M. G. DOME

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DEPOLARIZING EFFECTS OF QUANTUM FLUCTUATIONS AND THE
ACTION OF NON-LINEAR WIGGLERS ON EQUILIBRIUM POLARIZATION LEVEL

Caterina Biscari,
INFN, University of Naples, Italy

Jean Buon,
Laboratoire de l'Accélérateur linéaire, Orsay, France

Bryan W. Montague,
CERN, Geneva, Switzerland

Geneva, February 1983

1. INTRODUCTION

At very high energies the energy spread causes the spin tune spread in an electron storage ring to become comparable with the depolarization resonance spacing. The influence of the energy spread on the beam depolarization cannot be neglected and must be carefully studied.

The spin closed solution $\hat{n}$ in the presence of a smooth energy oscillation and in the vicinity of an integer resonance has been derived elsewhere¹⁾.

The purpose of this report is to add the effect of quantum fluctuations and to obtain the equilibrium polarization, through the calculation of the mean square of the spin-orbit coupling vector $\vec{T} = \gamma(\partial\hat{n}/\partial\gamma)$, taking into account the energy spread. The result will be an enhancement factor of depolarization due to this energy spread, with the appearance of depolarizing satellite resonances.

In Ref. 1 the energy γ of an electron (in units of the rest energy) and the spin tune ν are modulated by the synchrotron motion:

$$\gamma = \gamma_0 + \Delta\gamma \cos \Psi \quad (1)$$

($\Psi = Q_s \theta + \varphi$; Q_s = synchrotron tune, θ = azimuth variable, φ = synchrotron phase);

$$\nu = G\gamma = \nu_0 + a \cos \Psi \quad (a = G\Delta\gamma, \quad \nu_0 = G\gamma_0), \quad (2)$$

where $G = (g - 2)/2$ is the anomalous gyromagnetic factor.

Only the nearest integer resonance $\nu = k$ is considered, with a strength^{*)}

$$\varepsilon_k = Rg_k = \varepsilon, \quad (3)$$

where R is the mean radius of the ring, and g_k is the k^{th} harmonic of the radial field (due to closed-orbit displacements) exciting this resonance.

The periodic rotating reference frame^{2,3)} $(\hat{l}, \hat{m}, \hat{n})$, where the unit vector $\hat{n}$ is the spin closed solution corresponding to the unperturbed closed orbit, rotates around $\hat{n}$ such that the spin precession in this frame is uniform in the absence of perturbations. A new reference frame (n_1, n_2, n_3) , to which the components of the closed solution are referred, rotates around the axis $\hat{n}$ of the $(\hat{l}, \hat{m}, \hat{n})$ frame with angular velocity $k\theta$, its axis n_3 coinciding with $\hat{n}$.

*) In Appendix B the modulation of ε is shown to have a negligible influence.

Calling $\delta_0 = \nu_0 - k$ the mean distance between ν and the integer resonance k , the components of the spin closed solution, in the proximity of the resonance, may be written¹⁾ (to first order in ϵ):

$$\begin{cases} n_1 = \epsilon \sum_p J_p \left(\frac{a}{Q_s} \right) \frac{\cos \xi_p}{\delta_0 + pQ_s} \\ n_2 = -\epsilon \sum_p J_p \left(\frac{a}{Q_s} \right) \frac{\sin \xi_p}{\delta_0 + pQ_s} \\ n_3 = 1 \end{cases} \quad (4)$$

where J_p are the Bessel functions of integer order and

$$\xi_p = p(Q_s \theta + \varphi) - \frac{a}{Q_s} \sin(Q_s \theta + \varphi) . \quad (5)$$

We use a complex notation and define:

$$n_T = n_1 + i n_2 = \epsilon \sum_p J_p \left(\frac{a}{Q_s} \right) \frac{\exp(-i \xi_p)}{\delta_0 + pQ_s} \quad (6)$$

Quantum fluctuations introduce changes of the amplitude a and the phase ξ_p , proportional to the energy variation $\delta\gamma/\gamma$. The resulting spin-orbit coupling vector $\vec{\Gamma}$ comes through the spin closed solution variation:

$$d\vec{n} = \gamma \frac{\partial \vec{n}}{\partial \gamma} \frac{\delta\gamma}{\gamma} . \quad (7)$$

In particular its modulus is given by

$$|\vec{\Gamma}| = \left| \gamma \frac{\partial n_T}{\partial \gamma} \right| . \quad (8)$$

The consequent depolarization enhancement factor is large for a ring such as LEP. The use of a non-linear wiggler (dipole-quadrupole and dipole-octupole) is proposed in Section 4, as a possible way of reducing depolarization enhancement. Not all the particles contribute equally to it, but only those particles whose energy deviation exceeds a certain multiple of the r.m.s. energy deviation, σ_E . Such a deviation is, for LEP, of the order of about $3\sigma_E$ at about 50 GeV.

The non-linear wiggler acts on the distribution function of energy, narrowing or broadening its shape according to the wiggler parameters. With an adequate choice of these parameters it is possible to reduce the depolarization enhancement considerably.

An example of suitable dipole-quadrupole and dipole-octupole wiggler parameters for LEP is given.

2. SPIN-ORBIT COUPLING VECTOR

Let us first examine the energy dependence of a and Ψ . We consider the phase diagram of synchrotron oscillations (see Fig. 1).

If an electron, which is under the influence of synchrotron oscillations as described by Eq. (1), emits a photon of energy δE ($\delta E \ll E$), both the amplitude and the phase of its oscillation undergo a change:

$$\frac{\partial a}{\partial \gamma} \delta \gamma = \frac{a \cos \Psi}{\Delta \gamma} \delta \gamma = \frac{\nu_0 \cos \Psi}{\gamma_0} \delta \gamma = G \cos \Psi \delta \gamma, \quad (9)$$

$$\frac{\partial \Psi}{\partial \gamma} \delta \gamma = -\frac{1}{\Delta \gamma} \sin \Psi \delta \gamma. \quad (10)$$

As $\partial \hat{n} / \partial \gamma = \partial n_T / \partial \gamma$, the derivative of the spin closed solution is given by

$$\begin{aligned} \frac{\partial}{\partial \gamma} n_T = \varepsilon \sum_p \frac{\exp(-i \xi_p)}{\delta_0 + p Q_s} \left\{ \frac{G}{2 Q_s} \cos \Psi \left[J_{p-1} \left(\frac{a}{Q_s} \right) - J_{p+1} \left(\frac{a}{Q_s} \right) \right] \right. \\ \left. + i \frac{p}{\Delta \gamma} \sin \Psi J_p \left(\frac{a}{Q_s} \right) \right\}, \end{aligned} \quad (11)$$

and using the following property of Bessel functions⁴⁾,

$$p J_p(x) = \frac{x}{2} \left\{ J_{p-1}(x) + J_{p+1}(x) \right\}, \quad (12)$$

we may rewrite Eq. (11) as

$$\gamma \frac{\partial}{\partial \gamma} n_T = \frac{\nu_0 \varepsilon_0}{2 Q_s} \exp \left(i \frac{a}{Q_s} \sin \Psi \right) \sum_p \frac{C_p^- - C_p^+}{\delta_0 + p Q_s} \quad (13)$$

with

$$C_p^\pm = e^{-i(p \pm 1)\Psi} J_{p \pm 1} \left(\frac{a}{Q_s} \right). \quad (14)$$

The modulus of the spin orbit coupling vector $\vec{\Gamma}$ is then given by

$$|\vec{\Gamma}|^2 = \left(\frac{\nu_0 \varepsilon_0}{2 Q_s} \right)^2 \left[\sum_{p,q} \frac{C_p^- C_q^{*-} + C_p^+ C_q^{*+} - C_p^- C_q^{*+} - C_p^+ C_q^{*-}}{(\delta_0 + p Q_s)(\delta_0 + q Q_s)} \right], \quad (15)$$

where * indicates the complex conjugate.

3. DEPOLARIZING RESONANCES

Now we must add the cumulative effects of successive quantum fluctuations⁵⁾, which occur in a random fashion.

If the quantum fluctuations are sufficiently weak, the spin motion does not change appreciably during a few synchrotron periods, and successive crossings of a depolarizing resonance are completely correlated.

The condition for complete correlation of spin motion between successive crossings has been given by Derbenev, Kondratenko and Skrinskii⁶⁾.

Whenever an energy jump $\delta\gamma$ occurs, there is a consequent jump in spin tune $\delta\nu = \nu_0 (\delta\gamma/\gamma_0)$ and therefore a change in spin phase. Successive changes will accumulate in time, with an increasing spread in spin phase. If the total spread in spin phase between two successive crossings of one resonance is large, such crossings will be uncorrelated. The separation between them is of the order of one synchrotron period; the condition for correlation is that the average of $(\delta\varphi)^2$ ($\delta\varphi$ is the phase change due to the emission of a photon) in one synchrotron period should be much less than 1.

If the photon is emitted at the azimuth θ , $\delta\varphi$ is

$$\delta\varphi = \int_{\theta}^{\theta_0=1/Q_s} \delta\nu d\theta = G\delta\gamma(\theta_0-\theta) . \quad (16)$$

Averaging $(\delta\varphi)^2$ on time over $(0, \theta_0)$ and on all the emissions occurring in one synchrotron period, we have, with $\dot{N}$ in units of revolution frequency:

$$\dot{N} \overline{\left(\frac{\delta E}{E}\right)^2} \frac{1}{3} \frac{G^2 \gamma_0^2}{Q_s^3} \ll 1 . \quad (17)$$

Since⁷⁾

$$\dot{N} \overline{\left(\frac{\delta E}{E}\right)^2} = \frac{11}{9} \frac{1}{\tau_0} \quad (18)$$

the condition (17) may be written as

$$\frac{11}{27} \frac{\nu_0^2}{Q_s^2 \tau_0} \ll Q_s , \quad \text{or} \quad \frac{11}{27} \left(\frac{\nu_0}{Q_s}\right)^2 \frac{\tau_s}{\tau_0} \ll 1 , \quad (19)$$

where τ_s is the synchrotron period and τ_0 the characteristic polarization time of the Sokolov-Ternov effect. For LEP this condition is fulfilled for energies up to 50 GeV at least.

The rate of change of polarization is given by

$$\frac{dP}{dt} = -P \left\langle \tau_0^{-1} \left[1 - \frac{2}{9} (\hat{n} \cdot \hat{\beta})^2 + \frac{11}{18} \left(\gamma \frac{\partial \hat{n}}{\partial \gamma} \right)^2 \right] \right\rangle - \frac{8}{5\sqrt{3}} \left\langle \tau_0^{-1} \hat{b} \cdot \left(\hat{n} - \gamma \frac{\partial \hat{n}}{\partial \gamma} \right) \right\rangle, \quad (20)$$

where $\hat{\beta}$ is the unit vector in the direction of the velocity of the electron and $\hat{b}$ is the unit vector in the direction of the magnetic field. The average $\langle \rangle$ is taken both over the phase of synchrotron oscillation and over the equilibrium distribution of amplitudes a in the beam.

For $\hat{n}$ not too different from the magnetic field direction, the terms $(2/9) \times (\hat{n} \cdot \hat{\beta})^2$ and $\hat{b} \cdot (\gamma \partial \hat{n} / \partial \gamma)$ in Eq. (20) are negligible. So, the asymptotic degree of radiative polarization can be expressed in the form:

$$P = - \frac{8}{5\sqrt{3}} \frac{\langle \rho^{-3} \hat{n} \hat{b} \rangle}{\langle \rho^{-3} \left[1 + \frac{11}{18} \left(\gamma \frac{\partial \hat{n}}{\partial \gamma} \right)^2 \right] \rangle}. \quad (21)$$

In order to evaluate $\langle |\vec{I}|^2 \rangle$, we first average Eq. (15) over the synchrotron phase and obtain

$$\langle |\vec{I}|^2 \rangle_{ph} = v_0^2 \epsilon^2 \sum_p \frac{J_p^2(a/Q_s)}{[(\delta_0 - pQ_s)^2 - Q_s^2]^2}. \quad (22)$$

For the average over a we suppose a Rayleigh distribution of the amplitudes due to quantum fluctuations:

$$f(a) = \frac{2a}{\sigma_a^2} e^{-a^2/\sigma_a^2} \quad (23)$$

$[\sigma_a = \sqrt{2} (\sigma_E/E) v_0]$, and using⁸⁾

$$\int_0^\infty J_p^2 \left(\frac{a}{Q_s} \right) f(a) da = e^{-x} I_p(x), \quad (24)$$

where $x = \sigma_a^2 / 2Q_s^2$ and I_p is the modified Bessel function, we have

$$\langle |\vec{I}|^2 \rangle_{ph,a} = v_0^2 \epsilon_0^2 e^{-x} \sum_p \frac{I_p(x)}{[(\delta_0 - pQ_s)^2 - Q_s^2]^2}. \quad (25)$$

In the absence of synchrotron motion, Eq. (25) goes to the limit

$$\langle |\vec{I}|^2 \rangle \rightarrow \langle |\vec{I}|_0^2 \rangle = \frac{v_0^2 \epsilon_0^2}{\delta_0^4}. \quad (26)$$

So we can say that

$$\langle |\vec{I}|^2 \rangle = \langle |\vec{I}|_0^2 \rangle C, \quad (27)$$

where

$$C = \delta_0^4 \sum_P \frac{e^{-x} I_P(x)}{[(\delta_0 - pQ_s)^2 - Q_s^2]^2} \quad (28)$$

is the depolarization enhancement factor that takes into account the influence of the synchrotron motion on the spin-orbit coupling vector.

If we put Eq. (25) into Eq. (21) we see that the condition for obtaining polarization higher than 50% is

$$\frac{11}{18} v_0^2 \varepsilon_0^2 e^{-x} \sum_P \frac{I_P(x)}{[(\delta_0 - pQ_s)^2 - Q_s^2]^2} < 1. \quad (29)$$

This was the result given by Derbenev, Kondratenko and Skrinskii in ref. 6.

This condition is quite restrictive for LEP because of its large energy spread, given the value $Q_s \approx 0.07$.

The factor C given in Eq. (28) is plotted in Fig. 2 for LEP at 51.5 GeV, and its value is large even out of the resonances, which correspond to integer values of δ_0/Q_s .

In the next section we propose the use of a non-linear wiggler in order to improve conditions for polarization.

4. USE OF A NON-LINEAR WIGGLER FOR REDUCING THE DEPOLARIZATION ENHANCEMENT FACTOR

According to Hofmann and Jowett⁹⁾ and Jowett¹⁰⁾, if we use a combination of dipole-quadrupole and dipole-octupole wigglers, the damping partition number is

$$J_E = -\alpha + \beta \varepsilon^2, \quad (30)$$

where $\varepsilon = \Delta E/E_0$ is the instantaneous energy deviation divided by the nominal energy, and α and β are suitable dimensionless constants which characterize the strengths of the dipole-quadrupole and dipole-octupole wigglers. In the usual case, without wigglers, $\alpha = -2$ and $\beta = 0$.

The relative energy deviation ε is modulated by synchrotron oscillations:

$$\varepsilon = A \cos(Q_s \theta + \varphi). \quad (31)$$

The distribution function for A, under the action of the dipole-quadrupole and dipole-octupole wiggler, is

$$W(A) = \frac{1}{2\pi} \frac{E_0}{\sigma_E} \left(\frac{\beta}{2\pi} \right)^{\frac{1}{2}} \left\{ w \left[-i \frac{\alpha E_0}{(2\beta)^{\frac{1}{2}} \sigma_E} \right] \right\}^{-1} A \exp \left[\frac{\alpha E_0^2}{4\sigma_E^2} A^2 - \frac{\beta E^2}{32\sigma_E^2} A^4 \right], \quad (32)$$

where¹¹⁾

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz) \quad (33)$$

is the error function of a complex argument and σ_E is the normal standard energy deviation; $W(A)$ is normalized:

$$\int_0^\infty 2\pi W(A) dA = 1. \quad (34)$$

The relation between A and the amplitude a of the spin tune modulation is $a = A\nu_0$. Therefore the distribution for a is

$$f(a) = \left(\frac{\beta}{\pi} \right)^{\frac{1}{2}} \frac{\left\{ w \left[-i (\alpha \nu_0 / \sigma_a \sqrt{\beta}) \right] \right\}^{-1}}{2\pi \nu_0 \sigma_a} a \exp \left[\frac{\alpha}{2\sigma_a^2} a^2 - \frac{\beta}{16\nu_0^2 \sigma_a^2} a^4 \right]. \quad (35)$$

The function $f(a)$ tends to the Rayleigh function (Eq. 23) when $\alpha \rightarrow -2$ and $\beta \rightarrow 0$.

If $\alpha \approx -2$ for very small values of a , $f(a)$ behaves almost like the Rayleigh function, because the term proportional to a^4 in the argument of the exponential is negligible compared to the term proportional to a^2 .

Increasing the value of β , the damping effect becomes stronger for large a , as we can see in both Eq. (30) and Eq. (35).

In Fig. (3) we have plotted the Rayleigh function given in Eq. (23) and the function $[J_6^2(a/Q_s) + J_8^2(a/Q_s)]$, i.e. the dominant terms in the sum of Eq. (32) corresponding to the seventh satellite resonance ($\delta_0 \approx 0.5$). We can see that the overlap between the two functions corresponds to values around $4\sigma_a$, and therefore the enhancement factor C [Eq. (28)] is mostly due to particles whose amplitude of spin modulation is of the order of $4\sigma_a$. So if we choose the parameters α and β of the dipole-quadrupole and dipole-octupole wiggler in such a way that the new distribution function is negligible for $a > 3\sigma_a$, we will strongly reduce the factor C and therefore increase the polarization.

For LEP it will be useful for us to aim at having $\alpha = -2$ and $\beta \approx 10^6$, which can be obtained with the same field of the wiggler as in ref. 8 but with an increased dispersion $D \approx 3D_w$.

In Fig. 3 the function (35) is plotted for $\alpha = -2$ and $\beta = 10^6$. The overlap between the distribution function and the Bessel function is reduced with respect to the previous one, and the dominant term in the enhancement factor C is reduced to the order of unity. This example is given by way of illustration; the reduction in C depends fairly strongly on the beam energy spread, the synchrotron tune Q_s and the precise choice of the fractional spin tune δ_0 , which together determine the orders of the Bessel functions making the largest contributions. Furthermore, a complete picture requires taking into account also the satellites of the other adjacent $(k + 1)$ integer resonance, which has not been done in this paper.

5. CONCLUSIONS

Synchrotron oscillations produce satellites of integer resonances in spin motion, which decrease the polarization level; quantum fluctuations enhance the effect of these resonances, particularly at high energies where there is large energy spread.

Their effect may be compensated by correction procedures acting on integer resonances near the spin tune.

The depolarization enhancement due to the energy spread can be reduced by the use of non-linear wigglers, which will change the tails of the energy distribution⁹⁾ and eliminate, or at least strongly diminish, the region responsible for depolarization.

Fig. 1 Phase diagram of synchrotron oscillations

Fig. 2 Depolarization enhancement factor C

Fig. 3 Example of the effect of non-linear wiggler.
 - - - - - Rayleigh function.
 - - - - - Distribution function of a with wigglers.
 - - $J_6^2(a/Q_s) + J_8^2(a/Q_s)$.

Fig. 4 Modulation of ϵ

APPENDIX A

SPINOR ALGEBRA^{2,12)}

As the spin evolution of particles in accelerators and storage rings is a rotation in real three-dimensional space, it may be described by the algebra of two-component spinors and Pauli matrices.

The Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and they are used together with the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

They have the following properties:

$$\left. \begin{array}{l} \text{Tr } \sigma_j = 0 \\ \text{Det } \sigma_j = -1 \\ \sigma_j \sigma_j = I \\ \text{Unitarity: } \sigma_j \sigma_j^\dagger = I \\ \text{Hermiticity: } \sigma_j^\dagger = \sigma_j \end{array} \right\} j = 1, 2, 3$$

$$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i\sigma_3$$

$$\sigma_2 \sigma_3 = -\sigma_3 \sigma_2 = i\sigma_1$$

$$\sigma_3 \sigma_1 = -\sigma_1 \sigma_3 = i\sigma_2$$

In a real three-dimensional space the $\vec{\sigma}$ vector is defined as

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

and it has the properties:

$$\vec{\sigma}^\dagger = \vec{\sigma}$$

$$\vec{\sigma}^\dagger \cdot \vec{\sigma} = \vec{\sigma} \cdot \vec{\sigma}^\dagger = \vec{\sigma} \cdot \vec{\sigma} = 3I$$

$$\vec{\sigma} \times \vec{\sigma} = 2i\vec{\sigma}.$$

The spin state of a spin $\frac{1}{2}$ particle at rest is a superposition of two basic states: the "up" state and the "down" state, represented by the state vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The spin wave function or spinor is written as

$$\psi = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

c_1, c_2 complex

$$|c_1|^2 + |c_2|^2 = 1$$

The spin states corresponding to two different instants of time, t_0 and t_1 , are related by a rotation matrix 2×2 , D :

$$\psi(t_1) = D(t_1, t_0) \psi(t_0) .$$

The D matrices form the $SU(2)$ group, an irreducible representation of rotations, in which the infinitesimal generators of rotation are

$$\frac{1}{2} \sigma_1, \quad \frac{1}{2} \sigma_2, \quad \frac{1}{2} \sigma_3 .$$

This means that an infinitesimal rotation of an angle $\Delta\theta$ around an axis defined by the unit vector $\hat{n}$ is represented by

$$D_{\hat{n}}(\Delta\theta) = I - i(\Delta\theta/2) \vec{\sigma} \cdot \hat{n} ,$$

from which it is easy to show that infinitesimal rotations commute to first order. A finite rotation of an angle θ around the same axis is the product of successive infinitesimal rotations and is represented by

$$D_{\hat{n}}(\theta) = e^{-i(\theta/2) \vec{\sigma} \cdot \hat{n}} = I \cos \frac{\theta}{2} - i \vec{\sigma} \cdot \hat{n} \sin \frac{\theta}{2}$$

or

$$D_{\hat{n}}(\nu) = e^{-i\pi\nu \vec{\sigma} \cdot \hat{n}} = I \cos \pi\nu - i \vec{\sigma} \cdot \hat{n} \sin \pi\nu .$$

where $\nu = \theta/2\pi$ is the precession number.

This formula can be inverted:

$$\cos \pi\nu = \frac{1}{2} \text{Tr} (D)$$

$$\hat{n} = \frac{i}{2 \sin \pi\nu} \text{Tr} (\vec{\sigma} D) .$$

Therefore if we have the expression of the matrix D we can easily deduce the components of the direction around which the spin has rotated.

This formalism is more compact and elegant than the classical one in which rotations are described by linear transformations using real orthogonal 3×3 matrices: it uses only four parameters (three of which are independent) instead of the usual nine matrix elements.

MODULATION OF ε

We will include here the modulation of the strength ε , introducing a new frame in which this modulation vanishes.

The effect of synchrotron motion on δ and ε is described by

$$\begin{aligned}\delta &= \nu - k = \delta_0 + a \cos \Psi \\ \varepsilon &= \varepsilon_0 + \frac{\partial E}{\partial \nu} a \cos \Psi.\end{aligned}\tag{B.1}$$

For the description of rotations we use the algebra of two-component spinors and Pauli matrices (see Appendix A).

With the reference frame used in Ref. 1, in which the rotation of the spin is described by the transfer matrix Δ , δ moves along the third axis and ε along the first axis (see Fig. 4) and their modulations are correlated in phase.

The extremity of the rotation vector (ε, δ) of the closed solution moves along the segment AB, which forms an angle ϕ with the third axis:

$$\operatorname{tg} \phi = \frac{\partial \varepsilon}{\partial \nu} \approx \phi\tag{B.2}$$

because $\partial \varepsilon / \partial \nu$ is always small.

We introduce a new reference frame $S' = (\hat{1}', \hat{2}', \hat{3}')$ rotated with respect to the previous one by an angle ϕ around the 2nd axis, in such a way that the axis $3'$ is parallel to AB.

The equation for the Δ matrix¹⁾ in the frame $S = (\hat{1}, \hat{2}, \hat{3})$ was

$$\dot{\Delta} = -\frac{i}{2} [\delta \sigma_3 + \varepsilon \sigma_1] \Delta.\tag{B.3}$$

Let Δ' be the rotation matrix in the new system,

$$\Delta' = e^{i(\phi/2)\sigma_2} \Delta.\tag{B.4}$$

Using a commutation property of the Pauli matrices^{*)} and defining a new set of parameters

*) $e^{i(\phi/2)\sigma_2} \sigma_{3,1} e^{-i(\phi/2)\sigma_2} = \sigma_{3,1} e^{-i\phi\sigma_2}$

$$\begin{aligned}
 \text{a) } \delta'_0 &= \delta_0 \cos \phi + \varepsilon_0 \sin \phi \approx \delta_0 + \varepsilon_0 \phi \\
 \text{b) } \varepsilon'_0 &= \varepsilon_0 \cos \phi - \delta_0 \sin \phi \approx \varepsilon_0 - \delta_0 \phi \\
 \text{c) } a' &= a \cos \phi + \frac{\partial \varepsilon}{\partial \nu} a \sin \phi \approx a + \frac{\partial \varepsilon}{\partial \nu} a \phi \\
 \text{d) } \delta' &= \delta'_0 + a' \cos \psi
 \end{aligned} \tag{B.5}$$

we obtain from Eqs. (B.4) and (B.3) an equation for Δ' , referred to the system S' :

$$\dot{\Delta}' = -\frac{i}{2} (\delta' \sigma_3 + \varepsilon' \sigma_1) \Delta. \tag{B.6}$$

Since ϕ can be considered constant in time, Eq. (B.6) is completely analogous to Eq. (B.3), with the set of parameters (B.5).

Coming back to the system S :

$$\begin{cases} n_1 = n'_1 \cos \phi + n'_3 \sin \phi \approx n'_1 + \phi \\ n_2 = n'_2 \\ n_3 = n'_3 \cos \phi - n'_1 \sin \phi \approx n_3 = 1 \end{cases} \tag{B.7}$$

So

$$n_T = n_1 + i n_2 = n'_T + \phi. \tag{B.8}$$

Since ϕ does not depend on energy,

$$\frac{\partial \hat{n}}{\partial \gamma} = \frac{\partial n_T}{\partial \gamma} = \frac{\partial n'_T}{\partial \gamma}. \tag{B.9}$$

Abandoning the prime,

$$n_T = \varepsilon \sum_p \frac{J_p(a/Q_s)}{\delta_0 + p Q_s} \exp(-i \xi_p) + \phi. \tag{B.10}$$

Since ε is given by Eq. (B.5b), $\partial \varepsilon / \partial \gamma = 0$. This means that

$$\frac{\partial n}{\partial \gamma} = \frac{\partial n_T}{\partial a} \frac{\partial a}{\partial \gamma} + \frac{\partial n_T}{\partial \psi} \frac{\partial \psi}{\partial \gamma}. \tag{B.11}$$

Actually, if we estimate the order of magnitude of $\phi \approx \partial \varepsilon / \partial \nu$ we see that it is negligible^{*)}; therefore the set of parameters (B.5) are in fact equivalent to the original non-primed parameters.

*) For CESR II (50 GeV), $\phi \approx 6 \times 10^{-5}$.
For LEP (50 GeV), $\phi \approx 8 \times 10^{-5}$.

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