A STUDY OF ACTIONS WITH NEXT-TO-NEAREST NEIGHBOUR
INTERACTIONS IN 4-DIMENSIONAL Z_2 GAUGE THEORY.

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ABSTRACT

By Monte-Carlo simulation, we study a mixed action Z_2 theory in 4 dimensions. The action has terms involving planar 1×1 and 2×1 Wilson loops. We find a rich phase structure for this model, but no second order transitions. An investigation of a theory with bent 2×1 Wilson loops in the action was also done. This also revealed only first order transitions. The models we study have interesting frustrated phases with a strong dependence on the choice of boundary conditions.

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For various reasons^[1], it is interesting to generalize lattice gauge actions by including larger interaction loops. This is believed to improve the approach to continuum. Even for discrete gauge groups where such a limit does not usually exist, this action might produce interesting structures.

In this paper, we investigate the 4-dimensional Z_2 gauge theory with an extended action consisting of planar plaquettes with four and six links. This study has been motivated by several indications that this theory might have long range correlations^[2] and, possibly, a second order transition^[3]. In the two parameter space of couplings we investigate, we find only first order transitions. However, the phase structure of the model is quite complex and shows rather unusual frustrated phases.

The action of our theory is given by

$$S = \beta_1 \sum \sigma_{\square} + \beta_2 \sum \sigma_{\square\square}, \quad (1)$$

where $\sigma_{\square}$ is the product of four Z_2 link variables σ around a unit plaquette, and $\sigma_{\square\square}$ is the product of six link variables around a 2×1 planar Wilson loop. These (2×1) loops we will call "windows" for obvious reasons. For the moment, we assume periodic boundary conditions.

We define the two order parameters E_1 and E_2

$$E_1 = Z^{-1} \sum_{\{\sigma\}} \sigma_{\square} e^S = N_{\square}^{-1} \partial \ln Z / \partial \beta_1 = \langle \sigma_{\square} \rangle \quad (2a)$$

$$E_2 = Z^{-1} \sum_{\{\sigma\}} \sigma_{\square\square} e^S = N_{\square\square}^{-1} \partial \ln Z / \partial \beta_2 = \langle \sigma_{\square\square} \rangle \quad (2b)$$

$$\text{with} \quad Z = \sum_{\{\sigma\}} e^S \quad (2c)$$

where $N_{\square}$ and $N_{\square\square}$ are the number of plaquettes and windows respectively on the lattice.

In the limit $\beta_2 = 0$, the model reduces to the usual self-dual Wilson form of the Z_2 gauge theory with a first order transition at $\beta_1 = \pm 1/2 \ln(1+\sqrt{2})$. When $\beta_1 = 0$, the model becomes the one that was investigated by Edgar^[4], who showed that it also has a first order transition at $\beta_2 = 0.16 \pm 0.02$. In the $\beta_1\beta_2$ plane, the phase diagram must be symmetric under $\beta_1 \rightarrow -\beta_1$ because by a change of variables we can flip the sign of every unit plaquette^[5]. Similarly, there is a symmetry

$\beta_2 \rightarrow -\beta_2$, but only for $\beta_1 = 0$, obtained by flipping all window signs (multiplication of the configurations by a chessboard state described hereafter). At $\beta_2 = \infty$, all windows must equal unity. This means that, in every 2-dimensional plane, all the plaquettes must either be +1 or -1. Thus, given a configuration on a 4-dimensional hypercube h that satisfies the necessary constraints ($\prod \sigma_{\square} = 1$ for each of its 8 cubic faces), this requirement on each 2-plane determines the entire lattice configuration (by a recursive construction of the neighbouring hypercubes). Hence, these potentially disordering configurations on hypercubes (with some plaquettes equal to -1) have no entropy factor; among these 2^{17} configurations, the one with all plaquettes to +1 (resp. -1) is selected on energy basis as $\beta_1 > 0$ (resp. < 0). Therefore, there is no transition at any β_1 for $\beta_2 = \infty$, except $\beta_1 = 0$. This one point is a critical point across which E_1 flips from -1 to +1. Finally, the limit $\beta_2 = -\infty$ has every window equal to -1. Thus, there must be as many unit plaquettes which equal +1 as there are those which are -1, leading to "chessboard" configurations in each 2-plane¹. This means that the two order parameters are both determined, $E_1 = 0$, $E_2 = -1$. A similar construction of the 2^{17} configurations from a given hypercube configuration applies; but these states are now degenerate in energy, since $E_1 = 0$, leaving no scope for a phase transition, even now at $\beta_1 = 0$. Note that the limit $\beta_1 = \infty$ is trivial because $E_1 = 1$ implies $E_2 = 1$.

We have investigated the phase diagram of this model in the entire $\beta_1\beta_2$ plane by standard Monte-Carlo methods. Potential phase transition points were identified by hysteresis cycle runs and then investigated by iterations starting from ordered, disordered, frustrated and mixed phase lattice configurations to determine the order of the transition and the latent heat. We worked on a 4^4 lattice using the heat-bath algorithm of Creutz, Jacobs and Rebbi^[6], and did (typically) 100 thermalizing and 20 averaging iterations in hysteresis cycle runs, and 300 thermalizing and 200 averaging iterations in the runs to determine latent heats. A typical hysteresis cycle run at $\beta_1 = 1.0$ with varying

¹ Note that this is only possible with free boundary conditions, or periodic boundary conditions with even lattice sizes. In the xy plane, a configuration which realizes the condition $E_1 = 0$, $E_2 = -1$ is given in Figure 1. The case of other boundary conditions (periodic with odd sizes or skew periodic) always leads to violation of the chessboard scheme and then induces boundary effects.

β_2 is shown in Figure 2 for the order parameter E_2 . Two phase transitions are clearly evident in this figure. The complete phase diagram of the theory in the $\beta_1\beta_2$ plane is shown in Figure 3. The interesting question: "Do the lines AB and CD in Figure 3 collide or end before going out to $\beta_1 = \infty$, $\beta_2 = -\infty$ " is considered in the Appendix A. We have also measured the latent heat L at some critical points along the line AB of Figure 3. This is shown in Figure 4 as a function of β_1 . Notice that as β_1 increases, i.e. as we move down the transition line from A to B, the latent heat first decreases and then flattens out. This prevents the first order transition from melting into a second order one.

Finally, we looked at the model defined by

$$S_3 = \beta_3 \sum \sigma_{\text{chair}}, \quad (3)$$

where chair denotes a six-sided Wilson loop which looks like a chair. The order parameter of interest here is

$$E_3 = \langle \sigma_{\text{chair}} \rangle. \quad (4)$$

At $\beta_3 = \infty$, $E_3 = 1$ which is easy to achieve by setting up all links to +1. However, as $\beta_3 \rightarrow -\infty$, E_3 would like to be as negative as possible. E_3 is bounded from below in any number of dimensions, and this lower bound is not -1, even with periodic boundary conditions. It is easy to show on a three-dimensional cube that the most negative value of E_3 is -1/3. Hence $E_3 \geq -1/3$. The hope now is that, for negative β_3 , the phase transition one expects to find may be second order, because of this lower bound on E_3 which leads to frustration effects. Alas, it is not true. The model defined by equation (3) has transitions at $\beta_{3C}^+ = 0.045 \pm 0.005$ and $\beta_{3C}^- = -0.15 \pm 0.01$, which are both first order. Hence we expect that even models with mixed actions involving terms like S_3 may not show second order transitions.

Where does one go from here? A block spin, renormalization group type of investigation with several couplings should be attempted. At the very least, it would indicate what kind of Wilson loops are relevant in the search for a second order transition. More optimistically, such a study may reveal a second order fixed point and its critical exponents.

APPENDIX A

Theoretical considerations on the phase diagram.

For a given configuration, let us denote the fraction of windows containing both plaquettes to ± 1 by $p_{\pm}$. For these variables, the entropy (logarithm of the number of configurations for fixed p_{+} and p_{-}) per plaquette is written $\mathcal{J}(p_{+}, p_{-})$, while the energy per plaquette is

$$\mathcal{E} = \langle 2S / Nd(d-1) \rangle = \beta_1 E_1 + 2 \beta_2 E_2$$

$$\text{with } E_1 = p_{+} - p_{-}, \quad E_2 = 2 p_{+} + 2 p_{-} - 1$$

To determine the physical solution, the free energy

$$\mathcal{F} = \mathcal{E} + \mathcal{J}(p_{+}, p_{-})$$

is to be maximized. It is, however, difficult to give an explicit expression for $\mathcal{J}$. In the boundary regions of the phase diagram, the entropy is negligible with respect to the energy; as $\mathcal{E}$ is linear in p_{+} and p_{-} , only the edges of the allowed triangle (see Figure 5) will maximize $\mathcal{E}$. These are:

-point B (resp. D) if $|\beta_1| + 4 \beta_2 > 0$, $\beta_1 > 0$ (resp. $\beta_1 < 0$), corresponding to $E_1 = 1$ (resp. -1), $E_2 = 1$. All plaquettes are set to 1 (resp. -1).

-point C if $|\beta_1| + 4 \beta_2 < 0$, corresponding to $E_1 = 0$, $E_2 = -1$ (chessboard configurations).

Along the segment BD, there is only a finite number of possible configurations, as already stated in this paper; thus the entropy per plaquette vanishes. When β_1 changes its sign for large β_2 , the system will jump directly from B to D, yielding a first order phase transition.

This is not the case on BC (or on CD). States on this segment are characterized by the fact that any -1 plaquette is isolated, i.e. cannot form a window with another neighbouring -1 plaquette. They can be built from elementary excitations which are obtained by flipping the 2^{d-1} links pointing in a given direction of a given hypercube. The entropy $\mathcal{J}$ along the segment BC starts with an infinite slope (dilute gas of excitations) and may display one of the forms shown in Figure 6.

In figure 6a, the maximum of the free energy will flow continuously from B to D, yielding no phase transition. The two lines AB and CD of the phase diagram will stop; only a sharp crossover would be observed far in the phase diagram instead of the two first order transitions.

In Figure 6b, two jumps occur successively. This assumption is favoured by our Monte-Carlo analysis, although we were unable to prove it from a theoretical basis.

The mean field theory gives only partial indications in our case. Indeed, as the links are independent random variables in this approximation scheme, the parameters p_+ and p_- are correlated, and only a fraction of the total entropy along the dashed curve in figure 5 is taken into account. Multiplying the mean field selected configurations by a fixed state with all plaquettes to -1 (resp. all windows to -1) also gives the possibility of exploring, using this mean field method, the dashed-dotted (resp. dotted) curves. A comparison between these various assumptions leads to a phase diagram which is qualitatively correct, except that it misses the stretching of the disordered phase to infinity. Instead, the line CD is horizontal and meets AB at a triple point.

APPENDIX B

Monte-Carlo algorithms and pseudo-stabilization of states.

While exploring large β regions with a next-to-nearest neighbour action, one must be particularly careful with the Monte-Carlo updating procedures. We give here two examples leading to stabilize an unstable configuration.

i) The heat bath method updates the links in the order they are stored in the computer. This may be dangerous. Starting for instance from a state with all links to 1 at large negative β_2 , one sees that all links are successively flipped. The resulting configuration is gauge equivalent to the original one, instead of going toward a chessboard state. This disease may be cured using a random updating procedure.

ii) Near the far region $\beta_1 + 4 \beta_2 \sim 0$, the update of a single link costs a large energy $12 (\beta_1 + 3 \beta_2) \sim 3 \beta_1$. Flipping simultaneously the

eight parallel links on a given hypercube is much more energetically favoured; the cost is $48 (\beta_1 + 4 \beta_2) \sim 0$. Hence the usual link updating procedure stabilizes metastable phases, and delays the first order transitions in the BD region of the phase diagram. This is cured by using a procedure combining the updates of both links and groups of links, which clearly improves the hysteresis cycles as we checked.

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FIGURE CAPTIONS

- Figure 1 2-dimensional configuration with $E_1=0$, $E_2=-1$. Links denoted by straight lines are +1. Those with wiggly lines are -1. Shaded plaquettes are -1, non-shaded ones +1.
- Figure 2 Hysteresis cycle at $\beta_1=1.0$, varying β_2 in steps of 0.005 starting at -0.13 with an ordered lattice. At each point, 100 thermalizing iterations and 20 measuring iterations of the lattice were done.
- Figure 3 Phase diagram in $\beta_1\beta_2$ plane. All dashed lines are first order transitions. The values one would get for E_1 and E_2 in mean field theory for the various phases are also given.
- Figure 4 Latent heat along line AB of Figure 3. The curves are merely to guide the eye.
- Figure 5 The p_+p_- diagram. Shaded zones hopefully contain all

points maximizing the free energy; this disconnected form explain the measured phase diagram. Dashed, dashed-dotted and dotted lines are explored using the mean field technique.

Figure 6 Qualitative sketch of the possible forms of the entropy for $p_- = 0$. a) no transition b) two first order transitions. Dashed part of the curve corresponds to unphysical points, delimited by the dotted tangents.

- Figure 1 -

- Figure 2 -

- Figure 3 -

- Figure 4 -

- Figure 5 -

- Figure 6 -