Ref.TH.3522-CERN

Q² DUALITY AND RARE PION DECAYS

L. Bergström and E. Massó ^{*)}

CERN -- Geneva

and

Ll. Ametller and A. Bramon.

Departament de Física Teòrica
Universitat Autònoma de Barcelona, Spain

A B S T R A C T

Some peculiarities of the $\pi^0 \rightarrow e^+e^-$ amplitude are discussed. The equivalence between vector meson dominance and quark models is clearly established (Q^2 duality). The relevant mass scales in that decay imply some universal properties for the amplitude. The branching ratio $B^\pi \equiv \Gamma(\pi^0 \rightarrow e^+e^-)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ depends only on a characteristic cut-off and not on the details of the models. Our model-insensitive prediction is $B^\pi \sim 6 \times 10^{-8}$, far below the experimental data.

^{*)} On leave of absence from Departament de Física Teòrica, Universitat Autònoma de Barcelona, Bellaterra, Spain.

The $\pi^0 \rightarrow e^+e^-$ decay or, more precisely, the branching ratio $B^\pi \equiv \Gamma(\pi^0 \rightarrow e^+e^-)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ has been a subject of permanent theoretical interest for more than 20 years¹⁾⁻⁶⁾. To lowest order in QED, the branching ratio B^π is given by⁶⁾

$$B^\pi \equiv \frac{\Gamma(\pi^0 \rightarrow e^+e^-)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 2 \sqrt{1 - \frac{4m_e^2}{m_\pi^2}} \left(\frac{\alpha}{\pi} \frac{m_e}{m_\pi} \right)^2 |R|^2, \quad (1)$$

where $m_\pi^2 = q^2$ and $m_e^2 = p^2$ are the squared masses of the pion and the electron, and

$$R = \frac{2i}{\pi^2 m_\pi^2} \int d^4k \frac{[q^2 k^2 - (q \cdot k)^2] F_\pi[k^2, (q-k)^2]}{[k^2 + i\varepsilon][(q-k)^2 + i\varepsilon][(k-p)^2 - m_e^2 + i\varepsilon]} \quad (2)$$

contains the relevant, dynamical aspects of the $\pi^0 \rightarrow e^+e^-$ amplitude through the form factor $F_\pi[k^2, (q-k)^2]$ (see Fig. 1). The imaginary part of Eq. (2) is model- (i.e., form factor-) independent and turns out to be¹⁾⁻⁶⁾

$$\text{Im } R(m_\pi^2) = -\frac{\pi}{2\beta_\pi} \ln \left(\frac{1 + \beta_\pi}{1 - \beta_\pi} \right), \quad (3)$$

where $\beta_\pi^2 \equiv 1 - 4m_e^2/m_\pi^2$. This gives rise to the well-known unitary limit

$$B^\pi \geq B_{\text{unit}}^\pi = 4.75 \times 10^{-8}, \quad (4)$$

which has to be compared with the experimental data^{7),8)}

$$\begin{aligned} B_{exp}^{\pi} &= (1.82 \pm 0.61) \times 10^{-7} = (3.8 \pm 1.3) B_{unit}^{\pi} \\ B_{exp}^{\pi} &= \left(2.26 \begin{smallmatrix} +2.43 \\ -1.11 \end{smallmatrix} \right) \times 10^{-7} = \\ &= (4.8 \begin{smallmatrix} +5.2 \\ -2.4 \end{smallmatrix}) B_{unit}^{\pi}. \end{aligned} \quad (5)$$

One therefore requires a substantial contribution from the real part of the amplitude (2). A discussion of this real part is the main point of the present note.

In a previous paper⁹⁾, the authors have discussed the real part of this type of amplitude in the context of the (constituent) quark model, assuming that the relevant form factors are simply given by a triangular loop of quarks of mass M_Q . This is the same framework where the $\pi^0 \rightarrow \gamma\gamma$ decay was successfully evaluated and resembles that of Pratap and Smith⁵⁾, who worked with a triangular loop of baryons instead of quarks. We first begin to compute the real part of R , Eq. (2), in the soft pion limit, $q \rightarrow 0$. In this limit the two photon propagators in Eq. (2) become $1/(k^2)^2$, and our quark-loop form factor simplifies to

$$F_{QL}(k^2, k^2) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{2M_Q^2}{M_Q^2 - (\alpha+\beta)(1-\alpha-\beta)k^2} \quad (6).$$

Then one easily obtains

$$\text{Re } R_{QL}(q^2=0) = -3 \left[\ln \frac{M_Q}{m_e} + \frac{17}{12} + \dots \right], \quad (7)$$

where the dots refer to (negligible) terms of order $(m_e^2/M_Q^2) \ln(M_Q^2/m_e^2)$ and higher. For a (constituent) mass of non-strange quarks $M_Q = 300$ MeV, one finds

$$\text{Re } R_{QL}(q^2=0) = -23.4 \quad . \quad (8)$$

We will now compare these results with those emerging from the most traditional models¹⁾⁻⁴⁾ proposed to describe $\pi^0 \rightarrow e^+e^-$ and $\pi^0 \rightarrow \gamma\gamma$ decays, namely, the vector-meson dominance models (VMD). We will simultaneously consider three variants of the basic VMD form factor already discussed in the literature⁶⁾

$$F_{VMD1} = M_V^2 / (M_V^2 - k^2) , \quad (9a)$$

$$F_{VMD2} = M_V^4 / [(M_V^2 - k^2)(M_V^2 - (q-k)^2)] , \quad (9b)$$

$$F_{VMD3} = M_V^2 / [M_V^2 - k^2 - (q-k)^2] , \quad (9c)$$

where M_V is the relevant vector-meson mass, $M_V \approx M_{\rho,\omega} \approx 0.77$ GeV. Working in the soft pion limit, Eqs. (2) and (9) lead to

$$\text{Re } R_{VMD1}(q^2=0) = -3 \left[\ln\left(\frac{M_V}{m_e}\right) + \frac{5}{12} + \dots \right] \simeq -23.2 \quad (10a)$$

$$\text{Re } R_{VMD2}(q^2=0) = -3 \left[\ln\left(\frac{M_V}{m_e}\right) - \frac{1}{12} + \dots \right] \simeq -21.7 \quad (10b)$$

$$\text{Re } R_{VMD3}(q^2=0) = -3 \left[\ln\left(\frac{M_V}{\sqrt{2} m_e}\right) + \frac{5}{12} + \dots \right] \simeq -22.2 \quad (10c)$$

The numerical agreement between Eqs. (8) and (10) must be considered as a clear fulfilment of Q^2 duality¹⁰⁾. In our context, this implies the equivalence (in the average) between a VMD and a (constituent) quark model description of the e.m. form factors of hadrons. Equations (7) and (10) show that Q^2 duality is already satisfied at the analytical level, provided one uses $M_V \approx 2 M_Q$.

We will now show that this agreement between the VMD and quark-loop models is preserved when going from $q^2 = 0$ to on-shell amplitudes. The point is that, in our case, we cannot use the traditional smoothness assumption invoked when the soft pion limit is performed. Indeed, the branch cut coming from the two-photon intermediate state and starting at $q^2 = 0$ induces a strong q^2 dependence of the real part of the amplitude around the point $q^2 = 0$. To illustrate this fact, we have calculated $\text{Re } R(q^2)$ in the range $-m_\pi^2 \leq q^2 \leq m_\pi^2$ for the models so far discussed. In our quark-loop approach, we used the unsubtracted dispersion relation proposed in Ref. 5). In doing so, we have corrected a few computational errors contained in Ref. 5) and confirmed explicitly the adequacy of the dispersive treatment of Eq. (2) without subtractions⁹⁾. The results^{*)} have been plotted in Fig. 2. The on-shell value turns out to be

$$\text{Re } R_{QL}(q^2 = m_\pi^2) = + 8.44 \quad . \quad (11)$$

In order to perform the calculation in the VMD model, one requires a numerical integration of Eq. (2) or the use of a dispersion relation approach (which in this case needs one subtraction). Both methods give coincident results, which have been plotted in Fig. 2. For $q^2 = m_\pi^2$ we obtained

$$\text{Re } R_{VMD1}(q^2 = m_\pi^2) = + 8.8 \quad (12a)$$

$$\text{Re } R_{VMD2}(q^2 = m_\pi^2) = + 10.4 \quad (12b)$$

$$\text{Re } R_{VMD3}(q^2 = m_\pi^2) = + 9.7 \quad (12c)$$

^{*)} We have checked the values of $\text{Re } R_{QL}(q^2)$ for $q^2 < 0$ by means of a numerical integration of Eq. (2). Also, at $q^2 = 0$, we reproduced the result (8), coming from the analytical expression (7).

As anticipated, the equivalence between the VMD and quark-loop approaches is preserved even for our rapidly q^2 varying amplitude. Indeed, by comparing the different results plotted in Fig. 2, one sees the agreement between the two kinds of models. In particular, the on-shell values, Eqs. (11) and (12), turn out to be essentially the same.

Let us now rediscuss our findings in a dispersive framework. In general, we may write a once-subtracted dispersion relation for the real part of the amplitude

$$\begin{aligned} \text{Re } R(q^2) = & \text{Re } R(0) + \\ & + \frac{q^2}{\pi} \int_0^\infty dq'^2 \frac{\text{Im } R(q'^2)}{(q'^2 - q^2) q'^2} . \end{aligned} \quad (13)$$

On the other hand, the dynamical form factor $F_\pi(k^2, (q-k)^2)$ must have a structure such that the integral in Eq. (2) is convergent. In other words, any model must provide a cut-off Λ to the form factor in order to regularize the $\pi^0 \rightarrow e^+e^-$ amplitude, which in the case of a pointlike F_π , would be logarithmically divergent. These remarks allow us to write

$$\text{Re } R(0) = -3 \ln(\Lambda/m_e) + \text{const.} + \dots , \quad (14)$$

where only the constant term [and the higher ones of order $(m_e/\Lambda) \ln(\Lambda/m_e)$] depend on the details of F_π . Since in any realistic model one has $\Lambda \gg m_e$ (see below), we conclude that our amplitude, when calculated in the soft pion limit, is dominated by the log term in Eq. (14). The already discussed (analytical) Q^2 duality at $q^2 = 0$ follows nicely after the identifications $\Lambda \approx 2 M_Q$ and $\Lambda \approx M_V$, putting on a more rational basis the underlying constituent-quark picture of our quark-loop model. To follow the discussion, one should study the dispersive integral in Eq. (13), which describes how $R(q^2)$ extrapolates from $q^2 = 0$ to $q^2 = m_\pi^2$. As observed before, the crucial contribution from the (model-independent) $\gamma\gamma$ cut explains the peaked behaviour of the real part of the amplitude. Other additional intermediate states appear only for $q^2 \geq \Lambda^2$, and, therefore, their (model-dependent) contribution to the subtracted dispersive integral is suppressed, since one has $m_e^2 \ll m_\pi^2 \ll \Lambda^2$. Keeping only the $\gamma\gamma$ contribution, we find, for $q^2 \gg m_e^2$

$$\text{Re } R(q^2) \simeq -3 \ln(1/m_e) + \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right), \quad (15)$$

where $\beta = \sqrt{1-4m_e^2/q^2}$. We have neglected a constant term and smaller, $O[m_e^2/q^2 \ln(q^2/m_e^2)]$ terms in Eq. (15) ^{*)}. In Fig. 2, one can compare our Eq. (15) with $\Lambda = M_\rho$, with the results coming from the different models so far discussed, and convince oneself of the correctness of our claims. The emerging conclusion is that the real part of the $\pi^0 \rightarrow e^+e^-$ amplitude in the soft limit depends on the cut-off Λ of the form factor (but not on the details of the model) and that the extrapolation to the pion mass is performed in a model-independent way.

We now turn our attention to the comparison of our predictions with the experimental data. From the preceding discussion, it follows that any model with a cut-off Λ will have a real part of the $\pi^0 \rightarrow e^+e^-$ amplitude given by Eq. (15). Introducing in this equation $\Lambda = M_\rho$ and $t = m_\pi^2$, we find $\text{Re } R(m_\pi^2) = 9.2$, [which, of course, is essentially the value quoted in Eqs. (11) and (12)] and consequently, we have

$$B^\pi \simeq 1.3 B_{\text{unit}}^\pi = 6 \times 10^{-8} \quad (16)$$

which is considerably smaller than the experimental values, Eq. (5). Our prediction (16) is stable for small variations of t around the physical pion mass. As far as we know, there are two models [Refs. (5) and (11)] which predict values for B^π not far from the experimental ones. However, once the computational errors in the baryon-loop approach ⁵⁾ are corrected ⁹⁾, it shows the features discussed in the present note and in particular, predicts $B_{\text{BL}}^\pi(m_\pi^2) \simeq 1.1 B_{\text{unit}}^\pi \ll B_{\text{exp}}^\pi$. The approach followed in Ref. 11) is based on the hypothesis $\text{Re } R(0) = 0$. Our discussion following Eq. (14) excludes this possibility. In addition, the treatment in Ref. 11) has been shown to correspond to a highly unrealistic form factor ¹²⁾.

^{*)} The right-hand side of Eq. (15) is still a good approximation when $q^2 \rightarrow 0$. Indeed, the second term goes to zero in this limit [as it must; see the second term of the right-hand side of Eq. (13)] and the first term is dominating. We notice that, for $q^2 < m_e^2$, one must perform an analytical continuation of Eq. (15).

The claim that B^π is not sensitive to the peculiarities of the form factor F_π but only to the cut-off Λ [or, alternatively, the validity of Eq. (15)] is based on the fact that the scales involved in $\pi^0 \rightarrow e^+e^-$ are such that $m_e^2 \ll m_\pi^2 \ll \Lambda^2$. Because of this, we would like finally to discuss some aspects related to the scale Λ . First of all, this is the relevant scale in the $\pi^0 \rightarrow e^+e^- \gamma$ Dalitz decay. Unfortunately, the data on the slope of the form factor corresponding to that decay are not conclusive. However, by writing a dispersion relation in the off-shell photon variable, s , one sees that the first singularity appears at $s = 4m_\pi^2$ corresponding to a two-pion 1^{--} (p wave) intermediate state. This is easily found to be negligible when compared with the resonant ρ and ω contributions, which therefore set the scale for the form factor^{*)}. We can also investigate the closely related decays $\eta \rightarrow \mu^+\mu^- \gamma$ and $\eta \rightarrow \mu^+\mu^-$. One is able to describe consistently the experimental data for both decays in a VMD framework and in a quark-loop model, with $\Lambda_\eta \approx M_{\rho,\omega} \approx 2M_{u,d}$ ^{4),9),14)} which is the cut-off used in our prediction, Eq. (16). Finally, as an illustration of the insensitivity of our results on Λ , one can ask for the cut-off that would predict a theoretical value in agreement with the experimental result. By means of Eq. (15), one finds the huge value $\Lambda \approx 4 \times 10^5$ GeV. [With the same cut-off, this would imply $B^\eta \equiv \Gamma(\eta \rightarrow \mu^+\mu^-)/\Gamma(\eta \rightarrow \gamma\gamma) \approx 700 B_{\text{unit}}^\eta$, to compare with $B_{\text{exp}}^\eta \approx B_{\text{unit}}^\eta$ ¹⁵⁾]. We conclude that conventional theory is unable to explain the experimental results on $\pi^0 \rightarrow e^+e^-$. Further data on this decay are therefore desirable, and the possibility of measuring smaller values for B_{exp}^π - as has been the case for the analogous $\eta \rightarrow \mu^+\mu^-$ decay - is highly favoured from the theoretical point of view. Concerning the situation for the $\eta \rightarrow \mu^+\mu^-$ decay, (and for the e.m. part of the analogous $K_L \rightarrow \mu^+\mu^-$ process), the claims for a model-independent prediction are not as transparent as for $\pi^0 \rightarrow e^+e^-$. This is because $m_{\eta,K}^2$, M_ρ^2 and $4m_\mu^2$ are roughly of the same order, invalidating Eq. (15) for these decays. However, here also we find that Q^2 duality works remarkably well. Indeed, for all four models discussed above, we get (with $\Lambda \approx M_\rho$) the common prediction $B^\eta \approx (1.1-1.3) B_{\text{unit}}^\eta$, which moreover compares favourably with the experimental value¹³⁾. We note that the smearing of the form factor achieved by the loop integration in Eq. (2) seems to be sufficient for duality to work well also for $\eta, K_L \rightarrow \mu^+\mu^-$. We thus propose for these processes also model-insensitive predictions for the e.m. contributions close to the unitarity limits.

^{*)} In some of the older work this was not realized, since there were experimental data indicating at that time that the $\rho\pi\pi$ coupling was vanishing. With the more recent data, however, this approach has been shown to be beautifully consistent, and moreover it has been shown that QCD corrections change this picture very little¹³⁾.

Our analysis of the $\pi^0 \rightarrow e^+e^-$ amplitude has led to the following conclusions: i) Q^2 duality (i.e., the equivalence between VMD and quark models) has been shown to work extremely well for this process and also for the related $\eta \rightarrow \mu^+\mu^-$ decay; ii) the characteristic mass scales involved in this decay, $m_e^2 \ll m_\pi^2 \ll \Lambda^2$, where Λ is the characteristic cut-off of the form factor, imply universal results for the amplitude; iii) the branching ratio $B^\pi \equiv \Gamma(\pi^0 \rightarrow e^+e^-)/\Gamma(\pi^0 \rightarrow \gamma\gamma)$ does not depend on the detailed structure of the relevant form factor but only on the cut-off Λ ; and iv) all reasonable models consistently predict $B^\pi \approx 1.3 B_{\text{unit}}^\pi$, which is considerably smaller than the available experimental data.

ACKNOWLEDGEMENTS

We would like to thank D. Amati and J.S. Bell for useful discussions and for reading the manuscript. One of us (E.M.) is grateful to the "Caixa de Barcelona" for financial support.

REFERENCES

- 1) S.D. Drell, Nuovo Cimento 11 (1959) 693.
- 2) S.M. Berman and D.A. Geffen, Nuovo Cimento 18 (1960) 1192.
- 3) B.L. Young, Phys. Rev. 161 (1967) 1620.
- 4) C. Quigg and J.D. Jackson, UCRL Report No. 18487 (1968) (unpublished).
- 5) M. Pratap and J. Smith, Phys. Rev. D5 (1972) 2020.
- 6) L. Bergström, Z. Phys. C14 (1982) 129.
- 7) R.E. Mischke et al., Phys. Rev. Lett. 48 (1982) 1153.
- 8) J. Fischer et al., Phys. Lett. 73B (1978) 364; Phys. Lett. 76B (1978) 663 (E).
- 9) L.L. Ametller, L. Bergström, A. Bramon and E. Massó, CERN preprint TH.3501 (1983).
- 10) A. Bramon, E. Etim and M. Greco, Phys. Lett. 41B (1972) 609;
J.J. Sakurai, Phys. Lett. 46B (1973) 207.
- 11) G.B. Tupper and M.A. Samuel, Phys. Rev. D26 (1982) 3302.
- 12) L. Bergström and E. Ma, University of Hawaii at Manoa preprint UH-511-485-82 (1982).
- 13) M.A. Shifman and M.I. Vysotsky, Z. Phys. C10 (1981) 131.
- 14) A. Bramon and E. Massó, Phys. Lett. 104B (1981) 311;
K.S. Babu and E. Ma, Phys. Lett. 119B (1982) 449.
- 15) R.I. Dzhelyadin et al., Phys. Lett. 97B (1980) 471.

FIGURE CAPTIONS

Fig. 1 : Diagram for $\pi^0 \rightarrow e^+e^-$ with a generic form factor $F = F_\pi(k^2, (q-k)^2)$ governing the $\pi^0\gamma\gamma$ vertex.

Fig. 2 : The real part of R [Eq. (2)] as a function of q^2 (square of the pion mass), normalized to the model-independent values of $\text{Im } R(m_\pi^2)$ [Eq. (3)]. Curves are shown for the quark-loop form factor (marked with crosses), the formula Eq. (15) (filled circles) and the VMD2 form factor, Eq. (9b) (filled triangles). In all cases, the cut-off Λ is chosen to be $\Lambda \sim M_{\rho,\omega} \sim 0.77$ GeV. The results for the VMD1 and VMD3 models [Eqs. (9a) and (9c)] essentially coincide with the curve for Eq. (15). An arrow indicates the ratio $\text{Re}R/\text{Im}R$ at the physical pion mass.

Fig. 1

Fig. 2