Longitudinal Emittance Measurements at REX-ISOLDE

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Abstract

The rms longitudinal emittance at output from the REX-ISOLDE RFQ was measured as $0.34 \pm 0.08 \pi$ ns keV/u and at entry to the 7G3 as $0.36 \pm 0.04 \pi$ ns keV/u using the three gradient technique; systematic errors are not included but are thought to be approximately 10%. The 86% emittance was measured a factor of approximately 4.4 times larger than the rms emittance at $1.48 \pm 0.2$ and $1.55 \pm 0.12 \pi$ ns keV/u at the RFQ and 7G3, respectively. The REX-ISOLDE switchyard magnet was used as a spectrometer to analyse the energy spread as the beam was manipulated by changing the voltage of the ReB and 7G3 cavities operating at non-accelerating phases. The technique for measuring the energy spread was rigorously simulated and validated. A formalism was developed to accurately reconstruct the longitudinal beam parameters using a multi-gap constant-velocity bunching cavity when the voltage cannot be kept small. A silicon detector in its development phase was also exploited to measure the longitudinal beam properties. The measured longitudinal emittance is compatible with the acceptance of the HIE-ISOLDE superconducting linac upgrade.

Geneva, Switzerland
July 2011
## Contents

1 Introduction 3  

2 Experimental Procedure 4  
   2.1 Overview .................................................. 4  
   2.2 Theory of the Three Gradient Measurement ................. 6  
      2.2.1 Energy Spread Measurements in the Thin Buncher Approximation ........................................... 6  
      2.2.2 Bunch Length Measurements in the Thin Buncher Approximation ........................................... 7  
      2.2.3 Extension of the Thin Buncher Approximation for Multi-gap Bunching Cavities 7  
   2.3 The Spectrometer ............................................ 9  

3 TRAVEL Simulations of the Measurement 12  
   3.1 Effect of a Circular Aperture ........................................ 12  
   3.2 Effect of Perturbing the Switchyard Magnet ...................... 13  
   3.3 Effect of Losses ............................................. 13  
   3.4 Complete Measurement Simulations ......................... 14  

4 ReB Measurements 15  
   4.1 RFQ Energy Spread ............................................. 15  
   4.2 Spectrometer Measurement on 20° Beam Line .................... 15  
   4.3 Si Detector Measurement - $\Delta W$ ............................ 16  

5 7G3 Measurements 18  
   5.1 Spectrometer Measurement on 65° Beam Line ..................... 18  
   5.2 Si Detector Measurement - ToF ............................. 19  

6 Longitudinal Phase Space Distribution 21  

7 Summary of Results 26  

8 Conclusions 27  

Appendices 28  

A ReB and 7G3 Calibrations 28  

B Approximating the Longitudinal Transfer Matrix of a Multi-gap Buncher 30
1 Introduction

A campaign of emittance measurements has been undertaken in recent years in order to better understand the quality of beams delivered by REX-ISOLDE and to provide crucial information on the performance of the machine for ensuring its compatibility with the HIE-ISOLDE linac upgrade. The REX linac is shown next to the proposed HIE upgrade in Fig. 1 and included are the acronyms used throughout this report for each accelerating structure. For full details on the linac the reader is referred to [1].

Figure 1: The REX linac juxtaposed with the HIE linac: measurements were made before and after the IHS to assess its performance and to understand the longitudinal beam parameters at injection to the upgrade.

During various shutdown periods in 2006 and 2008 the transverse emittance was measured behind the RFQ, IHS, 7G3 and 9GP structures with a dedicated emittance rig, [18]. During 2010 and 2011 the longitudinal emittance was measured behind the RFQ and in front of the 7G3 at opportune moments during pauses in the physics programme.
2 Experimental Procedure

2.1 Overview

The longitudinal emittance was measured using the three gradient method at two locations along the linac; behind the RFQ and in front of the 7G3 in order to measure the emittance before and after the IHS. The bulk of the measurements used the switchyard magnet to measure the energy spread of the beam as a function of the voltage of the ReB and 7G3 cavities operating in a non-accelerating mode. The energy spread was inferred from measurements of the horizontal beam size in the dispersive region after the switchyard magnet. A silicon detector under development for the longitudinal diagnostic system of the HIE-ISOLDE linac was also exploited for both energy and timing measurements. The measurements with the silicon detector were found to be limited by the resolution of the system [21]. The silicon detector was placed in DB5.

The diagnostic system used to profile the low intensity beams at REX provides a good qualitative representation of the beam profile which is useful during beam tuning but is not well calibrated, [12]. It was therefore deemed more accurate to measure the energy spread by varying the dipole field of the switchyard magnet and scanning the beam across a slit in front of a Faraday cup located in the diagnostic boxes DB6 and DB7. A control software was custom-built for the measurement campaign in order that the beam current on the Faraday cup could be acquired as a function of the dipole field measured on a Hall probe inside the switchyard magnet. The measurements and the switchyard magnet were calibrated by assuming that the beam energy after the RFQ is 300 keV/u, which is a reasonable assumption to within the ±1.5 % energy spread of the beam [8].

![Figure 2: The longitudinal emittance at output from the RFQ, using PARMTEQM.](image-url)

In the absence of a dedicated offline ion source for machine development a beam consisting of residual gas from the REXEBIS was used for the measurements and a mass-to-charge state (A/q) of 4 was chosen to deliver beams composed dominantly of neon buffer gas (20Ne5+) leaked from the adjacent REXTRAP. Intensities from the ion source of over 100 pA were achieved by increasing the repetition rate of the ion source close to 50 Hz and increasing the pressure in the REXEBIS to a few 10⁻¹⁰ mbar. With the introduction of gas into the EBIS, which is usually done to produce high beam intensities (a few nA for all A/q states) for machine development, the electron beam becomes neutralised and the
source emittance increases as the radial potential trapping the ions is compensated. Nonetheless, the
effect of the ion source emittance on the longitudinal emittance at output from the RFQ was shown by
simulation to be weak, as shown in Fig. 2. The normalised transverse emittance was measured in similar
conditions as $< 0.30$ π mm mrad [18] at an estimated electron beam compensation of 10 %, which can
be compared to the 0.67 π mm mrad acceptance of the RFQ. The energy spread of the beam coming
from the source is estimated as $< 0.1$ % [19], which was also shown by simulation to have a negligible
effect on the longitudinal emittance developed in the RFQ [9].

At RFQ energy the transmission through the linac was typically as low as 50 % because of the large
beam size, misalignment of the linac [11] and lack of steerers. As experience was gained with tuning low
energy beams transmissions of up to 80 % were possible towards the end of the measurement campaign.

Attempts were made to directly profile the beam over a slit with a size smaller than the beam in front
of the Faraday cup, however, in order to attain a satisfactory signal-to-noise ratio, the size of the entrance
slit had to be increased and as a result the resolution of the spectrometer was compromised. The beam
current was very low and typically less than 10 pA after a 1 mm vertical slit was placed in DB5 at the
entrance to the spectrometer. This was most problematic on the more dispersive beam line where the
beam size is larger and the beam intensity is spread over a wider area at the Faraday cup. Instead, the
beam was accurately profiled using a slit wider than the horizontal beam size in front of the Faraday
cups in DB6 and DB7. The beam profile was reconstructed from the derivative of the beam current as
the beam was swept over each edge of the aperture, as described below and in [20]. An example profile
measurement using the 7G3 is shown in Fig. 3 alongside the reconstruction of the beam profile on each
side of the slit, which involved smoothing, fitting and differentiating the data.

![Raw Data](image1)

(a) Raw data with smoothed fit.

![Smoothing Spline Fit](image2)

(b) Derivative of the fit.

Figure 3: An example profile measurement made on the 65° beam line using a 1 mm vertical slit on
entry to the spectrometer and a 15 mm circular aperture in front of the Faraday cup.

The beam current measured on the Faraday cup was analysed to determine the rms energy spread.
From the derivative with respect to energy ($W$) of the current ($I$) measured on the cup, as shown in
Fig. 3(b), one can write,

$$\Delta W^2_{ rms } = \frac{ \int_{-\infty}^{\infty} (W - W_0)^2 \frac{dI(W)}{dW} dW }{ \int_{-\infty}^{\infty} \frac{dI(W)}{dW} dW },$$

(1)
where $W_0$ is the average energy of the distribution and is defined at the position of the centre of the slit after the switchyard magnet,

$$W_0 = \frac{\int_{-\infty}^{\infty} W \frac{dI(W)}{dW} dW}{\int_{-\infty}^{\infty} \frac{dI(W)}{dW} dW}. \quad (2)$$

A similar analysis was carried out to determine the beam size containing different fractions of the total beam intensity in order to study the distribution of the beam in longitudinal phase space up to a fraction containing 86 %, limited by the signal-to-noise ratio. As machine interventions had to be kept to a minimum the collimator wheels in front of the Faraday cups were not modified and in order to measure large beam sizes on the most dispersive beam line a circular aperture had to be used. The consequence of this and its mitigation are discussed below.

### 2.2 Theory of the Three Gradient Measurement

Only the most important results used in the analysis detailed in this report are listed below. Please see Appendix B for more details. In the thin lens approximation the linearised mapping of the longitudinal coordinates through a bunching cavity can be written,

$$\begin{pmatrix} \Delta t_1 \\ \Delta W_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -qV_{\text{eff}} \sin \phi_s & 1 \end{pmatrix} \begin{pmatrix} \Delta t_0 \\ \Delta W_0 \end{pmatrix}, \quad (3)$$

and expressed as,

$$X_1 = R_{\text{buncher}} X_0. \quad (4)$$

#### 2.2.1 Energy Spread Measurements in the Thin Buncher Approximation

The square of the energy spread measured after the buncher ($\Delta W^2$) can be written as a quadratic function of the effective voltage ($V_{\text{eff}}$) and the Twiss parameters ($\alpha_0$, $\beta_0$, $\gamma_0$ and $\epsilon_0$) immediately before the buncher,

$$\frac{\Delta W^2_1}{A^2} = \epsilon_0 \left[ \left( \frac{q}{A} \right)^2 \beta_0 \sin^2 \phi_s V_{\text{eff}}^2 + 2 \left( \frac{q}{A} \right) \alpha_0 \sin \phi_s V_{\text{eff}} + \gamma_0 \right], \quad (5)$$

where $\phi_s = \pm 90^\circ$ is the synchronous phase. The energy spread is of course independent of the drift distance $L$ after the buncher and can be measured anywhere along the transfer line. Experimentally the emittance can be determined from a quadratic fit of the form $y = a_2 x^2 + a_1 x + a_0$, where $x = V_{\text{eff}} \sin \phi_s$ and $y = \frac{\Delta W^2}{A^2}$. Therefore,

$$\epsilon_0 = \left( \frac{A}{q} \right) \sqrt{a_0 a_2 - \frac{a_1^2}{4}}. \quad (6)$$

From the same fit parameters $a_0$, $a_1$ and $a_2$ the other Twiss parameters can be written as,

$$\beta_0 = \left( \frac{A}{q} \right)^2 \frac{a_2}{\epsilon_0}, \quad (7)$$

$$\alpha_0 = \frac{1}{2} \left( \frac{A}{q} \right) \frac{a_1}{\epsilon_0}, \quad (8)$$

and,

$$\gamma_0 = \frac{a_0}{\epsilon_0}. \quad (9)$$
2.2.2 Bunch Length Measurements in the Thin Buncher Approximation

The square of the bunch length ($\Delta t^2$) measured at a distance ($L$) downstream of the buncher is also a quadratic function of the effective voltage and can be written,

$$\Delta t^2 = a_2 \sin^2 \phi_s V_{\text{eff}}^2 + a_1 \sin \phi_s V_{\text{eff}} + a_0,$$

where $\phi_s = \pm 90^\circ$. The fit parameters $a_0$, $a_1$ and $a_2$ can be expressed in terms of the Twiss parameters before the buncher and the beam velocity ($\beta$),

$$\frac{a_2}{\epsilon_0} = \left(\frac{q}{A}\right)^2 \left(\frac{\pi L}{\beta L \frac{W_0}{A}}\right)^2 \beta_0,$$

$$\frac{a_1}{\epsilon_0} = 2 \left(\frac{q}{A}\right) \frac{\pi L}{\beta L \frac{W_0}{A}} \left[\frac{\pi L}{\beta L \frac{W_0}{A}} \alpha_0 + \beta_0\right],$$

and,

$$\frac{a_0}{\epsilon_0} = \beta_0 + 2 \frac{\pi L}{\beta L \frac{W_0}{A}} \alpha_0 + \left(\frac{\pi L}{\beta L \frac{W_0}{A}}\right)^2 \gamma_0.$$

The expression for the bunch length is somewhat more complicated than that for the energy spread because it depends on the distance of the measurement downstream of the buncher. The emittance is given in terms of the fit parameters as,

$$\epsilon_0 = \left(\frac{A}{q}\right) \left(\frac{\beta L \frac{W_0}{A}}{\pi L}\right)^2 \sqrt{a_2 a_0 - \frac{a_1^2}{4}},$$

and Eqns. 11, 12 and 13 can be rearranged to calculate the other Twiss parameters from the experimental fit.

2.2.3 Extension of the Thin Buncher Approximation for Multi-gap Bunching Cavities

The ReB and 7G3 cavities have 3 and 7 accelerating gaps respectively, which limits the application of the thin approximation in the three gradient emittance measurement. The extent of the limitation was investigated and a formalism developed to extend the approximation to make accurate emittance measurements with multi-gap cavities.

The transfer matrix for an $N$-gap $\pi$-mode accelerating structure with a constant geometric velocity ($\beta_g$) and gap spacings of $\beta_g \lambda/2$ can be written in the form,

$$R_{\text{buncher}}^{N\text{gaps}} = \sum_{i=1}^{N} R_{i},$$

where,

$$R_{\text{buncher}}^{N\text{gaps}} = \sum_{i=1}^{N} \left( f_{i,11}(N) \left(\frac{\pi}{2} \sin \phi_s\right)^{i-1} \left(\frac{q V_{\text{eff}}}{W_0}\right)^{i-1} - \frac{\pi}{2 W_0} f_{i,12}(N) \left(\frac{\pi}{2} \sin \phi_s\right)^{i-2} \left(\frac{q V_{\text{eff}}}{W_0}\right)^{i-2} \right) \cdot \left( f_{i,21}(N) \left(\frac{\pi}{2} \sin \phi_s\right)^{i-1} \left(\frac{q V_{\text{eff}}}{W_0}\right)^{i-1} - f_{i,22}(N) \left(\frac{\pi}{2} \sin \phi_s\right)^{i-1} \left(\frac{q V_{\text{eff}}}{W_0}\right)^{i-1} \right),$$

with $f_{i,11}(N)$, $f_{i,12}(N)$, $f_{i,21}(N)$ and $f_{i,22}(N)$ representing the transfer matrices for each gap.
The beam velocity is assumed to be matched with the geometric velocity of the cavity so that the synchronous phase is the same in each gap and equal to the non-accelerating phases, i.e. $\phi_s = \pm 90^\circ$. The transfer matrix extends spatially from immediately before the first gap to immediately after the last gap and each gap is itself represented as a thin element. The matrix elements of the transfer matrix are finite polynomial expansions in $q V_{\text{eff}}/W_0$ with an order less than $N$ and with coefficients $f_{i,jk}$ that are simple functions of $N$. These coefficients depend on the voltage distribution across the gaps inside the cavity. Simple expressions for $f_{i,jk}$ have been derived for both a flat voltage distribution and a flat distribution with the end drift tubes grounded, see Appendix B. $q V_{\text{eff}}$ is the energy gain that an ion would achieve if accelerated at $\phi_s = 0^\circ$ and entering the cavity with an energy $W_0$. With $R_{\text{buncher}}^{N\text{ gaps}}$ expressed in this form one can truncate the polynomials and write approximate expressions for the transfer matrix of an $N$-gap buncher when $q V_{\text{eff}} \ll W_0$. The transfer matrix for an $N$-gap buncher can be approximated as,

$$R_{\text{buncher}}^{N\text{ gaps}} \approx R_1 + R_2 + R_3,$$

where $R_1$ represents the thin lens approximation and the next two matrices act as higher-order corrections, sufficient to calculate the beam parameters downstream up to second-order in $q V_{\text{eff}}/W_0$. By truncating the expansion at $i = 3$ the energy spread downstream of a multi-gap buncher can be written as a quartic function of $V_{\text{eff}}$,

$$\left( \frac{\Delta W^2}{A^2} \right) = a_4 \sin^4 \phi_s V_{\text{eff}}^4 + a_3 \sin^3 \phi_s V_{\text{eff}}^3 + a_2 \sin^2 \phi_s V_{\text{eff}}^2 + a_1 \sin \phi_s V_{\text{eff}} + a_0,$$

where,

$$\frac{a_4}{\epsilon_0} = \left( \frac{q}{A} \right)^4 \frac{\pi^2}{4W_0^2/A^2} (f_{2,21}^2 + 2f_{3,21}) \beta_0,$$

$$\frac{a_3}{\epsilon_0} = \left( \frac{q}{A} \right)^3 \frac{\pi}{W_0/A} \left[ f_{2,21} \beta_0 + \frac{\pi}{2W_0/A} (f_{2,21} f_{2,22} + f_{3,21} + f_{3,22}) \alpha_0 \right],$$

$$\frac{a_2}{\epsilon_0} = \left( \frac{q}{A} \right)^2 \left[ \beta_0 + \frac{\pi}{W_0/A} (f_{2,21} + f_{2,22}) \alpha_0 + \frac{\pi^2}{4W_0^2/A^2} (f_{2,22}^2 + 2f_{3,22}) \gamma_0 \right],$$

$$\frac{a_1}{\epsilon_0} = \left( \frac{q}{A} \right) \left[ 2 \alpha_0 + \frac{\pi}{W_0/A} f_{2,22} \gamma_0 \right],$$

and,

$$\frac{a_0}{\epsilon_0} = \gamma_0.$$

The fit parameters of the quartic polynomial characterise five independent simultaneous equations with only three independent and unknown Twiss parameters: $\alpha_0$, $\beta_0$ and $\epsilon_0$. The problem is over-constrained and a unique expression for the emittance does not exist as it did for the quadratic case. Nonetheless, one can implement the above relationships into a least-square fitting routine to solve for the beam parameters from experimental data. In some cases it may not be necessary to extend the expansion as far as to include quadratic terms in $q V_{\text{eff}}/W_0$ but only linear terms. The case where $R_{\text{buncher}}^{N\text{ gaps}}$ is expanded only as far as $i = 2$ is shown in Appendix B.

The need for these approximations becomes apparent when considering structures used to measure the emittance at REX, and especially when stable operation of the cavities is difficult at low voltages. The energy spread of the beam downstream of the ReB and 7G3 cavities is plotted using nominal beam parameters taken from simulation as a function of typical values of $q V_{\text{eff}}/W_0$ in Fig. 4. The $V_{\text{eff}}$ dependence of the complete transfer matrix $R_{\text{buncher}}^{N\text{ gaps}}$ is compared with the simple thin lens approximation and the extensions described above. Although the thin lens approximation is reasonable for the ReB, it is imperative that the thin lens approximation is extended in the case of the 7G3.
Figure 4: The energy spread calculated downstream of the ReB and 7G3 operating in bunching mode compared to the approximations discussed.

2.3 The Spectrometer

The basic parameters of the switchyard magnet are collected in Table 1.

<table>
<thead>
<tr>
<th>Beam Line</th>
<th>Edge Angle *</th>
<th>Bend Radius (m)</th>
<th>Max. Rigidity (Tm)</th>
<th>Gap Height (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(deg)</td>
<td></td>
<td>$B_{\text{max}} = 1.14$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>2.878</td>
<td>3.3</td>
<td>50</td>
</tr>
<tr>
<td>65</td>
<td>32</td>
<td>0.930</td>
<td>1.1</td>
<td>50</td>
</tr>
</tbody>
</table>

* No edge angle on entry.

The switchyard magnet was incorporated into a spectrometer system comprising the diagnostic boxes and quadrupole triplet magnets located behind it. The system was made point-to-point (the $R_{12}$ and/or $R_{34}$ transfer matrix elements set to zero) between the entrance slit of the system in DB5 and the exit slits in front of the Faraday cups in DB6 and DB7 by tuning each quadrupole gradient ($g$) using a TRACE3D model [2] shown in Fig. 5. The spectrometer settings are summarised in Table 2 for a beam with $A/q = 4$ at RFQ energy (300 keV/u) along with the transfer matrix elements consistent with the conventions of TRACE3D.

Profiling the beam by varying the dipole field has its drawbacks because the optics of the spectrometer system is perturbed, affecting the measurement through the various listed mechanisms:

- The dispersion function changes as the radius of the beam trajectory changes.
- The edge angle seen by the beam as it leaves the switchyard magnet varies.
- The point-to-point optics solution is mismatched because the quadrupole gradients were not varied as the dipole field was scanned.
- Some of the beam may be lost on the aperture of the quadrupole as the beam is moved.
Figure 5: The spectrometer in TRACE3D, giving a stigmatic focus \((R_{12} = R_{34} = 0)\) at RFQ energy with the first two quadrupole magnets powered.

Table 2: Spectrometer settings for \(A/q = 4\) and \(W_0 = 300\) keV/u.

<table>
<thead>
<tr>
<th>Beam Line (°)</th>
<th>Quad. Polarities</th>
<th>Integrated Quad. Grad.,(^{a,b}) (gL_{\text{eff}}) (T)</th>
<th>(R_{16}) (m)</th>
<th>Energy Res. (%) for a 1 mm slit</th>
<th>(R_{12}) (m)</th>
<th>(R_{34}) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>+, 0, 0 (astigmatic)</td>
<td>0.49, 0, 0</td>
<td>-0.53</td>
<td>0.17</td>
<td>0</td>
<td>3.80</td>
</tr>
<tr>
<td>65</td>
<td>+, -, 0 (stigmatic)</td>
<td>0.77, 1.85, 0</td>
<td>-0.86</td>
<td>0.17</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>+, 0, 0 (astigmatic)</td>
<td>0.57, 0, 0</td>
<td>-0.12</td>
<td>0.50</td>
<td>0</td>
<td>6.60</td>
</tr>
<tr>
<td>20</td>
<td>+, -, 0 (stigmatic)</td>
<td>1.22, 1.55, 0</td>
<td>-0.20</td>
<td>0.50</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^{a}\) External quad. cal.: \(gL_{\text{eff}}\) [T] \(\approx 0.018 I\) [A], [3].

\(^{b}\) Internal quad. cal.: \(gL_{\text{eff}}\) [T] \(\approx 0.031 I\) [A], [3].
The spectrometer was tuned to maximise the dispersion function at either DB6 and DB7, which minimises the change in the dipole field required to move the beam across the Faraday cup. The drawback of maximising the dispersion function is that the beam size at the Faraday cup is large, which limits the range of beam dispersion that can be measured with this technique; for a given slit size and dispersion function the range of energy spread measurable is limited by the requirement that the beam fits inside the slit at the Faraday cup. The optimum situation was found with the two external quadrupoles of the triplet powered with opposite polarity providing a stigmatic focus at the Faraday cup. As the polarity of the current leads did not facilitate this configuration the first two quadrupoles were powered instead. The quadrupole triplet magnets behind the switchyard magnet were removed from the beam line, calibrated and realigned when replaced [3].

Although the 65° beam line provides a resolution of almost three times that possible on the 20° beam line, the large edge angle on the 65° beam line strongly defocuses the beam in the dispersive plane and brings the beam envelope close to the aperture of the first quadrupole at 20 mm, as is evident in Fig. 5. Losses on the aperture of the quadrupole and the shape of the aperture in front of the Faraday cup can have a strong impact on the measurement.
3 TRAVEL Simulations of the Measurement

The TRAVEL code [14] was used to validate the technique employed to measure the energy spread. The input and batch files simulating the spectrometer using the TRAVEL code can be found on the CERN EDMS, [10].

3.1 Effect of a Circular Aperture

An example simulation is shown in Fig. 6 where the realistic particle distribution is tracked through the spectrometer on the 65° beam line as the dipole field is scanned and the beam is moved across the Faraday cup. A 1 mm slit was used at entry and a 15 mm vertical slit is compared to a 15 mm circular aperture at the Faraday cup in the stigmatic and astigmatic cases. In the astigmatic case only the first quadrupole is powered and the energy spread is measured considerably larger as a result of the aperture being circular. The effect of a circular aperture in front of the Faraday cup was shown to have a negligible effect on the measurement if a stigmatic focus was used and the vertical beam size made much smaller than the radius of the aperture.

![Graph](image1)

(a) astigmatic spectrometer.

![Graph](image2)

(b) $\sigma_{\text{rms, circ.}} = 0.53\%$ and $\sigma_{\text{rms, vert.}} = 0.25\%$.

![Graph](image3)

(c) stigmatic spectrometer.

![Graph](image4)

(d) $\sigma_{\text{rms, circ.}} = \sigma_{\text{rms, vert.}} = \sigma_{\text{rms, nom.}} = 0.25\%$.

Figure 6: TRAVEL simulations of the energy spread measurement on the 65° line with the 7G3 turned off. The rms energy spread of the simulated beam was $\sigma_{\text{rms, nom.}} = 0.25\%$. 
3.2 Effect of Perturbing the Switchyard Magnet

The variation in the $R_{11}$ and $R_{12}$ transfer matrix elements through the spectrometer was studied as the dipole field is perturbed. The calculations were done by launching eigenvectors made of 11 particles evenly spaced within the limits of $\pm 10$ mm and $\pm 10$ mrad in transverse phase space where the dipole field of the switchyard magnet is perturbed by $\pm 1.5\%$, as shown in Fig. 7. The beam dispersion was set to zero to negate the effect of the dispersion function.

The transfer matrix elements are represented by the gradients of the linear fits in Figs. 7(b) and 7(c) for a variation of $\pm 1.5\%$ in the dipole field of the switchyard magnet, sufficient to move the beam across a 15 mm aperture in the astigmatic case on the $65^\circ$ beam line. The dynamics is closely linear over the range of phase space probed by the eigenvectors, which is consistent with the nominal beam. The $R_{11}$ term changes by no more than $\pm 6\%$ which accounts for a very small change in the resolution if the entry position is restricted to $\pm 0.5$ mm with a 1 mm vertical slit. The $R_{12}$ term does vary from zero under the perturbation but at a level that doesn’t compromise the resolution attained from using a 1 mm vertical entrance slit and even for strongly divergent beams of 10 mrad. Finally, the dispersion function $R_{16} = -0.53$ m changes linearly with $\Delta B/B_0$ and the energy spread can be reconstructed to a few percent with this method.

3.3 Effect of Losses

Beam losses in the dispersive region after the switchyard magnet are the biggest limitation of the procedure employed. The effects of losses are complicated and it is not always the case that the longitudinal emittance is measured smaller as a result of losing beam on the aperture. In fact, with the method of reconstruction employed, simulations show that the energy spread can be measured larger with dispersion correlated losses because the circular aperture of the quadrupole can degrade the resolution in the same way that a circular aperture in front of the Faraday cup can. In addition, the beam size at the quadrupole triplet depends not only on the dispersion but also on the beam’s divergence at entry, which was not well known. To mitigate this effect the transmission was optimised, at well above 95 % through the spectrometer, between variations in the cavity voltage and before the dipole field was scanned each time. At higher beam energies transmission is increased through the spectrometer because the beam dispersion represents a lower fraction of the beam energy and the beam is less divergent due to the effect of adiabatic damping on the transverse betatron oscillations.
3.4 Complete Measurement Simulations

Simulations are presented in Fig. 8 of the ReB and 7G3 measurements on each beam line with a 5 mm vertical slit and a 15 mm circular aperture in front of the Faraday cups on the 20° and 65° beam lines, respectively. A 1 mm vertical slit was placed at the entrance to the spectrometer. The 65° beam line is suited for the 7G3 measurement because of the increased resolution, which is required to measure the small beam dispersion (as a fraction of the average energy) at 1.92 MeV/u. Losses are observed in the spectrometer on the 65° beam line for the ReB measurement at high voltages. The difference in resolution between the two beam lines has a negligible effect and the 20° beam line is better suited for the ReB measurement.

![Graphs showing ReB and 7G3 measurements](image)

(a) ReB measurement at 0.30 MeV/u.  
(b) 7G3 measurement at 1.92 MeV/u.

Figure 8: TRAVEL simulations of the measurement compared with the nominal energy spread.
4 ReB Measurements

Three gradient emittance measurements were made at various RFQ tank amplitudes and the nominal beam parameters at the output to the RFQ were found at a voltage corresponding to approximately 1980 mV on the pick-up. The ReB could not be operated stably below an amplitude corresponding to an effective buncher voltage of 35 kV. It was noted in [5] that due to mechanical problems in the manufacturing of the cavity the gaps were spaced slightly further apart than designed, giving a difference of $\pm 7.5^\circ$ in the synchronous phase seen by the beam in the external gaps, which was shown to have a negligible effect on the measurement. The energy spread was too large to take measurements at the debunching synchronous phase of $+90^\circ$. Although initial measurements were made on the $65^\circ$ beam line of the spectrometer, it was challenging to ensure transmission through the spectrometer as the dipole field was scanned and the $20^\circ$ beam line was preferred instead. The calibration of the ReB is shown in Appendix A and the raw data from the measurements can be found on the CERN EDMS, [10].

![Energy spread profiles.](image1)

![Analysis.](image2)

Figure 9: The energy spread at output of the RFQ as a function of the RFQ pick-up voltage.

4.1 RFQ Energy Spread

The energy spread at the output of the RFQ was found to be very sensitive to the voltage of the RFQ. The energy spread measured as a function of the voltage measured on the RFQ pick-up is plotted in Fig. 9, alongside the measured beam profiles. The emittance measurement was carried out with a voltage of 1982 mV measured on the RFQ pick-up, which gives the beam parameters expected from simulation. When the RFQ is over or under-powered by $\approx 1\%$ the beam parameters are observed to change significantly.

4.2 Spectrometer Measurement on $20^\circ$ Beam Line

The emittance was inferred from a quadratic fit to the data plotted in Fig. 11 using Eqn. 5. The least-squares fit, which was weighted with the random errors shown by the error bars in Fig. 11, gave an rms emittance of $0.34 \pm 0.08 \pi \text{ ns keV/u}$ with a 1 mm entry slit. The measured beam parameters, which are summarised in Table 3, are affected by the resolution but are still in good agreement with simulation. The raw data and the energy spread profiles are shown in Fig. 10 for a range of voltages with a 5 mm
vertical slit placed in front of the Faraday cup. An estimate of the systematic error arising from the contribution to the resolution of the finite width of the entry slit was made by comparing the measured beam profiles with different slit sizes, as is shown explicitly in Fig. 12 for the 7G3 measurement. By assuming that the beam distribution is uniform across the slit one can write the resolution contribution per mm of entry slit at a given effective voltage as,

$$\sigma_{\text{res/mm}} = \frac{1}{4} \sqrt{\Delta_{\text{3 mm}}^2 - \Delta_{\text{5 mm}}^2},$$

(23)

where $\Delta_{\text{3 mm}}$ and $\Delta_{\text{5 mm}}$ are the measured energy spreads with 3 and 5 mm slits, respectively. The resolution from the 1 mm slit was estimated at $0.4 \pm 0.1$ keV/u, which corresponds to a resolution of 0.13 % and is considerably lower than the estimated resolution of 0.50 %. After the subtraction of the resolution the emittance can be estimated as $0.29 \pm 0.07 \pi$ ns keV/u. The voltage required to minimise the energy spread is consistent with simulation.

Figure 10: The raw data and energy spread profiles for a range of effective buncher voltages measured at 300 keV/u on the 20$^\circ$ beam line of the spectrometer.

4.3 Si Detector Measurement - $\Delta W$

In accordance with an investigation of the energy resolution of the solid-state diagnostic system it was also used to measure the longitudinal emittance. The rms emittance measured with the silicon detector is a factor of 4.6 times larger than the measurement with the spectrometer on the 20$^\circ$ beam line and a factor of 6 times larger than simulation. The system performs well in comparison to other solid-state systems used to measure the longitudinal emittance. During the commissioning of the new injector at LNL direct emittance measurements using correlated energy and time signals from a silicon detector lead to a measurement a factor of 18 times larger than simulation, [15]. ToF measurements at LNL using the three gradient method gave a factor of 2.5 times the expected value. At GSI, a solid-state system was used in a direct measurement which attained an rms emittance some 7.5 times larger than simulation [13]. The limited energy resolution makes accurate measurements of the beam energy spread challenging with the solid-state system. Nonetheless, the system can still be used to provide important
information regarding the average beam energy required for tuning the cavities of the HIE-ISOLDE linac. The value of 3.9 keV/u/ns for the ratio of $\alpha/\beta$, which defines the minimum of the parabola in Fig. 11, agrees closely with theory. The resolution of the diagnostic system was estimated as 1.4 % rms [21] and, after subtraction in quadrature, the energy spread agrees closely with both measurement with the spectrometer and simulation, as shown in Fig. 11.

Figure 11: Emittance measurement with the ReB using the spectrometer and the Si detector, where the Si detector results are plotted with the estimated resolution subtracted.
5 7G3 Measurements

In nominal operation the beam has an energy of 1.9 MeV/u at the 7G3 and a reduced velocity of 6.4 %, compared to the geometric velocity of the 7G3 cavity of 6.6 % [17]. The effect of the variation of the synchronous phase in each gap as a result of the mismatched velocity was investigated by simulating the 7G3. The phase deviates by no more than 20° from the non-accelerating phases in the first and last gaps and the emittance can be reliably reconstructed to within 12 %, [6]. The calibration of the 7G3 can be found in Appendix A and the raw data from the measurements can be found on the CERN EDMS, [10].

5.1 Spectrometer Measurement on 65° Beam Line

The energy spread was measured on the 65° beam line as a function of the effective voltage of the 7G3 and is presented in Fig. 12 with three different sized vertical slits placed at entry to the spectrometer system to illustrate the effect of the entry slit on the resolution of the measurement.

![Figure 12: Emittance measurement with the 7G3 and spectrometer with different entry slit sizes.](image)

The energy spread was reconstructed from scans across a 15 mm circular aperture in front of the Faraday cup. The error bars on each data point reflect the standard deviation of three scans made at each voltage, corresponding to an ensemble of six measurements of the energy spread. The non-quadratic behaviour of the 7G3 is evident through the asymmetry of the data about the minimum and quartic fits were fitted to the data, giving emittances of $0.36 \pm 0.05$, $0.42 \pm 0.04$ and $0.51 \pm 0.02 \pi$ ns keV/u for 1, 3 and 5 mm entry slits, respectively. At low voltages the resolution contribution from a 1 mm slit is estimated as $1.0 \pm 0.2$ keV/u, which is negligible when compared to the random error involved in
the measurement. This corresponds to a resolution of 0.05 %, considerably lower than the estimated resolution of 0.17 % but a similar factor lower than was estimated for the ReB measurement on the 20° beam line. The results are collected in Table 4.

5.2 Si Detector Measurement - ToF

The bunch length was measured by Time of Flight (ToF) as a function of the effective voltage using a silicon detector in DB5 located 11.4 m downstream of the 7G3, as shown in Fig. 13. The data was fitted with a quadratic polynomial and the beam parameters calculated using Eq. 10. The position of the detector in DB5 was selected primarily to favour ease of access and development of the solid-state diagnostic system away from possible sources of noise in the linac and not specifically for an emittance measurement. Nonetheless, after a drift of over 11 m to the focal plane at the detector in DB5, the bunches were resolvable at 101.28 MHz. Unfortunately, as a result of the flight distance, the voltage required to focus the bunch in time was small and the low-level rf feedback control of the cavity’s amplitude and phase was not always possible, which would have increased the time-averaged emittance. More importantly, the bunches started to coalesce at DB5 making the analysis of the rms bunch length difficult, as shown in Fig. 14. The data was divided into 9.87 ns windows and the rms of each peak calculated inside these windows. The tails of the adjacent bunches inevitably biased the measurements and the rms emittance was measured a factor of 5.8 times larger than the spectrometer measurement and a factor of 6.6 times larger than simulation at 2.1π ns keV/μ. The reconstructed beam parameters are still qualitatively consistent with simulation and the spectrometer measurements, see Table 4 and Fig. 18. The time resolution of the solid-state diagnostic system is estimated as < 0.1 ns rms and the
system would provide a useful tool for bunch length and emittance measurements if the time-structure of the beam is modified and the bunch spacing increased by chopping the beam, as is in discussion for HIE-ISOLDE upgrade.

Figure 14: Bunch length distributions measured at DB5 during the ToF emittance measurement.
6 Longitudinal Phase Space Distribution

The distribution of each energy spread scan was analysed and the emittance containing a given fraction of the beam calculated by applying the three gradient method in the same way as was described above for the rms emittance. This indirect measurement of the distribution of the beam in longitudinal phase space is summarised in Fig. 15. The signal-to-noise ratio allowed the distribution to be probed reliably up to an emittance containing 86 % of the beam. The core of the beam is closely Gaussian but particles are populated further from the core than is described by the Gaussian distribution; low energy tails were observed as was shown in Fig. 10(b). The ellipses containing different beam fractions were interpolated in the longitudinal phase plane and compared to simulation, as shown in Fig. 16 at a position directly in front of the ReB and after being tracked back 1.026 m to the exit of the RFQ in Fig. 17. The beam parameters appear more convergent than simulation at exit from the RFQ, which can be attributed to the finite energy resolution of the measurement discussed in [6].
Figure 16: A comparison between the simulated and measured phase space distribution at entry to the ReB (colour bars: (a) particle density and (b) beam fraction).

Figure 17: A comparison between the simulated and measured phase space distribution at exit from the RFQ (colour bars: (a) particle density and (b) beam fraction).
The same procedure was repeated for the 7G3 measurement and presented at a position directly in front of the 7G3 in Fig. 18. The interior ellipses are rotated slightly with respect to those on the exterior which causes some irregularities in the interpolation in the phase space.

The phase space distribution was also directly reconstructed by tomography using a computer program developed at LANL and PSI, see [16] for details. The program takes the energy projections of the beam measured at different orientations in phase space after the cavity and reconstructs the two dimensional distribution based on an algorithm that maximises the entropy of the resulting distribution, creating the simplest phase space possible with the given input. The output of the program for the ReB measurement is shown in Figs. 19 and 20. The results confirm that the distribution is closely Gaussian and the 86 % emittance calculated by the code is in agreement with the result presented in Table 3.

![Simulated phase space distribution.](image1)

![Measured (with spectrometer and silicon detector) phase space distribution.](image2)

Figure 18: A comparison between the simulated and measured phase space distribution in front of the 7G3 (colour bars: (a) particle density and (b) beam fraction).
Figure 19: Output overview from MENT Beam Tomography Program [16] for the ReB measurement.
Figure 20: Contour plot from the MENT Beam Tomography Program [16] for the ReB measurement.
The results of the measurement campaign are summarised in Tables 3 and 4 along with an analysis of data taken during commissioning of the REX front-end [6, 7, 5]. It could not be clarified what beam fraction was used for the data set taken during commissioning and a 95% fraction was assumed.

### Table 3: Summary table of the longitudinal emittance measurements with ReB at 300 keV/u.\(^a\)

<table>
<thead>
<tr>
<th>Measurement</th>
<th>(\alpha) (ns / keV/u)</th>
<th>(\beta) (keV/u / ns)</th>
<th>(\alpha/\beta) (π ns keV/u)</th>
<th>(\epsilon_{\text{rms}}) (π ns keV/u)</th>
<th>(\epsilon_{86%}) (π ns keV/u)</th>
<th>Trans. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectrometer (20°)</td>
<td>1.58</td>
<td>0.41</td>
<td>3.85</td>
<td>0.34±0.08</td>
<td>1.48±0.2</td>
<td>60</td>
</tr>
<tr>
<td>Si Detector (ΔW)</td>
<td>0.29</td>
<td>0.074</td>
<td>3.92</td>
<td>1.56±0.17</td>
<td>5.44±0.7</td>
<td>60</td>
</tr>
<tr>
<td>Commissioning</td>
<td>0.87</td>
<td>0.23</td>
<td>3.78</td>
<td>n/a</td>
<td>3.3±0.4</td>
<td>95</td>
</tr>
<tr>
<td>Simulation</td>
<td>2.25</td>
<td>0.60</td>
<td>3.75</td>
<td>0.26</td>
<td>0.96</td>
<td>99.9</td>
</tr>
</tbody>
</table>

\(^a\) Beam parameters given in front of the buncher cavity without any correction for the measurement resolution.

### Table 4: Summary table of the longitudinal emittance measurements with 7G3 at 1.92 MeV/u.\(^a\)

<table>
<thead>
<tr>
<th>Measurement</th>
<th>(\alpha) (ns / keV/u)</th>
<th>(\beta) (keV/u / ns)</th>
<th>(\epsilon_{\text{rms}}) (π ns keV/u)</th>
<th>(\epsilon_{86%}) (π ns keV/u)</th>
<th>Trans. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectrometer (65°)</td>
<td>0.06</td>
<td>0.02</td>
<td>0.36±0.05</td>
<td>1.55±0.12</td>
<td>80</td>
</tr>
<tr>
<td>Si Detector (ToF)</td>
<td>0.81</td>
<td>0.013</td>
<td>2.09</td>
<td>n/a</td>
<td>80</td>
</tr>
<tr>
<td>Simulation</td>
<td>0.05</td>
<td>0.008</td>
<td>0.32</td>
<td>1.03</td>
<td>99.8</td>
</tr>
</tbody>
</table>

\(^a\) Beam parameters given in front of the buncher cavity without any correction for the measurement resolution.
8 Conclusions

The rms longitudinal emittance at output from the RFQ was measured as $0.34 \pm 0.08 \pi \text{ ns keV/u}$ and at entry to the 7G3 as $0.36 \pm 0.04 \pi \text{ ns keV/u}$, indicating a small growth of emittance in the IHS. After the subtraction of the estimated resolution from the 1 mm slit in front of the spectrometer the rms emittance measured before the IHS is $0.29 \pm 0.07 \pi \text{ ns keV/u}$, which gives an emittance growth in the IHS, 7G1 and 7G2 cavities of approximately 20 %, consistent with simulation. The 86 % emittance was measured a factor of approximately 4.4 times larger than the rms emittance at $1.48 \pm 0.2$ and $1.55 \pm 0.12 \pi \text{ ns keV/u}$ at the RFQ and 7G3, respectively. Systematic errors are not included but thought to be no greater than 10 %, as discussed. A silicon detector in its development phase was also exploited to measure the longitudinal beam properties.

Figure 21: The REX beam compared with the longitudinal acceptance of the HIE-ISOLDE linac.

An emittance after the RFQ of $2 \pi \text{ ns keV/u}$ is consistent with simulation, the measurements made during commissioning a decade ago and the recent measurement campaign. The emittance can be accepted into the longitudinal acceptance of the superconducting HIE-ISOLDE linac, as shown in Fig. 21 for two stages of the upgrade; after the IHS and 9GP structures, respectively. The realistic particle distribution at the RFQ output was tracked through the REX linac up to the first cavity of the HIE linac in stages 1 and 2b of the upgrade. Simulations show that the 9GP cavity increases the rms longitudinal emittance by approximately a factor of 2.
Appendices

A ReB and 7G3 Calibrations

The pick-up ports of the ReB and 7G3 cavities were calibrated by measuring the average beam energy as a function of the voltage signal on the pick-up ($A_{pu}$) with the cavity operating at a synchronous phase of $0^\circ$. The calibration curves are plotted in Figs. 22(a) and 22(b) using a beam of $A/q = 4$ and can be written for the ReB as,

$$V_{\text{eff,ReB}} [\text{kV}] = (0.0442 \pm 0.004)A_{\text{pu,ReB}} [\text{mV}],$$

and for the 7G3 as,

$$V_{\text{eff,7G3}} [\text{kV}] = (0.739 \pm 0.015)A_{\text{pu,7G3}} [\text{mV}].$$

The switchyard magnet was used to measure the average beam energy with an error close to that of the intrinsic energy spread of the beam. The variations in the calibration data about the linear fit for the 7G3 arise from the dependence of the synchronous phase on the $A_{pu}$.

The cavity being calibrated was first turned off, the reference energy corresponding to zero energy gain recorded and then the cavity was powered and the phase rotated to bring the beam back to the reference energy close to either $\phi_s = \pm 90^\circ$, with respect to the maximum of the energy gain. The energy gain as a function of phase taken during a calibration for the 7G3 is shown in Fig. 23, where the change in the phase of the maximum energy gain is plotted at two different values of $A_{pu}$. The plots are fitted with the second-order approximation for the energy gain, [4]. The synchronous phase was set at a moderate value of $A_{pu}$.
Figure 22: Calibration curves for the ReB and 7G3 pick-up ports.

Figure 23: Phasing the 7G3 with the beam.
B Approximating the Longitudinal Transfer Matrix of a Multi-gap Buncher

The longitudinal transfer matrix \( R_{\text{buncher}}^{N_{\text{gaps}}} \) describing the dynamics between the first and last gaps of an \( N \)-gap constant-velocity buncher operating in \( \pi \)-mode can be approximated by,

\[
R_{\text{buncher}}^{N_{\text{gaps}}} = R_{1,\text{buncher}}R_{1,\text{drift}}R_{2,\text{buncher}}R_{2,\text{drift}} \cdots R_{N-1,\text{buncher}}R_{N-1,\text{drift}}R_{N,\text{buncher}},
\]

where the \( i^{\text{th}} \) gap is approximated by a thin element,

\[
R_{i,\text{buncher}} = \begin{pmatrix} 1 & 0 \\ -qV_{i,\text{eff}} \sin \phi_{i,s} & 1 \end{pmatrix},
\]

and \( \phi_i = \pm 90^\circ \). The effect of the drift with length \( \beta_g \lambda/2 \) can be written as,

\[
R_{i,\text{drift}} = \begin{pmatrix} 1 & -\frac{\pi}{2W_0} \\ 0 & 1 \end{pmatrix},
\]

where the beam energy \( W_0 \) is assumed matched to the geometric velocity of the cavity. At non-accelerating phases the beam velocity remains constant between the gaps and the synchronous phase is the same in each gap. \( V_{i,\text{eff}} \) is defined such that,

\[
V_{\text{eff}} = \sum_{i=1}^{N} V_{i,\text{eff}}.
\]

One can show by expanding out Eq. 26 that each element of \( R_{\text{buncher}}^{N_{\text{gaps}}} \) consists of a finite polynomial in \( qV_{\text{eff}}/W_0 \) of an order less than \( N \):

\[
R_{\text{buncher}}^{N_{\text{gaps}}} = \sum_{i=1}^{N} \left( f_{i,11}(N) \left( \frac{\pi}{2} \sin \phi_s \right)^{i-1} \left( \frac{qV_{\text{eff}}}{W_0} \right)^{i-1} - \frac{\pi}{2W_0} f_{i,12}(N) \left( \frac{\pi}{2} \sin \phi_s \right)^{i-2} \left( \frac{qV_{\text{eff}}}{W_0} \right)^{i-2} \right),
\]

and more concisely as,

\[
R_{\text{buncher}}^{N_{\text{gaps}}} = \sum_{i=1}^{N} R_{i},
\]

where \( R_1 \) represents the thin lens approximation of the cavity and the first set of coefficients \( f_1 \) are independent of \( N \),

\[
f_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

Depending on the voltage distribution across the gaps simple expressions for the coefficients \( f_{i>1} \) can be derived. For a flat voltage distribution the voltage on each gap is identical,

\[
V_{i,\text{eff}} = \frac{V_{\text{eff}}}{N},
\]

and the next sets of coefficients can be expressed as,

\[
f_2(N) = \begin{pmatrix} N-1 \\ \frac{N^2-1}{6N} \\ \frac{N-1}{2} \end{pmatrix},
\]
and

\[ f_3(N) = \begin{pmatrix}
\frac{(N-2)(N^2-1)}{24N} & \frac{(N-2)(N-1)}{(N-1)^2} \\
\frac{6}{24N} & \frac{(N-2)(N-1)}{(N-1)^2}
\end{pmatrix} \]  \tag{34}

In the case of a flat voltage distribution with grounded external drift tubes, the first and last gaps have a voltage,

\[ V_{1,\text{eff}} = V_{N,\text{eff}} = \frac{V_{\text{eff}}}{2(N-1)}, \]  \tag{35}

and the internal gaps,

\[ V_{2,\text{eff}} = V_{3,\text{eff}} = \ldots = V_{N-1,\text{eff}} = \frac{V_{\text{eff}}}{N-1}, \]  \tag{36}

and the next sets of coefficients can be expressed as,

\[ f_2(N) = \begin{pmatrix}
\frac{N-1}{3-4N+2N^2} & \frac{N-1}{12(N-1)} \\
\frac{N}{12(N-1)^2} & \frac{N}{12(N-1)^2}
\end{pmatrix}, \]  \tag{37}

and

\[ f_3(N) = \begin{pmatrix}
\frac{N(N-2)}{(N-1)^2} & \frac{N(N-2)}{24(N-1)^2} \\
\frac{24}{120(N-1)^2} & \frac{6}{24N}
\end{pmatrix} \]  \tag{38}

If \( qV_{\text{eff}} \ll W_0 \) then \( R_{\text{buncher}}^{N \text{ gaps}} \) can be approximated and the polynomial series truncated to a given order. \( R_{\text{buncher}}^{N \text{ gaps}} \) must be expanded up until \( i = 3 \) to include terms up to second-order in \( \frac{qV_{\text{eff}}}{W_0} \):

\[ R_{\text{buncher}}^{N \text{ gaps}} \approx R_1 + R_2 + R_3, \]  \tag{39}

where,

\[ R_1 = R_{\text{buncher}} = \begin{pmatrix}
1 & 0 \\
-qV_{\text{eff}} \sin \phi_s & 1
\end{pmatrix}, \]  \tag{40}

and the next two matrices in the expansion are,

\[ R_2 = \begin{pmatrix}
f_{2,11}(N) \left( \frac{\pi}{2} \sin \phi_s \right) \left( \frac{qV_{\text{eff}}}{W_0} \right) & -\frac{\pi}{2W_0} f_{2,12}(N) \\
-qV_{\text{eff}} \sin \phi_s f_{2,21}(N) \left( \frac{\pi}{2} \sin \phi_s \right) \left( \frac{qV_{\text{eff}}}{W_0} \right) & f_{2,22}(N) \left( \frac{\pi}{2} \sin \phi_s \right) \left( \frac{qV_{\text{eff}}}{W_0} \right)
\end{pmatrix}, \]  \tag{41}

and,

\[ R_3 = \begin{pmatrix}
f_{3,11}(N) \left( \frac{\pi}{2} \sin \phi_s \right)^2 \left( \frac{qV_{\text{eff}}}{W_0} \right)^2 & -\frac{\pi}{2W_0} f_{3,12}(N) \left( \frac{\pi}{2} \sin \phi_s \right) \left( \frac{qV_{\text{eff}}}{W_0} \right)^2 \\
-qV_{\text{eff}} \sin \phi_s f_{3,21}(N) \left( \frac{\pi}{2} \sin \phi_s \right)^2 \left( \frac{qV_{\text{eff}}}{W_0} \right)^2 & f_{3,22}(N) \left( \frac{\pi}{2} \sin \phi_s \right)^2 \left( \frac{qV_{\text{eff}}}{W_0} \right)^2
\end{pmatrix}. \]  \tag{42}

By truncating the expansion at \( i = 2 \), such that \( R_{\text{buncher}}^{N \text{ gaps}} = \sum_{i=1}^{2} R_i = R_1 + R_2 \), the energy spread downstream of a multi-gap buncher can approximated to orders linear in \( \frac{qV_{\text{eff}}}{W_0} \) as a cubic function of \( V_{\text{eff}} \),

\[ \left( \frac{\Delta W^2}{A^2} \right) = a_3 \sin^3 \phi_s V_{\text{eff}}^3 + a_2 \sin^2 \phi_s V_{\text{eff}}^2 + a_1 \sin \phi_s V_{\text{eff}} + a_0, \]  \tag{43}
where,

$$\frac{a_3}{\epsilon_0} = \left( \frac{q}{A} \right)^3 \frac{\pi}{W_0/A} f_{2,21} \beta_0$$

(44)

$$\frac{a_2}{\epsilon_0} = \left( \frac{q}{A} \right)^2 \left[ \beta_0 + \frac{\pi}{W_0/A} (f_{2,21} + f_{2,22}) \alpha_0 \right].$$

(45)

$$\frac{a_1}{\epsilon_0} = \left( \frac{q}{A} \right) \left[ 2\alpha_0 + \frac{\pi}{W_0/A} f_{2,22} \gamma_0 \right],$$

(46)

$$\frac{a_0}{\epsilon_0} = \gamma_0.$$  

(47)

This analytic approximation can be used to describe the effect on the beam of an $N$-gap buncher when the beam velocity is matched to the geometric velocity of the structure.
References


[21] F. Zocca, M.A. Fraser, E. Bravin, M. Pasini, D. Voulot, and F. Wenander. A silicon detector under development for the longitudinal diagnostic system of hie-isolde was also exploited for both energy and timing measurements. Internal Note CERN-BE-Note-2011, to be published., CERN, 2011.