}(i) \cdot \vec{s}(i) \hspace{1cm} ; \hspace{1cm} \vec{s}(i) = (\sigma(i),\vec{\pi}(i)) \hspace{1cm} (4)$$

The Hamiltonian of the system is given by

$$H = g \sum_{x} c^*(x) + \sum_{x} \left( \sum_{a=1}^{2} \Pi_a(x) \Delta^a \Pi_a(x) - \sigma(x) \Delta^* \sigma(x) \right)$$  \hspace{1cm} (5)$$
where $C^2$ is the Casimir operator of $O(2,1)$ and $\Delta^\alpha A(x) = A(x+\alpha) - A(x)$.

Had the nearest neighbour potential been obviously bounded, as in the $O(3)$ case\textsuperscript{11}, a meaningful strong coupling expansion could have been applied. For $g$ equal strictly to infinity, the system would reduce to many decoupled quantum mechanical systems. The energy levels of each such system would correspond to the eigenvalues of $O(2,1)$ Casimir operating on functions defined on the $O(2,1)/O(2)$ manifold\textsuperscript{16}. There exists a singlet state; however, it is not normalizable, the excited states having eigenvalues of the form $1/4+\alpha^2$, $\alpha$ being a continuous real variable. These states are still not perfectly normalizable, but their divergences are much milder than those of the singlet state. The absence of a gap in the spectrum of these states will be a further obstacle in the detection of a transition. However, since the kinetic part is not obviously bounded, it may well be that, just as in the free field theory, the strong coupling expansion has zero radius of convergence\textsuperscript{17}.

On the lattice one can also apply numerical methods to analyze the system. We have thus performed a Monte Carlo (MC) simulation of the $O(2,1)/O(2)$ model as given by Eq. 4). In order to avoid perturbative infra-red divergences, we study, as would be suggested by an extension of Eliitzur's theorem\textsuperscript{18} to non-compact symmetries, only $O(N,1)$ invariant quantities.

We have thus calculated the energy, the specific heat and the two- and four-point invariant correlation functions which are defined, respectively, by

$$ E = \frac{1}{V} \sum_{i,\xi_f} \bar{S}(i) \cdot \cdot \cdot \cdot S(i+\xi_f) $$

$$ C_v = \frac{\partial}{\partial q} E(q) = \frac{1}{q} \left( \langle E^2 \rangle - \langle E \rangle^2 \right) $$

$$ S(i) = \langle \bar{S}(i) \cdot \cdot \cdot \cdot S(i+\xi) \rangle $$

$$ G(i) = \langle \bar{S}(i) \cdot \cdot \cdot \cdot S(i+\xi_f) \bar{S}(i+\xi) \cdot \cdot \cdot \cdot \bar{S}(i+\xi+\xi_f) \rangle - \langle \bar{S}(i) \cdot \cdot \cdot \cdot S(i+\nu_1) \rangle \langle \bar{S}(i+\xi_f) \cdot \cdot \cdot \cdot \bar{S}(i+\xi+\nu_1) \rangle $$

(6)
In all these quantities we search for either an indication of a bonafide phase transition or a cross-over away from the perturbative regime. The behaviour of the two- and four-point correlation functions should uncover a mass generation in the system. We have no evidence for any of the above phenomena. The results of the Monte Carlo simulations are displayed in Figs. 1)-4). The runs were done mainly on a $40 \times 40$ lattice; runs done on a $20 \times 20$ lattice have led to essentially the same results. The number of iterations per coupling constant was about 5000.

The Monte Carlo sampling was done by the Metropolis method. The sampling should be symmetric with respect to the stability group in each point on the hyperbolic manifold; in our case, the stability group is $O(2)$. Thus, if at a given site the value of the spin is $\mathbf{s}_i$, the new spin $\mathbf{s}'_i$ is chosen in an $O(2)$ symmetric manner inside a generalized circle centered at $\mathbf{s}_i$ and with a given fixed radius, that is $\mathbf{s}'_i$ belongs to the set:

$$A = \{ \mathbf{s}'_i | \mathbf{s}'_i \cdot \mathbf{s} \leq \cosh(\tau_{\text{max}}) \}$$

where $\tau_{\text{max}}$ is a parameter that changes according to the value of the coupling constant. Typical values for $\tau_{\text{max}}$ are 1.0 for $g \sim 0.1$ and 10.0 for $g \sim 100.0$.

Practically, the new spin was chosen in the following way. We first chose a random spin $\mathbf{s}_i$ in the generalized circle with the centre at the origin - $\mathbf{s}_i = (1,0,0)$. This is done by selecting a random angle $\theta$ uniformly distributed in the segment $[0,2\pi]$ and a random "rapidity" $\tau$ uniformly distributed in the segment $[0,\tau_{\text{max}}]$. The spin $\mathbf{s}_i$ is then defined as

$$\mathbf{s}_i = (\cos \theta \cosh \tau, \sin \theta \sinh \tau, \sinh \tau \cosh \theta)$$

This sampling is clearly $O(2)$ symmetric. We then boost $\mathbf{s}_i$ to the required $\mathbf{s}'_i$ that belongs to the set $A$, by performing on it the "Lorentz transformation" that connects $\mathbf{s}_i$ (the origin) with $\mathbf{s}'_i$ (the given spin). Since the scalar product is invariant under the $O(2,1)$ transformation, the sampling is still $O(2)$ symmetric around $\mathbf{s}'_i$. The average nearest neighbour interaction is shown in Fig. 1).

The weak coupling results are in very good agreement with lattice perturbation theory. The strong coupling expansion is not easy to perform due to the complexity of the irreducible representations of $O(2,1)$. These results, as
well as the specific heat shown in Fig. 2), do not seem to vindicate the existence of a non-trivial fixed point or any dramatic deviation from perturbation theory. In Fig. 3) we show the unsubtracted two-point invariant correlation function for a 40 x 40 lattice. It has been plotted for three representative values of the coupling constant; for small \( g \) there is very good agreement with lattice perturbation theory. Note that the increase with distance of the correlation functions signals in fact a loss of correlation. That the function is increasing at all is due to the hyperbolic nature of the manifold; that the function does not approach a limiting value is due to the fact that \( \sigma \) is free to wander over the unbound manifold almost independent of the initial conditions. This results in an increasing of \( S(\vec{r}) \). The ill-defined expectation value of \( \sigma \) is indeed essentially unbounded, while that of the fields \( \vec{\Pi} \) is zero. In order to obtain the more conventional description of the correlation function, we have calculated the subtracted four-point invariant Green's function \( G(\vec{r}) \) of Eq. 6). In this case, the correlation function indeed decreases. For small \( g \) the decrease could be fitted by the expected perturbative power-like decrease. For the value \( g = 1 \) the correlation function shown in Fig. 4) could be fitted by a power law after 200,000 iterations. The large number of iterations is necessary in order to get significant results for at least three non-trivial points. For large \( g \), the small value of \( G(3) \) does not allow a meaningful fit. We thus do not know from studying \( G(\vec{r}) \) if a mass is formed for larger values of \( g \).

Returning to \( s(\vec{r}) \) shown in Fig. 3), in perturbation theory \( S(\vec{r}) \) behaves as

\[
S(\vec{r}) \sim 1 + \frac{g}{\mu} \ln r + \mathcal{O}(g^2)
\]

(7)

We thus fitted \( S(\vec{r}) \) for all couplings by a power law of the form

\[
S(\vec{r}) \approx r^{-\alpha(g)}
\]

(8)

and we plot \( \alpha(g) \) in Fig. 5). No change in structure is observed as a function of \( g \). For small \( g \), \( \alpha \approx g \) as expected; for larger \( g \) there is indeed, as shown, a deviation from perturbation theory; yet nothing but the existence of essentially massless particles can be deduced from these results. We note, however, that the calculations for large \( g \) (approaching the light-cone) converge very slowly. For example, in the more economical 0(1,1) model \( S(\vec{r}) \) did not converge for \( g = 100 \), even after 100,000 iterations. The time scales involved become prohibitive.
The fact that the field manifold is unbounded may lead one to worry that, even for invariant functions, convergence, in general, is very hard to achieve. For most plotted quantities and values of the coupling constant, no such problems seem to occur. To check the problem further, we have studied free field theory on the lattice. We have calculated the propagator

$$G'(r) = \langle \cos(y_{i+\ell}) - y_{i+\ell}\rangle$$

whose large $r$ behaviour is given by

$$G'(r) \propto \frac{1}{r^{3/2\beta}}$$

The results converged after a few hundred iterations to the analytic result. In $G'(r)$ is plotted versus $\ln r$, for various values of $g$, in Fig. 6. The calculations were done on a $20 \times 20$ lattice. Non-invariant quantities, such as $\langle \Phi \rangle$, did not converge. Models with no phase transition were endowed with one\(^{19}\) by complicating the topological structure of the field manifold. In this case, it is tantamount to studying

$$S = \sum \beta_c \sum_i \bar{s}(i)S(i-\ell) + \beta_a \sum |\bar{s}(i)\bar{s}(i-\ell)|^2$$

Even for $N = \infty$, one can show, using the methods of Ref. 20), that no phase transition will occur for $\beta_c = 0$ and $\beta_a \neq 0$. This is due to the fact that, contrary to the compact $O(N)$ model, $\langle S(i)S(i+\ell)\rangle (\beta_c, \beta_a = 0)$ is increasing as $\beta_c$ tends to zero. Other methods for mixing the forward and backward hyperboloid cannot be studied by MC procedures. We have added to the Lagrangian [Eq. (3)] mass terms which break the symmetry explicitly. Switching these terms on and off has not revealed any interesting structure.

The $O(N,1)/O(N)$ model can be formally obtained from the $O(N+1)/O(N)$ model by reversing the sign of the coupling $g$ and analytically continuing some of the fields. If this mapping had any non-perturbative significance, we would expect the $O(1,1)$ model to have some features in common with the $O(2)$ model. We have calculated the four quantities appearing in Eq. (6) in the $O(1,1)$ model. Results were consistent with the $O(1,1)$ model being essentially a free field theory for all couplings.

An attractive way in which infinite dimensional irreducible representations may manifest themselves is by a topological classification. Indeed, the classical
equations of motion of the $O(2,1)/O(2)$ model can be mapped\textsuperscript{5}) into those of a different model (on the quantum level) studied by Getmanov\textsuperscript{21}). (A mass term has to be added in a limiting procedure.) Unfortunately, there is no evidence for the quantum stability of these solitons, and once again one is left with no compelling evidence in favour of the existence of infinite dimensional representations.

In conclusion, in the Monte Carlo simulations of the $O(2,1)/O(2)$ the order parameters we have used have not revealed a significant deviation from the perturbative results. The same holds for other ideas and methods presented. It may well be that we have not chosen the correct model to study; it could be that supersymmetric models in four dimensions would have less difficulties in producing a scale. In any case, we believe that the problem of dynamical linear realizations of non-compact symmetries merits future research.

After this work had been completed, we learned that another group\textsuperscript{22}) had studied the supersymmetric quantum mechanical $O(2,1)/O(2)$ model.

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FIGURE CAPTIONS

Fig. 1: The energy $E$ as a function of $g$ is plotted on a logarithmic scale. Results obtained are from a 40 x 40 lattice; the typical number of iterations for each coupling was 5000. On a linear scale, the graph is almost linear.

Fig. 2: The specific heat as a function of $g$. The results were obtained by measuring the fluctuations of the energy and by taking the derivative of graph 1).

Fig. 3: The two-point correlation function $S(r) = \langle \tilde{S}(0), \tilde{S}(r) \rangle$ as a function of $r$ is plotted on a logarithmic scale. The results are for a 40 x 40 lattice. The behaviour is consistent with a power law increasing function for all couplings. The scale on the left is used for $g = 0.1, 1.0$ and the one on the right is used for $g = 10.0$. The slope $\alpha$ is indicated for each coupling.

Fig. 4: The four-point invariant function $G(r) = \langle \tilde{S}(i), \tilde{S}(j), \tilde{S}(i+r), \tilde{S}(j+r) \rangle$ is plotted as a function of $r$ on a logarithmic scale. The results shown are for $g = 1.0$, obtained after 200,000 iterations on a 20 x 20 lattice. The value of $G(4)$ is not significantly different from 0. The results are consistent with a power law decay with slope $\alpha = 3.3$.

Fig. 5: The power $\alpha(g)$ describing the power law behaviour of the two-point invariant function $S(r)$. The straight line describes the perturbation theory behaviour $\alpha = g/2\pi$. For strong coupling, a deviation from perturbation theory is seen, but no interesting structure emerges.

Fig. 6: The correlation function $G(r) = \langle \cos(\varphi(i) - \varphi(i-r)) \rangle$ for free field theory plotted as a function of $r$ on a logarithmic scale. Results are for a 20 x 20 lattice and obtained after 500 iterations. The theoretical values of the slopes are: for $g^2 = 0.25$, $\alpha = 3.98 \times 10^{-2}$; for $g^2 = 0.5$, $\alpha = 7.96 \times 10^{-2}$. 
Fig. 4