Ref.TH.3491-CERN

STOCHASTIC PROCESSES IN LATTICE (EXTENDED) SUPERSYMMETRY

S. Cecotti

Scuola Normale Superiore, Pisa
INFN, Sezione di Pisa, Italy

and

L. Girardello

CERN -- Geneva

and

Istituto di Fisica, Università degli Studi, Milano
INFN, Sezione di Milano, Italy

A B S T R A C T

We show that ungauged $N = 2$ supersymmetric models can be put on the (Hamiltonian) lattice in such a way as to preserve a subalgebra of supersymmetry large enough to ensure the existence of the Nicolai mapping. The models can be interpreted as stochastic systems described by Langevin equations. We describe both Wilson and Susskind versions of the model.

In two dimensions everything seems fine, but in 4D, one expects - on general grounds - that the continuum limit will be either trivial or non-Lorentz invariant.

Ref.TH.3491-CERN

17 December 1982

1. - INTRODUCTION *)

A stochastic interpretation has been introduced ^{1),2)} for the supersymmetric field theories having a "local" Nicolai mapping ¹⁾⁻⁵⁾. In these cases, there exists a change of the bosonic variables φ_i in the functional integral (Nicolai mapping)

$$\varphi_i(x) \rightarrow h_i[\varphi] \quad (1.1)$$

such that

- (i) the functional $h_i = h_i(\varphi)$ is "local", in the sense that only a finite number of derivatives of φ_i appear ;
- (ii) $\mathcal{L}_{\text{BOS}}(\varphi) = \frac{1}{2} \sum_i (h_i)^2 + \text{surface terms}$, where $\mathcal{L}_{\text{BOS}}$ is the bosonic part of the Euclidean Lagrangian $\mathcal{L} = \mathcal{L}_{\text{BOS}} + \bar{\psi} D[\varphi] \psi$;
- (iii) the Jacobian of the transformation (1.1) is the fermionic determinant.

From (ii) and (iii), it follows that - after the integration over the fermions - the functional integral is Gaussian in the fields h_i with covariance 1. As discussed in Refs. 1) and 2), this allows for an interpretation of Eq. (1.1) as a stochastic differential equation.

It is easy to see that ***) the general ansatz for the mapping is

$$h_i = \dot{\varphi}_i + \frac{\delta W}{\delta \varphi_i} \quad (1.2)$$

(Langevin equations) - up to a redefinition of h_i fields - where W is a functional of the fields and of their space derivatives. Then (ii) requires W to be a local solution of the classic Euclidean zero-energy Hamilton-Jacobi equation

$$H\left(\frac{\delta W}{\delta \varphi_i}; \varphi_i\right) = 0 \quad (1.3)$$

where H is the Euclidean Hamiltonian ²⁾. Only if the Hamiltonian satisfies certain integrability conditions does a local solution to Eq. (1.3) exist. For instance, for the trivial model in 2D: $\mathcal{L} = \frac{1}{2} [\dot{\varphi}^2 + (\partial_1 \varphi)^2] + \bar{\psi} \partial \psi$ the local $W[\varphi]$ would have the property $\delta W / \delta \varphi_x = \pm (\partial_1 \varphi)_x$ which is not consistent with the fact that

*) The main results of this paper were presented at the 21st International Conference on High Energy Physics, Paris (July 26-31, 1982). See : L. Girardello, Proceedings of the Paris Conference, eds. P. Petiau and M. Porneuf (1982), p. 323.

**) If $\mathcal{L}$ is quadratic in $\dot{\varphi}_i$ and does not contain terms linear in $\dot{\varphi}_i$.

$\delta^2 W / \delta \phi_x \delta \phi_y$ is a symmetric operator. One could write down the integrability conditions starting from Eq. (1.2), however, we prefer to discuss them from an algebraic point of view (see Section 2).

In this paper we use the knowledge of the integrability conditions for the construction of supersymmetric models on the lattice having a local Nicolai mapping, i.e., with a dynamics governed by stochastic differential equations. We shall use a Hamiltonian lattice (continuous time)⁶⁾, which is suitable for our purposes for two reasons : it allows explicit invariance with respect to a subalgebra of supersymmetry^{7),8),*)} and, moreover, it makes it possible to describe the evolution (in time) in terms of stochastic differential equations. On the lattice there seems to be a very deep interplay between the integrability conditions and fermion doubling¹⁰⁾. As a matter of fact, all known models with a local mapping have $N = 2$ supersymmetry, namely Dirac (as opposed to Weyl) fermions in the continuum case. This will be discussed in Section 2.

If a (continuous) model admits a local Nicolai mapping, the lattice version can be obtained simply by discretizing it in space. The lattice mapping will be just the discretized version of the continuum mapping. Thus the only thing we need in order to construct a model is a suitable functional W .

Given a continuous Nicolai mapping, there are many inequivalent ways to discretize it. They correspond to different choices for the fermionic Lagrangian (with different properties with respect to chiral transformations, for instance). Also the doubling of the degrees of freedom (fermionic and bosonic) will depend on the particular discretization used, in a way consistent with the general arguments¹⁰⁾.

We will present two particularly relevant cases corresponding to Wilson¹¹⁾ and Susskind¹²⁾ fermions, respectively. In particular, Susskind fermions are very natural in the present context ; they are the more straightforward realization of the integrability conditions for the (discretized) Hamilton-Jacobi equation.

All the Lagrangians have the correct Lorentz invariant form up to a bosonic spurious term which is, formally, a total divergence. The fate of this term in the continuum limit is expected to be very different in two and four space-time dimensions. Indeed, whereas in two continuous dimensions there are

*) This subalgebra will be the same that solves the integrability conditions (see Section 2) for $W[\varphi]$. Moreover it will allow the use of index theory^{9),2)} to control the number of fermionic modes⁸⁾.

many interesting models of the class here considered ($N = 2$ ungauged models ^{*)}), in four dimensions it seems ¹⁾ that there is only one consistent example: the free massless theory. Consequently, one thinks that in two dimensions the spurious term goes into a total divergence in the continuum limit, whereas in four it survives and gives non-Lorentz invariant contributions in the bosonic Lagrangian, except in the free massless case in which it is zero from the very beginning. We will elaborate more on this in Sections 3, 4 and 5.

However, even in four dimensions, we think that our results are interesting. For one thing they give further insights on the problem of lattice fermions. It is surprising that a construction based on classic ideas leads automatically to fermion fields which have the correct spin and Lagrangian, even if - from the stochastic point of view - they are introduced as ghosts, in order to represent a determinant.

Hopefully, this study is preliminary to the case of more interesting (gauged) models; it may be interesting in view of some recent speculation ¹³⁾ concerning $N = 4$ super-Yang-Mills. After all, the fact that $N = 2$ free theory has a stochastic structure in 4D is not a trivial fact, and a deeper result can be hidden behind it.

The paper is organized as follows: in Section 2 we study the integrability conditions from a general algebraic point of view. In Section 3 we present the 2D models with Wilson fermions; in Section 4, we present the Susskind version, and in Section 5 we present the particular results in 4D. Finally, in Section 6 we give our conclusions.

2. - INTEGRABILITY CONDITIONS AND SUPERSYMMETRY ALGEBRA ⁴⁾

2.1. - 1D supersymmetry

Consider a theory with K bosonic $(x_i, p_j, i, j = 1, \dots, K)$ and K fermionic $(\chi_A^i, A = 1, 2, i = 1, \dots, K)$ degrees of freedom

$$[x_i, p_j] = i \delta_{ij} \quad \{ \chi_A^i, \chi_B^j \} = 2 \delta_{AB} \delta^{ij} \quad (2.1)$$

^{*)} Without central charges.

and assume that there exist two Hermitian operators Q_A , linear in the canonical fermionic operators χ so that the 1D supersymmetric algebra

$$\{Q_A, Q_B\} = 2 \delta_{AB} H \quad (2.2)$$

holds with $H = \frac{1}{2} \sum_i p_i^2 + U(x_i, \chi_A^i)$. If $\det[\eta_{(A,i)(B,j)}] \neq 0$ [where $\eta_{(A,i)(B,j)}$ is the matrix defined by $Q_A = \eta_{(A,i)(B,j)} p_i \chi_B^j + \text{terms independent of } p$], there exists a function $W(x)$ such that ^{*} we have

$$\sqrt{2} Q_A = \chi_A^i p_i + \epsilon_{AB} \chi_B^i \partial_i W \quad (2.3)$$

$$H = \frac{1}{2} \sum_i [p_i^2 + (\partial_i W)^2] + \frac{i}{2} \sum_{ij} \partial_i \partial_j W \chi_1^i \chi_2^j \quad (2.4)$$

Indeed, without loss of generality ^{**}, we can write

$$\sqrt{2} Q_A = \chi_A^i p_i + \chi_A^i R_i^{(A)}(x) + \epsilon_{AB} \chi_B^i S_i^{(A)}(x) \quad (2.5)$$

and then

- 1) Q_A^2 does not contain terms linear in p_i , requires $R_i^{(A)} = 0$,
- 2) $Q_1^2 = Q_2^2$ implies $S_i^{(A)}(x) = S_i(x)$,
- 3) $0 = \{Q_1, Q_2\} = \frac{1}{2}(\chi_1^i \chi_1^j - \chi_2^i \chi_2^j)(\partial_i S_j - \partial_j S_i)$ $S_j = \partial_j W$. This completes the argument.

The Euclidean Lagrangian corresponding to the Hamiltonian (2.4) is

$$L = \frac{1}{2} \sum_i [\dot{x}_i^2 + (\partial_i W)^2] + \sum_{ij} \bar{\xi}_i [\delta_{ij} d_t + \partial_i \partial_j W] \xi_j \quad (2.6)$$

where $\bar{\xi}_i$ and ξ_j are Grassman variables. The functional mapping

$$h_i = \dot{x}_i + \partial_i W \quad (2.7)$$

^{*}) After a suitable redefinition of χ .

^{**}) There are some subtleties which do not play any rôle in the case we are interested in.

transforms the (periodic) functional integral

$$Z = \int [dx_i d\bar{\xi}_i d\xi_i]_{\text{per.}} \exp\left(-\int_0^\beta d\tau L(x, \bar{\xi}, \xi)\right) \quad (2.8)$$

into a Gaussian one with covariance 1. Indeed

$$\frac{1}{2} \sum_i h_i^2 = L_B + \frac{dW}{dt}$$

and the Jacobian is

$$\begin{aligned} \det \left[\frac{\delta h_i}{\delta x_j} \right]_{\text{per.}} &= \det \left[\delta_{ij} d_t + \partial_i \partial_j W \right]_{\text{per.}} = \\ &= \int [d\bar{\xi} d\xi]_{\text{per.}} \exp\left(-\int_0^\beta d\tau L_F(\bar{\xi}, \xi)\right) \end{aligned} \quad (2.9)$$

From Eq. (2.3), we see that the relevant integrability condition (namely the existence of W) is the algebra of Eq. (2.2). Hence the models of Ref. 7) have a mapping.

2.2. - Extension to higher dimension

2.2.a - The strategy

The idea of the extension is very simple. From the considerations of Refs. 2) and 3), we know that we must put the theory in an Euclidean box with periodic boundary conditions. Then (in the continuum case) we decompose the fields into Fourier components (in space only). With a suitable UV regularization (see later), we have a finite number of degrees of freedom, as in subsection 2.1, but with a major difference : the superalgebra is not Eq. (2.2), but

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta} H - 2\vec{\alpha}_\alpha \cdot \vec{P} \quad (2.10)$$

and the argument of the previous section does not work. Our strategy is then the following ^{4),8)}. First, algebra (2.10) can be brought into the form of Eq. (2.2) (this is possible if $N = 2$). Then, using the arguments following Eq. (2.5), we have an effective W which substituted into Eq. (2.6), gives us the Nicolai mapping for the regularized version of the theory.

Let us comment on the regularization. One expands in Fourier series the superfields and then just cuts off the integration over modes with $|\vec{K}| < \Lambda$. Note that it is crucial that the cut-off is done directly on the superfields because the analogous procedure after the elimination of the auxiliary fields would

be non-supersymmetric ^{*)} and we could not use the arguments of subsection (2.1).

2.2.b - N = 2 in two dimensions

In a Majorana basis, we have

$$\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\delta_{\alpha\beta} H - \sigma_{\alpha\beta}^3 P) \quad (2.11)$$

(a,b,α,β = 1,2). Let us call $\tilde{Q}_A = 2^{-\frac{1}{2}}(Q_1^A + Q_2^A)$. From Eq. (2.11)

$$\{\tilde{Q}_A, \tilde{Q}_B\} = 2\delta_{AB} H \quad (2.12)$$

The matrix $\eta_{(A,i)(B,j)}$ is determined only by the free part of the Noether's charge and we see that Eq. (2.5) holds. Hence, using the results of subsection (2.1), we have a function W such that the mapping is given by Eq. (2.7) ^{**)} . For explicit examples, see Refs. 1) and 2). In the continuum case, one can easily see that they are UV finite ²⁾. They belong to the class analyzed in Ref. 14).

2.2.c - N = 2 in four dimensions

We have

$$\{Q_\alpha^a, Q_\beta^b\} = 2\delta^{ab}(\delta_{\alpha\beta} H - \vec{\alpha}_{\alpha\beta} \cdot \vec{P}) \quad (2.13)$$

(a,b = 1,2, α,β = 1,-,4). We use the Majorana basis :

$$\alpha_1 = \begin{pmatrix} 0 & -\sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \quad (2.14)$$

Let us consider $\tilde{Q}_A = 2^{-\frac{1}{2}}(Q_A^1 + Q_{A+2}^2)$. Using Eqs. (2.13) and (2.14) we have

$$\{\tilde{Q}_A, \tilde{Q}_B\} = 2\delta_{AB} H \quad (2.15)$$

Again, the non-singularity of the relevant η can be checked in the free case,

$$W(\varphi_a) = \int d^3x \frac{1}{2} \varphi_a \eta_{ab}^i \partial_i \varphi_b$$

^{*)} Supersymmetry is realized linearly only if we do not eliminate auxiliary fields.

^{**)} By comparison of Eqs. (2.7) and (1.2), we see that if the mapping exists, then we need the algebra (2.2) which is not a subalgebra of N = 1 supersymmetry. Hence we must have N = 2 supersymmetry (if the model is Lorentz invariant).

($i = 1, 2, 3$; $a, b = 1, \dots, 4$) and η_{ab}^i is the 't Hooft tensor. Unfortunately, there are problems with the interactions, as explained in the Introduction.

We have shown how the integrability conditions have a very simple algebraic interpretation : there must be a subalgebra isomorphic to Eq. (2.2), and this is automatic with $N = 2$ supersymmetry. One of the reasons for a Hamiltonian lattice is to make the H operator in Eq. (2.2) to coincide with the generator of the (continuous) time translations. Then an $N = 2$ model on the lattice will become a 1D supersymmetric model (in the sense of Section 2.1) described by the Langevin Eq. (2.7).

3. - 2D, $N = 2$ LATTICE SUPERSYMMETRY WITH WILSON FERMIONS

In Refs. 1) and 2) it was shown that continuum 2D, $N = 2$ supersymmetric models admit a local Nicolai mapping whose functional W 2) is

$$W[\varphi_i^*, \varphi_i] = \int dx \left[\frac{i}{2} \sum_i \varphi_i^* \overleftrightarrow{\partial}_1 \varphi_i + V(\varphi_i) + V(\varphi_i^*) \right] \quad (3.1)$$

where V (the superpotential) is a real analytic function of φ_i .

The results of Ref. 8) suggest that a discretization of Eq. (3.1) corresponding to Wilson action for fermions must exist. The Wilson version exists also for the $N = 1$ case ⁸⁾ in which case consistency with Ref. 10) requires real fermions. This means that in order to have a Wilson-like Lagrangian, we have to look for a discretization of Eq. (3.1) which explicitly breaks the complex structure of W (corresponding to chirality for fermions). Writing $\varphi_x = 2^{-\frac{1}{2}}(A_x + iB_x)$ we make the following ansatz for W (one multiplet for simplicity)

$$\begin{aligned} W[A_x, B_x] &= \\ &= \sum_x \left\{ A_x [B_{x+1} - B_x] + 2 \operatorname{Re} V[A_x + iB_x] \right\} \end{aligned} \quad (3.2)$$

Then the bosonic Lagrangian is [see Eq. (2.6)]

$$\mathcal{L}_{\text{Bos}} = \frac{1}{2} \sum_x (\dot{A}_x^2 + \dot{B}_x^2) + \frac{1}{2} \sum_x \left(B_{x+1} - B_x + 2 \frac{\partial \operatorname{Re} V}{\partial A_x} \right)^2 + \quad (3.3)$$

$$\begin{aligned}
 & + \frac{1}{2} \sum_x \left(A_{x-1} - A_x + 2 \frac{\partial \text{Re} V}{\partial B_x} \right)^2 = \\
 & = \frac{1}{2} \sum_x \left\{ \dot{A}_x^2 + \dot{B}_x^2 + (B_{x+1} - B_x)^2 + (A_{x+1} - A_x)^2 + \right. \\
 & \left. + \left(2 \frac{\partial \text{Re} V}{\partial A_x} \right)^2 + \left(2 \frac{\partial \text{Re} V}{\partial B_x} \right)^2 + \sum_x \left\{ (B_{x+1} - B_x) \frac{\partial \text{Re} V}{\partial A_x} + (A_{x-1} - A_x) \frac{\partial \text{Re} V}{\partial B_x} \right\} \right\} \quad (3.3) \\
 & \text{Contd.}
 \end{aligned}$$

The Cauchy-Riemann equations allow us to rewrite the last term in Eq. (3.3) (spurious term) as

$$\begin{aligned}
 & (B_{x+1} - B_x) \frac{\partial \text{Re} V}{\partial A_x} + (A_{x-1} - A_x) \frac{\partial \text{Re} V}{\partial B_x} = \\
 & = (B_{x+1} - B_x) \frac{\partial \text{Im} V}{\partial B_x} + (A_x - A_{x-1}) \frac{\partial \text{Im} V}{\partial A_x} \rightarrow \partial_1 \text{Im} V \quad (3.4)
 \end{aligned}$$

which is a total divergence in the naive continuum limit.

The fermionic piece of the Lagrangian can be read in Eq. (2.6) :

$$\begin{aligned}
 L_F &= \sum_x \bar{\xi}_x^A \Delta^+ \xi_x^B + \sum_x \bar{\xi}_x^B \Delta^- \xi_x^A + \\
 & 2 \frac{\partial^2 \text{Re} V}{\partial A_x^2} \sum_x \bar{\xi}_x \sigma_3 \xi_x + 2 \frac{\partial^2 \text{Re} V}{\partial A_x \partial B_x} \sum_x \bar{\xi}_x \sigma_1 \xi_x \quad (3.5)
 \end{aligned}$$

where

$$\Delta^+ \xi_x = \xi_{x+1} - \xi_x \quad \Delta^- \xi_x = \xi_x - \xi_{x-1} \quad (3.6)$$

The Fermionic Hamiltonian ($\xi^{A,B} \rightarrow \chi^\alpha$) in the momentum space, reads

$$\begin{aligned}
 H_F &= \chi_\alpha^+(p) \left\{ \sigma_{\alpha\beta}^1 (1 - \cos p) + \sigma_{\alpha\beta}^2 \sin p + \right. \\
 & \left. + 2 \frac{\partial^2 \text{Re} V}{\partial A^2(p)} \sigma_{\alpha\beta}^1 + 2 \frac{\partial^2 \text{Re} V}{\partial A(p) \partial B(p)} \sigma_{\alpha\beta}^3 \right\} \chi_\beta(p) \quad (3.7)
 \end{aligned}$$

and, after diagonalization of the kinetic terms, it reads :

$$H_F = \chi_\alpha^\dagger(p) \left\{ \sigma_{\alpha\beta}^1 (1 - \omega p) + \sigma_{\alpha\beta}^3 \sin p + \right. \\ \left. + 2 \frac{\partial^2 \text{Re } V}{\partial A^2(p)} \sigma_{\alpha\beta}^1 - 2 \frac{\partial^2 \text{Re } V}{\partial A(p) \partial B(p)} \sigma_{\alpha\beta}^2 \right\} \chi_\beta(p) \quad (3.8)$$

Just as in Ref. 8), the Hamiltonian (3.8) contains a suitable Wilson term to prevent the doubling and to make the fermionic dispersion relation to coincide with the bosonic one as required by supersymmetry and index theory ⁹⁾.

The spurious term (3.4) may cause some problems in the continuum limit, but probably not very serious ones. The situation was discussed in part in Ref. 8) (also in the present models one can show that there are no Lorentz-breaking induced Yukawa couplings). In a sense, the situation is better than in Ref. 8) ; those models were super-renormalizable whereas ours are UV finite in the continuum case.

There is another argument in favour of the Lorentz invariance of the continuum limit. A Langevin equation like (2.7) corresponds to a Fokker-Planck equation $-\partial_t \psi = L_W^* \psi$ where the adjoint diffusion operator L^* is

$$L_W^* = - \sum_i \frac{\partial}{\partial x_i} \left[\frac{\partial}{\partial x_i} + 2 \frac{\partial W}{\partial x_i} \right] \quad (3.9)$$

We see that the condition for having $L_W^* = L_W$ is

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} W \equiv 0 \quad (3.10)$$

$L_W^* = L_W$ means that the forward Fokker-Planck and backward Fokker-Planck ^{**)} are actually the same (up to a change $t \rightarrow -t$). This means that the condition (2.10) corresponds to a peculiar symmetry for time-inversion. By studying how this symmetry is realized on fermions ^{2), 16)}, we see that it is a remainder of the 4D

*) Our fermions differ for normalization from those of Ref. 8).

**) For the definition, see for instance Ref. 15).

PCT symmetry (the model can be considered as a dimensional reduction of 4D Wess-Zumino models). Then, because the kinetic term of W (or if you prefer of Q_A) is not renormalized, this symmetry implies that the induced interactions (in the continuum limit) are given by a superpotential V which is a holomorphic function of $A + iB$. This suffices to prove the existence of the necessary Cauchy-Riemann identities needed to cancel the spurious term, as in Ref. 2).

An alternative Wilson discretization is

$$W[A_x, B_x] = \sum_x \left\{ A_x [B_{x+1}, -B_x] + \right. \\ \left. + \text{Re} V(A_x + i B_x) + \text{Re} V(A_x - i B_x) \right\} \quad (3.11)$$

which has the merit of having an exact parity symmetry

$$A_x \rightarrow A_{-x} \quad B_x \rightarrow -B_{-x+1} \quad (3.12)$$

4. - 2D N = 2 LATTICE SUSY : SUSSKIND VERSION

If one naively discretizes the continuum W , Eq. (3.1), in such a way as to keep the complex structure explicit, one ends up with a Susskind version in which the bosonic fields are doubled, because the odd and even sites are distinguished. This is just the "superpartner" of the usual fermionic doubling. Hence we have twice the number of fields we need in the continuum. This effect can be corrected by simply considering real instead of complex scalars. Then the integrability conditions are satisfied and we have the algebra Eq. (2.2).

W reads

$$W(\varphi_x) = \sum_x \frac{(-)^x}{2d} \varphi_x \varphi_{x+1} + \sum_x (-)^x V(\varphi_x) \quad (4.1)$$

Substituting Eq. (4.1) into (2.6), we obtain the Lagrangian

$$\begin{aligned}
 L = & \frac{1}{2} \sum_x \left\{ \dot{\varphi}_x^2 + \frac{1}{4d^2} (\varphi_{x+1} - \varphi_{x-1})^2 + \left(\frac{\partial V}{\partial \varphi_x} \right)^2 \right\} + \\
 & + \sum_x \frac{\partial V}{\partial \varphi_x} \frac{(\varphi_{x+1} - \varphi_{x-1})}{2d} + \\
 & + \sum_x \bar{\xi}_x \dot{\xi}_x + \sum_x \frac{(-)^x}{2d} (\bar{\xi}_{x+1} - \bar{\xi}_{x-1}) \xi_x + \sum_x \frac{\partial^2 V}{\partial \varphi^2} \bar{\xi}_x \xi_x
 \end{aligned} \tag{4.2}$$

with the redefinition of the fermionic fields

$$\phi_x = (i)^x (-)^{\left[\frac{x}{2}\right]} \xi_x \quad \phi_x^+ = (-i)^x (-)^{\left[\frac{x}{2}\right]} \bar{\xi}_x \tag{4.3}$$

We rewrite the fermionic piece of Eq. (4.3) in the Susskind form

$$\begin{aligned}
 L_F = & \sum_x \phi_x^+ \dot{\phi}_x + \frac{i}{2d} \sum_x (\phi_{x+1}^+ \phi_x - \phi_x^+ \phi_{x+1}) + \\
 & + \sum_x \frac{\partial^2 V}{\partial \varphi^2} \phi_x^+ \phi_x
 \end{aligned} \tag{4.4}$$

and we see that the models coincide with those of Ref. 7).

The model is fine, except for the presence of the spurious term $\sum_x (\partial V / \partial \varphi_x) (\varphi_{x+1} - \varphi_{x-1}) / 2d$. Because the odd and even site bosonic fields are different in the continuum limit, the spurious term does not have the structure of a total derivative.

However, from the continuum case and the Wilson example, we have learned that an expression of the form $(\partial V / \partial A) \partial B - (\partial V / \partial B) \partial A$ becomes a total derivative because we have a complex structure (namely $N = 2$ SUSY) and hence Cauchy-Riemann equations. Then the relevant question is if and how the model (4.1) can recover a complex structure in the continuum limit. For fermions the complex structure is the same as the chiral structure. On a Susskind lattice, we have only discrete γ_5 symmetries - which correspond to odd lattice translations - but it is generally believed that a continuous chiral invariance could be recovered in the continuum limit. In this case a complex structure would be also recovered. This could produce the relevant Cauchy-Riemann identities.

Hence the whole story boils down to understanding how continuous chirality is recovered in these Susskind models. Of course, if there is a meaningful continuum limit for these models they must correspond to the models of Refs. 1) and 2).

5. - FOUR DIMENSIONS

Guided by the structure of the integrability conditions, we make the following ansatz for $W(\varphi_{\vec{r}})$ ($\varphi_{\vec{r}}$ real variable)

$$\begin{aligned} W(\varphi_{\vec{r}}) = & \sum_{\vec{r}} \frac{(-)^x}{2d} \varphi_{\vec{r}} \varphi_{\vec{r}+\vec{e}_x} + \sum_{\vec{r}} \frac{(-)^{x+y}}{2d} \varphi_{\vec{r}} \varphi_{\vec{r}+\vec{e}_y} + \\ & + \sum_{\vec{r}} \frac{(-)^{x+y+z}}{2d} \varphi_{\vec{r}} \varphi_{\vec{r}+\vec{e}_z} + \sum_{\vec{r}} (-)^{x+y+z} V(\varphi_{\vec{r}}) \end{aligned} \quad (5.1)$$

where $\vec{r} = (x, y, z)$ is a site index and $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are the unit vectors along the axis.

Now using the standard formula, Eq. (2.6), we have

$$\mathcal{L}_B = L_B + \Delta L \quad (5.2)$$

where *)

$$L_B = \frac{1}{2} \sum_{\vec{r}} \left\{ \dot{\varphi}_{\vec{r}}^2 + (\Delta_x \varphi_{\vec{r}})^2 + (\Delta_y \varphi_{\vec{r}})^2 + (\Delta_z \varphi_{\vec{r}})^2 + V'(\varphi_{\vec{r}}) \right\} \quad (5.3)$$

$$\begin{aligned} \Delta L = & \sum_{\vec{r}} \left\{ (-)^{y+z} V'(\varphi_{\vec{r}}) [\Delta_x \varphi_{\vec{r}}] + (-)^z V'(\varphi_{\vec{r}}) [\Delta_y \varphi_{\vec{r}}] + \right. \\ & + V'(\varphi_{\vec{r}}) [\Delta_z \varphi_{\vec{r}}] + (-)^y [\Delta_x \varphi_{\vec{r}}] [\Delta_y \varphi_{\vec{r}}] + (-)^{y+z} (\Delta_x \varphi_{\vec{r}}) \cdot \\ & \cdot [\Delta_z \varphi_{\vec{r}}] + (-)^z [\Delta_y \varphi_{\vec{r}}] [\Delta_x \varphi_{\vec{r}}] \left. \right\} \end{aligned} \quad (5.4)$$

*) $\Delta_x \varphi_{\vec{r}} = (\varphi_{\vec{r}+\vec{e}_x} - \varphi_{\vec{r}-\vec{e}_x})/2d$ and analogously for $\Delta_y \varphi_{\vec{r}}$ and $\Delta_z \varphi_{\vec{r}}$.

L_B is the "Susskind" form of the discretized bosonic Lagrangian and ΔL is a spurious term. The part of ΔL which does not contain V' contains products of even site or odd site bosons and hence it is expected to be (in the limit) a surface term ⁷⁾. The terms proportional to V' are definitely not surface terms, as it is obvious from the fact that there is no renormalizable continuum $N = 2$ supersymmetric model with $V' \neq 0$.

On the other hand, in four dimensions, a Wilson version (with a local Nicolai mapping) is possible only if $V' = 0$. The reason is that there is no generalization to 4D of the 2D Cauchy-Riemann equations we used in Section 3. The same problem is encountered in the continuum case ¹⁾.

Nevertheless, the fact that these models have a stochastic structure is interesting by itself. For one thing, they have a completely acceptable fermionic Hamiltonian which - after a redefinition of the fermionic fields similar to that of Eq. (4.3) - is

$$\begin{aligned}
 H_F = i \sum_{\vec{n}} \frac{1}{2d} (\phi_{\vec{n}}^+ \phi_{\vec{n}+\vec{e}_x} - \phi_{\vec{n}+\vec{e}_x}^+ \phi_{\vec{n}}) - \sum_{\vec{n}} \frac{(-)^{x+y}}{2d} (\phi_{\vec{n}}^+ \phi_{\vec{n}+\vec{e}_y} + \\
 + \phi_{\vec{n}+\vec{e}_y}^+ \phi_{\vec{n}}) + i \sum_{\vec{n}} \frac{(-)^{x+y}}{2d} (\phi_{\vec{n}}^+ \phi_{\vec{n}+\vec{e}_z} - \phi_{\vec{n}+\vec{e}_z}^+ \phi_{\vec{n}}) + \\
 - \sum_{\vec{n}} V''(\varphi_{\vec{n}}) \phi_{\vec{n}}^+ (-)^{x+y+z} \phi_{\vec{n}}
 \end{aligned} \tag{5.5}$$

The Hamiltonian (5.5) is just the free Hamiltonian for spin $\frac{1}{2}$ fermions [compare with Eq. (3.5) of Ref. 12)] plus an interaction term which - comparing with Eq. (4.17) of Ref. 12), reads $\sum_{\vec{r}} V''(\varphi_{\vec{r}}) (\bar{q}q)_{\vec{r}}$ which is Lorentz and isospin invariant (the model describes two Dirac fermions).

This is an improvement with respect to Ref. 17) where also the Yukawa couplings were not Lorentz invariant.

6. - CONCLUSIONS

We have seen that if we have a local Nicolai mapping we can very easily put the model on a lattice without spoiling the stochastic structure, and a fortiori, in an explicitly supersymmetric way.

Unfortunately, we do not know any non-trivial 4D Nicolai mapping. Nevertheless the fact that for some (pseudo)-model there is a stochastic structure seems interesting in itself because of the connection with relevant features of fermions on a lattice.

It seems that in 4D what is relevant for the existence of a local mapping is duality. The obvious speculation is $N = 4$. Nicolai¹³⁾ has given topological evidence for this speculation. Note that $N = 4$ would have two Dirac fermions, as the minimal Susskind doubling¹²⁾.

ACKNOWLEDGEMENTS

We would like to thank E. Rabinovici, G. Parisi, I. Stamatescu and S. Yankielowicz for discussions.

REFERENCES

- 1) G. Parisi and N. Surlas - Nuclear Phys. B206 (1982) 321.
- 2) S. Cecotti and L. Girardello - Annals of Physics 145 (1983) 81.
- 3) S. Cecotti and L. Girardello - Phys.Letters 110B (1982) 39.
- 4) S. Cecotti and L. Girardello - Trieste ICTP Preprint IC/82/105 (1982).
- 5) H. Nicolai - Phys.Letters 89B (1980) 341 ; Nuclear Phys B193 (1981) 173.
- 6) J. Kogut - Revs.Modern Phys. 51 (1979) 659 and references therein.
- 7) V. Rittenberg and S. Yankielowicz - CERN Preprint TH. 3263 (1982).
- 8) S. Elitzur, E. Rabinovici and A. Schwimmer - Phys.Letters 119B (1982) 165.
- 9) E. Witten - Nuclear Phys. B202 (1982) 253.
- 10) H.B. Nielsen and M. Ninomiya - Nuclear Phys. B185 (1981) 20 ; Nuclear Phys. B193 (1981) 173.
- 11) K. Wilson - Erice Lectures (1975).
- 12) L. Susskind - Phys.Rev. D16 (1977) 3031.
- 13) H. Nicolai - Phys.Letters 117B (1982) 408.
- 14) P.C. Davies, P. Salomonson and J.W. van Holten - Nuclear Phys. B208 (1982) 484.
- 15) C. De Witt-Morette and K.D. Elworthy - New Stochastic Methods in Physics, Physics Reports 77 (1981) No 3.
- 16) S. Cecotti - Tesi Scuola di Perfezionamento, Scuola Normale Superiore, Pisa (in preparation).
- 17) T. Banks and P. Windey - Nuclear Phys. B198 (1982) 226.