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MINIMIZATION OF THE $SU(5)$ INVARIANT SCALAR POTENTIAL
FOR THE 45- DIMENSIONAL REPRESENTATION *)

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A B S T R A C T

We show that non-minimal $SU(5)$ models involving a 45-plet of scalars with non-vanishing expectation value are consistent with an unbroken $SU(3)_C \times U(1)_{em}$ as far as minimization of the potential in tree approximation is concerned.

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45-dimensional representations of scalars have been added to the minimal SU(5) model of "grand unification" mainly for the following reasons :

- to avoid ¹⁾ the fermion mass relations $m_{Q=1} = m_{Q=\frac{1}{3}}$ that stem from the SU(4) symmetry ²⁾ left unbroken by the usual 5-plet in the minimal model ;
- to stabilize the proton à la Jarlskog ³⁾ ;
- to solve the strong CP problem ⁴⁾.

Even though Sulak's experiment gives a new limit to the proton lifetime of 6×10^{31} years in the pion-positron channel thus ruling out minimal SU(5), we still cannot find non-minimal SU(5) appealing because of the drastically increased number of arbitrary parameters.

On the other hand, the minimization of the scalar potential for the 45-plet has, to our knowledge, never been carried out. As we shall see, there are reasons beforehand to expect that no vacuum expectation value leaving colour SU(3) and Maxwell's U(1) unbroken would exist. However, at the end, a tiny piece of the parameter space does survive such that non-minimal SU(5) is not ruled out. Moreover, we show that introducing the 24-dimensional scalar multiplet does not spoil this conclusion.

Let us denote by $H_Y^{\alpha\beta}$ ($\alpha, \beta, \gamma = 1, \dots, 5$) the elements of the complex 45-dimensional representation of SU(5), antisymmetric and traceless

$$\begin{aligned} H_Y^{\alpha\beta} &= -H_Y^{\beta\alpha} \\ H_Y^{\alpha\beta} &= 0 \end{aligned} \quad (1)$$

The most general real SU(5) -invariant polynomial of degree 4 ("Higgs potential") in the $H_Y^{\alpha\beta}$ is

$$\begin{aligned} 4V(H) &= -2v^2 H_Y^{\alpha\beta} H_{\alpha\beta}^{\gamma\delta} + \lambda_1 (H_Y^{\alpha\beta} H_{\alpha\beta}^{\gamma\delta})^2 \\ &+ \lambda_2 H_Y^{\alpha\beta} H_{\alpha\beta}^{\delta\epsilon} H_Y^{\kappa\lambda} H_{\kappa\lambda}^{\gamma\delta} + \lambda_3 H_Y^{\alpha\beta} H_{\alpha\beta}^{\delta\epsilon} H_Y^{\kappa\gamma} H_{\kappa\delta}^{\lambda\epsilon} \\ &+ \lambda_4 H_Y^{\alpha\delta} H_{\alpha\beta}^{\gamma\epsilon} H_Y^{\kappa\gamma} H_{\kappa\delta}^{\lambda\epsilon} + \lambda_5 H_Y^{\alpha\delta} H_{\alpha\beta}^{\gamma\epsilon} H_Y^{\kappa\delta} H_{\kappa\gamma}^{\lambda\epsilon} \\ &+ \lambda_6 H_Y^{\alpha\gamma} H_{\alpha\beta}^{\delta\epsilon} H_Y^{\kappa\epsilon} H_{\kappa\gamma}^{\lambda\delta} \end{aligned} \quad v^2, \lambda_1, \dots, \lambda_6 \in \mathbb{R} \quad (2)$$

We use the same notation as Frampton, Nandi and Scanio¹⁾, suppressing, however, the λ_7 and λ_8 quartic invariants which are respectively identical to the λ_6 and to the negative of the λ_4 terms^{*)}. It is then apparent that their extremal constraint (32) contains a numerical error.

Let us call $h \in 45$ the tensor with four non-vanishing components:

$$\begin{aligned} h_i^{i5} &= -h_i^{5i} = k & i=1,2,3 \\ h_4^{45} &= -h_4^{54} = -3k \end{aligned} \quad (3)$$

This tensor is left invariant by the $SU(3)_C \times U(1)_{em}$ subalgebra of $SU(5)$. Our task is to find a choice of the seven parameters $v, \lambda_1, \dots, \lambda_6$ such that the corresponding potential $V(H)$ has a minimum at h . It is not at all obvious whether such a choice is possible. The following solution was derived algebraically and numerically on a computer.

A first necessary condition for the existence of the minimum is that the partial derivatives of the potential with respect to all 90 real variables vanish at h . This is the case if and only if

$$v^2 = \left(24\lambda_1 + 14\lambda_2 + 7\lambda_3 + \frac{19}{2}\lambda_4 - 6\lambda_5 - \frac{7}{2}\lambda_6 \right) k^2 \quad (4)$$

Multiplying now all parameters with a real non-zero number just rescales the whole potential and the positions of the extrema remain unchanged. Therefore, our entire parameter space is the five-dimensional real projective plane RP_5 .

Secondly, we have to require that the 90×90 matrix of second derivatives evaluated at h be positive semi-definite. Thanks to the $SU(3)_C \times U(1)_{em}$ symmetry, this matrix is almost diagonal. Its eigenvectors and eigenvalues in the different charge sectors are given in the Table. The requirement of positive semi-definiteness is equivalent to the nine inequalities :

$$48\lambda_1 + 28\lambda_2 + 14\lambda_3 + 19\lambda_4 - 12\lambda_5 - 7\lambda_6 \geq 0 \quad (5a)$$

$$-4\lambda_2 - 2\lambda_3 - \lambda_4 + \lambda_6 \geq 0 \quad (5b)$$

$$-4\lambda_2 - 2\lambda_3 - \lambda_4 + 4\lambda_5 + \lambda_6 \geq 0 \quad (5c)$$

*) The existence of six independent invariants can be shown by decomposing the product 45×45 .

$$- 24 \lambda_2 - 11 \lambda_3 - 17 \lambda_4 + 12 \lambda_5 + 6 \lambda_6 \geq 0 \quad (5d)$$

$$- 8 \lambda_2 - \lambda_3 - 3 \lambda_4 + 4 \lambda_5 + 2 \lambda_6 \geq 0 \quad (5e)$$

$$12 \lambda_2 - 10 \lambda_3 + 3 \lambda_4 + 12 \lambda_5 + 5 \lambda_6 \geq 0 \quad (5f)$$

$$-28 \lambda_2 + 8 \lambda_3 - 9 \lambda_4 - 4 \lambda_5 - \lambda_6 \geq 0 \quad (5g)$$

$$-28 \lambda_2 - 17 \lambda_4 + 12 \lambda_5 + 7 \lambda_6 \geq 0 \quad (5h)$$

$$8 \lambda_2 - 3 \lambda_3 - 17 \lambda_4 + 12 \lambda_5 + 14 \lambda_6 \geq 0 \quad (5i)$$

The subset of RP_5 described by these inequalities is non-empty. Its relative volume calculated by Monte Carlo is 1.2%. Warning : more than 3% of the potentials with the right minimum, i.e., satisfying the inequalities (5a-i), are unbounded from below.

Assuming the existence of h as a minimum of the pure 45-potential $V(H)$, Kalyniak and Ng⁵⁾ have considered the full scalar potential containing a 45-plet as well as a 24-plet ϕ_β^α ($\phi_\alpha^\alpha = 0$)

$$V = V(H) + V(\phi) + V(H, \phi)$$

with

$$\begin{aligned} V(\phi) = & -\frac{\mu^2}{2} \phi_\beta^\alpha \phi_\alpha^\beta + \frac{a}{4} (\phi_\beta^\alpha \phi_\alpha^\beta)^2 + \frac{b}{2} \phi_\beta^\alpha \phi_\gamma^\beta \phi_\delta^\gamma \phi_\alpha^\delta \\ V(H, \phi) = & \alpha (H_\gamma^{\alpha\beta} H_{\alpha\beta}^\gamma) \phi_\epsilon^\delta \phi_\delta^\epsilon + \beta_1 H_\gamma^{\alpha\beta} \phi_\alpha^\delta H_{\delta\epsilon}^\gamma \phi_\beta^\epsilon \\ & + \beta_2 H_\gamma^{\alpha\beta} \phi_\rho^\gamma H_{\alpha\epsilon}^\delta \phi_\delta^\epsilon + \beta_3 H_\gamma^{\alpha\beta} \phi_\delta^\gamma H_{\alpha\epsilon}^\delta \phi_\beta^\epsilon \\ & + \beta_4 H_\gamma^{\alpha\beta} \phi_\alpha^\delta H_{\epsilon\beta}^\gamma \phi_\delta^\epsilon + \beta_5 H_\gamma^{\alpha\beta} \phi_\delta^\gamma H_{\alpha\beta}^\epsilon \phi_\epsilon^\delta \end{aligned} \quad (6)$$

The most general $SU(3)_C \times U(1)_{em}$ invariant vacuum expectation value for the ϕ multiplet is

$$\langle \phi \rangle = \begin{pmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & 0 & & -\frac{3}{2}(1+\epsilon) & \\ & & & & -\frac{3}{2}(1-\epsilon) \end{pmatrix} \delta \quad (7)$$

In particular, the pure 24-potential $V(\phi)$ admits the two well-known minima ⁶⁾

$$\begin{aligned} \text{if } b > 0 \quad \epsilon = 0 \quad & SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \\ \text{if } b < 0 \quad \epsilon = \pm \frac{5}{3} \quad & SU(5) \rightarrow SU(4) \times U(1) \end{aligned} \quad (8)$$

Imposing now the partial derivatives of the full potential with respect to δ and ϵ to be zero at the minimum, we get the following cubic equation for the ϵ parameter :

$$\begin{aligned} \epsilon(3\epsilon+5)(3\epsilon-5) = \frac{k^2}{b\delta^2} \left\{ (5-66\epsilon-3\epsilon^2)\beta_1 - \frac{15}{2}(\epsilon-1)(3\epsilon+5)\beta_2 \right. \\ \left. + (25-26\epsilon-15\epsilon^2)\beta_3 + \frac{1}{2}(-15+38\epsilon+9\epsilon^2)\beta_4 + (45+14\epsilon-27\epsilon^2)\beta_5 \right\} \end{aligned} \quad (9)$$

by eliminating μ^2 from the stationary equations. This equation is the analogue of equation (21b) in Magg and Shafi's analysis of the potential in the minimal model ⁷⁾. In the approximation of a large hierarchy $(k^2/\delta^2) \ll 1$, we find again three solutions :

$$\begin{aligned} \epsilon &\simeq -\frac{k^2}{5b\delta^2} \left[\beta_1 + \frac{15}{2}\beta_2 + 5\beta_3 - \frac{3}{2}\beta_4 + 9\beta_5 \right] \\ \epsilon &\simeq \pm \frac{5}{3} \end{aligned} \quad (10)$$

They are respectively associated to the symmetry breaking patterns

$$\begin{aligned} SU(5) &\xrightarrow{\text{strong}} SU(3) \times SU(2) \times U(1) \xrightarrow{\text{weak}} SU(3) \times U(1) \\ SU(5) &\xrightarrow{\text{strong}} SU(4) \times U(1) \xrightarrow{\text{weak}} SU(3) \times U(1) \end{aligned} \quad (11)$$

The strong symmetry breaking suggested in Ref. 6)

$$SU(5) \xrightarrow{\text{strong}} SU(3) \times U(1) \times U(1) \quad (12)$$

is therefore excluded by Eq. (9). Moreover, contrary to claims in Ref. 6), we find that the breaking

$$SU(5) \xrightarrow{\text{strong}} SU(4) \times U(1) \xrightarrow{\text{weak}} SU(4) \quad (13)$$

induced by the adjoint with fundamental representations ^{7),8)}, is ruled out by the vacuum expectation of a 45-plet whose maximal little groups are only $SU(3) \times U(1)$ and $SU(2) \times SU(2) \times U(1)$. As an additional indication, we see that an exact solution $\epsilon = -\frac{5}{3}$ for Eq. (9) already requires an unnatural cancellation among the β_i couplings :

$$4\beta_1 + \beta_3 - \beta_4 - 2\beta_5 = 0 \quad (14)$$

We conclude that non-minimal $SU(5)$ models are compatible with tree-level minimization of the scalar potential. However, to the long list of arbitrary Yukawa couplings that have to be adjusted very accurately by hand in these models ^{1),3),4)}, we add a set of scalar couplings to be adjusted by hand equally accurately. We have nothing to say about the stability of this tuning under higher order contributions.

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$U(1)_{em} \times SU(3)_C$	Eigenvectors	Eigenvalues (in units of k^2)
$(\frac{4}{3}, \bar{3})$	$H_i^{45} + H_{i5}^4$	0
	$H_i^{45} - H_{i5}^4$	$2(12\lambda_2 - 10\lambda_3 + 3\lambda_4 + 12\lambda_5 + 5\lambda_6)$
$(1, 1)$	H_k^{k4}	0
$(1, 8)$	$H_i^{j4} - \frac{1}{3} \delta_i^j H_k^{k4}$	$3(-8\lambda_2 - \lambda_3 - 3\lambda_4 + 4\lambda_5 + 2\lambda_6)$
$(\frac{2}{3}, 3)$	H_5^{i4}	$-28\lambda_2 + 8\lambda_3 - 9\lambda_4 - 4\lambda_5 - \lambda_6$
$(0, 1)$	$\text{Im } H_k^{k5}$	0
	$\text{Re } H_k^{k5}$	$8(48\lambda_1 + 28\lambda_2 + 14\lambda_3 + 19\lambda_4 - 12\lambda_5 - 7\lambda_6)$
$(0, 8)$	$(H_j^{i5} - H_{j5}^i) - \frac{1}{3} \delta_j^i (H_k^{k5} - H_{k5}^k)$	$6(-4\lambda_2 - 2\lambda_3 - \lambda_4 + 4\lambda_5 + \lambda_6)$
	$(H_j^{i5} + H_{j5}^i) - \frac{1}{3} \delta_j^i (H_k^{k5} + H_{k5}^k)$	$4(-4\lambda_2 - 2\lambda_3 - \lambda_4 + \lambda_6)$
$(-\frac{1}{3}, 3)$	$\epsilon^{kli} \epsilon_{kmn} H_1^{mn} - 6H_4^{4i}$	0
	$\epsilon^{kli} \epsilon_{kmn} H_1^{mn} + \frac{4}{3}H_4^{4i}$	$11(-8\lambda_2 - \lambda_3 - 3\lambda_4 + 4\lambda_5 + 2\lambda_6)$
$(-\frac{1}{3}, \bar{5})$	$\epsilon_{kli} H_j^{kl} + \epsilon_{klj} H_i^{kl}$	$-24\lambda_2 - 11\lambda_3 - 17\lambda_4 + 12\lambda_5 + 6\lambda_6$
$(-\frac{2}{3}, \bar{3})$	$\epsilon_{ikl} H_5^{kl}$	$-28\lambda_2 - 17\lambda_4 + 12\lambda_5 + 7\lambda_6$
$(-\frac{5}{3}, \bar{3})$	$\epsilon_{ikl} H_4^{kl}$	$8\lambda_2 - 3\lambda_3 - 17\lambda_4 + 12\lambda_5 + 14\lambda_6$

Table - Eigenvectors and eigenvalues of the 90×90 matrix of second derivatives of the potential with respect to $H_Y^{\alpha\beta}$, evaluated at the $SU(3)_C \times U(1)_{em}$ minimum h ($i, j, k, l, m, n = 1, 2, 3$ are the colour indices).

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