Ref.TH.3526-CERN

EXPLICIT EVALUATION OF PHYSICAL QUANTITIES AND SUPERSYMMETRY PROPERTIES
OF LATTICE $O(N)$ σ MODELS AT LARGE N

P. Di Vecchia

Physics Department, University of Wuppertal
D-5600 Wuppertal 1, West Germany

R. Musto, F. Nicodemi and R. Pettorino

Istituto di Fisica Teorica dell'Università di Napoli
and INFN, Sezione di Napoli

P. Rossi

CERN - Geneva

and

P. Salomonson

Institute of Theoretical Physics, Göteborg, Sweden

A B S T R A C T

We explicitly show how to extract physical quantities of the continuum theory from two lattice versions of the non-linear σ model, that are exactly solvable for $N \rightarrow \infty$. We also build a lattice action which is supersymmetric in the large N limit and suggest how to recognize supersymmetry restoration in the continuum even when starting from a non-supersymmetric lattice formulation.

Ref.TH.3526-CERN

February 1983

A very convenient way to study non-perturbative phenomena in gauge theories is to introduce a lattice structure in the Euclidean spacetime ¹⁾. The quantities of the continuum theory can then be determined by sending the lattice distance a to zero (this corresponds, in an asymptotic free theory, to the limit of small coupling constant) and by extracting, in this limit, a behaviour in agreement with the prediction of the renormalization group. This procedure has been used to compute the string tension ²⁾ and the glueball mass ³⁾ in lattice Yang-Mills theory by means of the Monte Carlo technique. Other quantities like the topological charge fluctuation ⁴⁾ and the gluon condensate ⁵⁾ have also been computed with the same technique ; however, in this case the subtraction of a perturbative tail was necessary before one could isolate the renormalization group behaviour. Of course, such a tail depends on the particular definition used for the quantities on a lattice, while their non-perturbative behaviour in the scaling region is universal.

In order to show explicitly how to extract physical quantities of the continuum theory from a lattice theory, we consider in this letter the asymptotically free $O(N)$ non-linear σ model, that can be explicitly solved for large N both in the lattice and in the continuum theory. In particular, we show that physical quantities as the magnetic susceptibility, the correlation length and $\langle (1/2) \partial_\mu \vec{S}_\mu \partial_\mu \vec{S}_\mu \rangle$ can uniquely be extracted from the lattice theory and that they are independent from the particular lattice version used.

In the second part of this letter we study the problem of supersymmetry on a lattice. In particular, we consider two lattice versions of the $O(N)$ non-linear σ model with fermions, that are supersymmetric in the naïve continuum limit, but not on the lattice. We show that one of these versions has no phase transition and its leading order in N is supersymmetric, while the other has a phase transition and stays not supersymmetric also for large N . However, both of them in the scaling region show the existence of a supersymmetric continuum theory, whose physical quantities can be computed using any of the two lattice versions.

We start from two different lattice formulations of the $O(N)$ non-linear σ model, described by the following actions :

$$\begin{aligned} -S_2 &= \frac{1}{2} \sum_{n,\mu} \vec{S}_{n+\mu} \cdot \vec{S}_n \\ -S_4 &= \frac{1}{2g} \sum_{n,\mu} (\vec{S}_{n+\mu} \cdot \vec{S}_n)^2 \end{aligned} \tag{1}$$

together with the constraint $\vec{S}_n^2 = 1$.

In order to perform the $1/N$ expansion of these theories we introduce in S_4 an auxiliary real field $\lambda_{n,\mu}$, that makes it quadratic in the fields S_n , we use for the δ function of the constraint the usual exponential representation and, after a rescaling of the fields $\vec{S}_n \rightarrow \sqrt{g} \vec{S}_n$, $\alpha_n \rightarrow (\alpha_n/g)$ we can perform the Gaussian functional integration over $\vec{S}_n$ getting the following effective actions :

$$-\frac{S_2}{N}^{\text{eff}} = -i\beta \sum_n \alpha_n - \frac{1}{2} \text{Tr} \text{Ln} \left(\sum_\mu \cos p_\mu \delta(p+q) + i\alpha(p+q) \right) \quad (2)$$

$$-\frac{S_4}{N}^{\text{eff}} = -i\beta \sum_n \alpha_n - \frac{1}{2} \beta \sum_{n,\mu} \lambda_{n,\mu}^2 - \frac{1}{2} \text{Tr} \text{Ln} \left(i\alpha(p+q) + \sum_\mu \lambda_\mu(p+q) e^{ipr} \right) \quad (3)$$

where we have introduced the Fourier transforms of the fields α and λ :

$$\alpha_n = \sum_p e^{ipn} \alpha(p) \quad \lambda_{n,\mu} = \sum_p e^{ipn} \lambda_\mu(p) \quad (4)$$

and $\beta = 1/gN$. For the sake of simplicity we use the same symbols g, β for the coupling in both actions.

At large N and fixed β we can restrict ourselves to constant fields :

$$\alpha(p) = i\alpha \delta(p) \quad \lambda_\mu(p) = \lambda_\mu \delta(p) \quad (5)$$

and we get the following saddle point condition :

$$2\alpha\beta = f_0\left(\frac{1}{\alpha}\right) \quad (6)$$

for the quadratic action S_2 and

$$2\alpha\beta = f_0\left(\frac{\lambda}{\alpha}\right) \quad (7)$$

$$\alpha - 2\lambda^2 = \frac{1}{2\beta} \quad (8)$$

for the quartic action S_4 . $f_0(\gamma)$ is given by

$$f_0(\gamma) = \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{1}{1 - \gamma \sum_\mu \cos p_\mu} \quad (9)$$

In two dimensions $f_0(\gamma) = 2/\pi K(2\gamma)$ where K is the complete elliptic function of the first kind.

The saddle point condition (6) describes a unique phase, while the conditions (7) and (8) imply a first-order phase transition occurring for $2\beta \sim 0.956$ for the quartic action. In the strong coupling phase we have

$$\alpha = \frac{1}{2\beta} \quad \lambda = 0 \quad (10)$$

while in the weak coupling phase α and λ are both different from zero. The details of this phase transition are described in a future publication ⁶⁾; here we are only interested in studying the transition to the continuum theory, which is obtained by staying in the weak coupling phase, since the σ model is asymptotically free.

The spontaneously generated mass of the continuum theory can be obtained from (6) and (7) for large β (or equivalently for $\gamma \rightarrow \frac{1}{2}$) by using the asymptotic formula

$$f_0(\gamma) \xrightarrow{\gamma \rightarrow \frac{1}{2}} \frac{1}{\pi} \ln \frac{16}{1-4\gamma^2} + O(1-2\gamma) \quad (11)$$

Inserting (11) into (6) and (7) we get the following asymptotic behaviours :

$$\frac{1}{\alpha} \xrightarrow{\beta \rightarrow \infty} \frac{1}{2} - 4e^{-4\pi\beta} \quad \frac{\lambda}{\alpha} \xrightarrow{\beta \rightarrow \infty} \frac{1}{2} - 4e^{-2\pi\beta[1+\sqrt{1-\frac{1}{\beta}}]} \rightarrow \frac{1}{2} - 4e^{\pi} e^{-4\pi\beta} \quad (12)$$

for the two theories in (1), respectively.

The mass (i.e., the inverse correlation length) is defined in the two theories by the relations :

$$am = ch^{-1}(\alpha-1) \quad am = ch^{-1}\left(\frac{\alpha}{\lambda}-1\right) \quad (13)$$

and in the continuum theory this reduces to

$$a^2 m^2 = 2(\alpha-2) \quad a^2 m^2 = 2\left(\frac{\alpha}{\lambda}-2\right) \quad (13')$$

Inserting finally the asymptotic formulas (12) in (13) we get :

$$a^2 m^2 \xrightarrow{\beta \rightarrow \infty} 32e^{-4\pi\beta} \quad a^2 m^2 \xrightarrow{\beta \rightarrow \infty} 32e^{\pi} e^{-4\pi\beta} \quad (14)$$

We find therefore the right renormalization group behaviour, but the constant in front of the exponential is different. This is, however, due to the fact that $\Lambda_2^{(\text{lattice})} \neq \Lambda_4^{(\text{lattice})} \neq \Lambda^{(\text{continuum})}$. A relation between the various Λ 's can be obtained from the requirement that we renormalize the theory keeping a physical parameter fixed as, for instance, the mass. In this case, if we define in the different regularizations the corresponding Λ parameters as follows :

$$\Lambda_2^{(\text{lattice})} = \frac{1}{a} e^{-2\pi\beta} \quad \Lambda_4^{(\text{lattice})} = \frac{1}{a} e^{-\pi\beta} \quad \Lambda^{(\text{continuum})} = \Lambda e^{-2\pi\beta} \quad (15)$$

we get from (14) that the mass spontaneously generated is given by the following formulas in terms of the various Λ 's :

$$m = \sqrt{32} \Lambda_2^{(\text{lattice})} = \sqrt{32} e^{\frac{\pi}{2}} \Lambda_4^{(\text{lattice})} = \Lambda^{(\text{continuum})} \quad (16)$$

Remember that the saddle point condition corresponding to (6) and (7) in the continuum theory, after a Pauli-Villars regularization, is given by

$$\beta = \frac{1}{4\pi} \ln \frac{\Lambda_{(\text{cont})}^2}{m^2} \quad (17)$$

We now compute the two-point function for the field ϕ_n^i for N large in the two theories defined by the actions in (1). An easy calculation gives the following result :

$$\langle S_n^i S_m^j \rangle = \frac{1}{2\alpha\beta N} \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \frac{e^{ip(n-m)}}{1 - \gamma \sum_{\mu} \omega_{\mu} p_{\mu}} \delta^{ij} \quad (18)$$

where $\gamma = 1/\alpha(\lambda/\alpha)$ in the case of the quadratic (quartic) action. The magnetic susceptibility can easily be obtained from (18) by summing over i and m :

$$\sum_{i=1}^N \langle S_n^i \sum_m S_m^i \rangle = \frac{1}{2\alpha\beta} \frac{1}{1-2\gamma} \quad (19)$$

By using the asymptotic formulas (12), we find that (19) behaves in the limit $\beta \rightarrow \infty$ as required by the renormalization group. Then, remembering the relations (15) and (16) we get in this limit the following value for the magnetic susceptibility

$$\sum_{i=1}^N \langle \vec{s}_i \cdot \sum_m \vec{s}_m \rangle = \frac{1}{a^2 m^2 \beta} \quad (20)$$

in complete agreement with the result of the continuum theory.

Finally, we want to show how to extract from the lattice theory the non-perturbative renormalization group invariant quantity $\langle (1/2) \vec{S} \square \vec{S} \rangle$ of the continuum theory, which corresponds to the gluon condensate in Yang-Mills theory. It can easily be computed in the continuum theory for large N and one gets :

$$\langle \frac{1}{2} \vec{S} \square \vec{S} \rangle = \frac{1}{2} m^2 \quad (21)$$

The continuum theory is defined by the following Lagrangian

$$L = \frac{1}{2g} \partial_\mu \vec{S} \cdot \partial_\mu \vec{S} \quad (22)$$

with the constraint $\vec{S} \cdot \vec{S} = 1$.

In the lattice theory, the quantity (21) can be extracted by computing the internal energy which is given by :

$$\langle \sum_v (\vec{S}_{m+v} \cdot \vec{S}_m - 1) \rangle = \alpha - 2 - \frac{1}{2\beta} \quad (23)$$

$$\langle \frac{1}{2} \sum_v ((\vec{S}_{m+v} \cdot \vec{S}_m)^2 - 1) \rangle = \lambda^2 - 1 \quad (24)$$

for the quadratic and the quartic actions, respectively. On the right-hand side, we have written the expressions obtained for large N . By using the asymptotic formulas (12) together with the identity :

$$\lambda^2 = \frac{1}{8\gamma^2} \left[1 + \sqrt{1 - \frac{4\gamma^2}{\beta}} - \frac{2\gamma^2}{\beta} \right] \quad \gamma = \frac{\lambda}{\alpha} \quad (25)$$

we obtain the following behaviour for large β for (23) and (24) :

$$\langle \sum_v (\vec{S}_{m+v} \cdot \vec{S}_m - 1) \rangle \xrightarrow{\beta \rightarrow \infty} -\frac{1}{2\beta} + 16 e^{-4\pi\beta} + \dots \quad (26)$$

$$\langle \frac{1}{2} \sum_v ((\vec{S}_{m+v} \cdot \vec{S}_m)^2 - 1) \rangle \xrightarrow{\beta \rightarrow \infty} \left[-\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{1}{\beta}} - \frac{1}{4\beta} \right] + 16e^{\pi} e^{-4\pi\beta} + \dots \quad (27)$$

Therefore we get a perturbative tail which is dependent on the particular regularization procedure and which has no meaning in the continuum theory. In addition we get an exponential behaviour which is in agreement with the renormalization group prediction. The use of Eqs. (16) tells us that the non-perturbative part of (26) and (27) is universal and is indeed equal to the result (21) of the continuum theory.

We have also computed (26) and (27) by using a mean field approximation. We have found that a naïve ⁷⁾ and a more sophisticated (1/d expandable) version ⁸⁾ of the mean field theory reproduces exactly the perturbative tails of (26) and (27). In addition, the one-loop correction in the expansion around the mean field gives an additional contribution to the free energy thus matching its large N perturbative value, and leaves the internal energy unchanged. It is clear from this example that a mean field theory does not give any information on the quantities of the continuum theory. Our example suggests also that a mean field approximation can be used in order to compute the perturbative tail ⁹⁾.

In the second part of this letter, we introduce fermions in the actions in (1) with the aim of studying supersymmetry on the lattice. The supersymmetric O(N) σ model in the continuum theory is described by the following free action ¹⁰⁾ :

$$S = \int d^2x \left\{ \frac{1}{2} (\partial_\mu \vec{S})^2 - \frac{i}{2} \vec{\Psi} \gamma^\mu \partial_\mu \vec{\Psi} - \frac{1}{2} (\vec{F})^2 \right\} \quad (28)$$

with the addition of the constraints

$$\vec{S}^2 = 1 \quad \vec{S} \cdot \vec{\Psi} = 0 \quad \vec{F} \cdot \vec{S} = \frac{i}{2} \vec{\Psi} \gamma_5 \vec{\Psi} \quad (29)$$

Both the action (28) and the constraints (29) are left invariant by the following supersymmetry transformations :

$$\delta \vec{S} = i\epsilon \vec{\Psi} \quad , \quad \delta \vec{\Psi} = (\gamma^\mu \partial_\mu \vec{S} + \gamma_5 \vec{F})\epsilon \quad , \quad \delta \vec{F} = i\epsilon \gamma_5 \gamma^\mu \partial_\mu \vec{\Psi} \quad (30)$$

It is straightforward to write a lattice version of the free action (28)

$$-S_2 = \frac{1}{4g} \left\{ \sum_{n,\mu} \left[(\bar{S}_{n+\mu} \cdot \bar{S}_{n-\mu} - 1) + i \bar{\psi}_n \gamma^\mu (\bar{\psi}_{n+\mu} - \bar{\psi}_{n-\mu}) \right] + 2 \sum_n \bar{F}_n^2 \right\} \quad (31)$$

where the bosonic part contains fields located at points separated by two links like the fermions in order to have the same number of bosonic and fermionic degrees of freedom as is necessary in a supersymmetric theory¹¹⁾. In Eq. (31), we have added a field independent term which is essential if we are interested in computing the vacuum energy. We have omitted this term in the purely bosonic Lagrangian, since its presence was inessential for the considerations of the first part of this letter. The supersymmetry transformations on the lattice written for fields in momentum space are :

$$\begin{aligned} \delta S(p) &= i \epsilon \psi(p) \quad , \quad \delta \psi(p) = (i \gamma^\mu \sin p_\mu S(p) + \gamma_5 F(p)) \epsilon \\ \delta F(p) &= -\epsilon \gamma_5 \gamma^\mu \sin p_\mu \psi(p) \end{aligned} \quad (32)$$

It is easy to check that (31) is left invariant by these transformations. The constraints

$$\bar{S}_n^2 = 1 \quad \bar{S}_n \bar{\psi}_n = 0 \quad (33)$$

are, however, not preserved by the transformations (32) and therefore the theory described by (31) and (33) is not supersymmetric. It becomes, however, naïvely supersymmetric in the continuum limit.

Using the constraint (33) on F we can eliminate the field F from the action (31) and get a quartic term for the fermions :

$$-S_2 = \frac{1}{4g} \left\{ \sum_{n,\mu} \left[(\bar{S}_{n+\mu} \cdot \bar{S}_{n-\mu} - 1) + i \bar{\psi}_n \gamma^\mu (\bar{\psi}_{n+\mu} - \bar{\psi}_{n-\mu}) \right] - \frac{1}{2} \sum_n (\bar{\psi}_n \gamma_5 \bar{\psi}_n)^2 \right\} \quad (34)$$

S_2 should be supplemented with the constraints :

$$\bar{S}_n^2 = 1 \quad \bar{S}_n \bar{\psi}_n = 0 \quad (35)$$

As in the case of the purely bosonic theory we can write another version of the lattice $O(N)$ σ model described by the following action :

$$-S_4 = \frac{1}{4g} \left\{ \sum_{n,\mu} \left[\left(\frac{\vec{S}_{n+\mu} \cdot \vec{S}_{n-\mu}}{2} - 1 \right) + i \bar{\Psi}_n \gamma^\mu (\vec{\Psi}_{n+\mu} - \vec{\Psi}_{n-\mu}) \right] - \frac{1}{2} \sum_n (\bar{\Psi}_n \gamma_5 \vec{\Psi}_n)^2 \right\} \quad (36)$$

together with the constraints (35).

The $1/N$ expansion of the two previous theories can be performed by introducing the auxiliary fields ϕ_n and $\lambda_{n,\mu}$ which make the actions (34) and (36) quadratic in $\vec{\Psi}$ and $\vec{S}$ and by exponentiating the two constraints (35). We can then integrate over the fields $\vec{S}$ and ψ and we get the following effective Lagrangians (with the fermionic multiplier $C = 0$) :

$$-S_2^{eff} = -\beta N \sum_n \left(i\alpha_n + \frac{1}{2} \phi_n^2 \right) - \frac{N}{2} \text{Tr} \text{Ln} 2 \left(i\alpha(p+q) + \sum_\mu \cos p_\mu \delta(p+q) \right) + \frac{N}{2} \text{Tr} \text{Ln} \left(i\gamma_5 \phi(p+q) + \sum_\mu 2\gamma^\mu \sin p_\mu \delta(p+q) \right) \quad (37)$$

$$-S_4^{eff} = -\beta N \sum_n \left(i\alpha_n + \frac{1}{2} \phi_n^2 + \frac{1}{2} \sum_\mu \lambda_{n,\mu}^2 \right) - \frac{N}{2} \text{Tr} \text{Ln} 2 \left(i\alpha(p+q) + \sum_\mu \lambda_\mu(p+q) \cos 2p_\mu \right) + \frac{N}{2} \text{Tr} \text{Ln} \left(i\gamma_5 \phi(p+q) + \sum_\mu 2\gamma^\mu \sin p_\mu \delta(p+q) \right) \quad (38)$$

respectively, for the actions (34) and (36), where $\beta = 1/4gN$.

For large N it is sufficient to consider constant fields

$$\alpha(p) = i\alpha \delta(p) \quad \phi(p) = \phi \delta(p) \quad \lambda_\mu(p) = \delta(p) \lambda_\mu \quad (39)$$

and we get the following saddle point conditions

$$8\beta\alpha = f_0\left(\frac{1}{\alpha}\right) \quad 8\beta \frac{\phi^2 + 4}{2} = f_0\left(\frac{2}{\phi^2 + 4}\right) \quad (40)$$

$$8\beta\alpha = f_0\left(\frac{\lambda}{\alpha}\right) \quad 8\beta \left(\frac{\phi^2 + 4}{2}\right) = f_0\left(\frac{2}{\phi^2 + 4}\right) \quad (41)$$

$$\alpha - 2\lambda^2 = \frac{1}{2\beta}$$

respectively, for the two theories (34) and (36).

The two conditions (40) imply a relation between α and ϕ

$$\alpha = \frac{\phi^2 + 4}{2} \quad (42)$$

which, when it is inserted in (37) with the constant fields (39), gives a vanishing vacuum energy. This fact and the equality between the fermionic and bosonic correlation lengths implied by (42) for any β suggests that the large N limit of the quadratic theory is supersymmetric. This is true in fact because the action (37) rewritten before integrating over the fields $\vec{S}$ and $\vec{\psi}$ for $C = 0$ and constant α and ϕ , is invariant under the transformations (32) as it has already been proved.

In the case of the quartic action there is no relation between the two conditions in (41) : therefore in general the bosonic and fermionic correlation lengths are different even for $N \rightarrow \infty$ and the theory is not supersymmetric. This symmetry is, however, restored in the limit $\beta \rightarrow \infty$. In this limit, in fact, the bosonic and fermionic correlation lengths become equal as can be seen by using the asymptotic formulas (11) and (12). Their value is also equal to the common correlation length which is extracted from the theory with the quadratic action reproducing also the result of the continuum theory.

In conclusion, it is nice to have a supersymmetric lattice theory, but if we want to use the lattice only as a way to regularize the quantum theory, it does not matter whether supersymmetry is lost together with Lorentz and translation invariance. If it is not spontaneously broken, its existence will anyway be recognized for sufficiently large β .

As a consequence of our analysis, the restoration of supersymmetry in the continuum limit for more complicated theories like four-dimensional lattice gauge theories can be checked, for example by means of the Monte Carlo technique or of strong coupling series, by showing that in the scaling region the fermionic and the bosonic glueballs have the same mass.

REFERENCES

- 1) R. Wilson - Phys.Rev. D10 (1974) 2445.
- 2) M. Creutz - Phys.Rev.Letters 43 (1979) 553 ; Phys.Rev. D21 (1980) 2308 ;
E. Pietarinen - Nuclear Phys. B190 (1981) 349 ;
For a recent review, see :
B. Berg - Lecture Notes presented at the Johns Hopkins Workshop, Florence
(1982), CERN Preprint TH. 3327 (1982) ;
See also :
K.H. Mütter and K. Schilling - Phys.Letters 117B (1982) 75 and Wuppertal
Preprint WUB 82-14 (1982).
- 4) P. Di Vecchia, K. Fabricius, G.C. Rossi and G. Veneziano - Nuclear Phys.
B192 (1981) 392 ; Phys.Letters 108B (1982) 323 ;
K. Fabricius and G.C. Rossi - Rome Preprint (1983) ;
K. Ishikawa, G. Schierholz, H. Schneider and M. Teper - LAPP-TH-71 (1983).
- 5) A. Di Giacomo and G.C. Rossi - Phys.Letters 100B (1981) 481 ;
T. Banks, R. Horsley, H.R. Rubinstein and U. Wolff - Nuclear Phys. B190
(1981) 692 ;
A. Di Giacomo and G. Paffuti - Phys.Letters 108B (1982) 327.
- 6) P. Di Vecchia, R. Musto, F. Nicodemi, R. Pettorino, P. Rossi and
P. Salomonson - in preparation.
- 7) R. Balian, J.-M. Drouffe and C. Itzykson - Phys.Rev. D10 (1974) 3375.
- 8) J.-M. Drouffe - Nuclear Phys. B170 (1980) 211 ;
E. Brezin and J.-M. Drouffe - Nuclear Phys. B200 (1982) 93.
- 9) Such an idea is being applied to compute the gluon condensate in Yang-Mills
theory by :
K. Fabricius, O. Hann and H. Schlichting - to appear.
- 10) P. Di Vecchia and S. Ferrara - Nuclear Phys. B130 (1977) 93 ;
E. Witten - Phys.Rev. 16 (1977) 2991.
- 11) We follow here the same procedure as in the paper by :
T. Banks and P. Windey - Nuclear Phys. B198 (1982) 226.