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MIXED ABELIAN AND NON-ABELIAN BRS TRANSFORMATIONS
AND THE OCCURRENCE OF CHERN-SIMON THREE-FORMS

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A B S T R A C T

We give a geometrical interpretation of Becchi-Rouet-Stora (BRS) equations which occur upon the quantization of the unified $n = 1$ supergravity and $n = 1$ super Yang-Mills theory in $d = 10$ dimensions. The relevant Abelian and non-Abelian coupled BRS transformations are derived in a straightforward fashion from the horizontality condition of a non-Abelian field strength two-form and of an Abelian Chern-Simon three-form, the exterior derivative of which is the usual Abelian anomaly. Our result illustrates quite well a direct passage from the geometry to the full quantum Lagrangian.

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In this note, we present a geometrical construction of BRS and anti-BRS equations associated with the gauge invariance recently introduced by Chapline and Manton in order to unify the $n = 1$ supergravity and $n = 1$ super Yang-Mills theory in $d = 10$ dimensions¹⁾. Chapline and Manton have been led to introduce the following gauge transformations, acting on the Abelian skew-tensor gauge field $B_{\mu\nu}$, a component of the supergravity multiplet, and on a non-Abelian Yang-Mills field A_μ , taking its values in the algebra $\mathcal{G}$ of some Lie group G

$$\begin{aligned} \delta A_\mu &= D_\mu \epsilon \equiv \partial_\mu \epsilon + [A_\mu, \epsilon] \\ \delta B_{\mu\nu} &= \partial_{[\mu} \epsilon_{\nu]} + \epsilon \cdot \partial_{[\mu} A_{\nu]} \end{aligned} \quad (1)$$

Here ϵ and ϵ_μ are, respectively, scalar and vector infinitesimal local parameters. ϵ^μ is Abelian, and ϵ takes its values in the Lie algebra $\mathcal{G}$. The symbol $[,]$ denotes the Lie bracket in $\mathcal{G}$, and for Lie algebra valued p -forms (including 0-forms like ϵ), we have defined the $\cdot$ symbol by $X \cdot Y = \text{Tr}(X \wedge Y)$. Here and elsewhere, the antisymmetrization operation on Lorentz indices acts as an idempotent, and all coupling constants are set equal to one.

A striking feature of the gauge transformations (1) is that they mix Abelian and non-Abelian parameters. As a consequence, it is non-trivial to fix the gauge in the Chapline-Manton Lagrangian while imposing BRS invariance. In a previous paper²⁾, we solved this problem by adopting the general method described in Ref. 3). As a first step, we found the BRS and anti-BRS equations associated with the classical transformations Eq. (1). Then, the nilpotency relation, satisfied by the BRS equations, allows a trivial derivation of a gauge fixed and BRS invariant Lagrangian.

The result of Ref. 2) is that one must introduce a pair of Abelian anti-commuting vector ghosts C_μ and $\bar{C}_\mu$, and a pair of $\mathcal{G}$ -valued anticommuting scalar ghosts C and $\bar{C}$. Obviously C_μ , $\bar{C}_\mu$ and C , $\bar{C}$ are respectively associated with the gauge parameters ϵ_μ and ϵ in Eq. (1). In addition, three Abelian commuting scalar ghosts, $\bar{M}$, N and M must be introduced in order to define closed BRS equations. In short, the ghost spectrum introduced in Ref. 2) is as follows

$$\begin{array}{c} A_\mu [0] \\ \swarrow \quad \searrow \\ \bar{C} [-1] \quad C [1] \end{array} \qquad \begin{array}{c} B_{\mu\nu} [0] \\ \swarrow \quad \searrow \\ \bar{C}_\mu [-1] \quad C_\mu [1] \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \bar{M} [-2] \quad N [0] \quad M [2] \end{array} \quad (2)$$

Besides, the BRS symmetry is defined by the action of its generators s and $\bar{s}$ on all fields of Eq. (2) according to

$$\begin{aligned} s A_\mu &= D_\mu C & \bar{s} A_\mu &= D_\mu \bar{C} \\ s C &= -\frac{1}{2} [C, C] & \bar{s} \bar{C} &= -\frac{1}{2} [\bar{C}, \bar{C}] \\ s \bar{C} + \bar{s} C &= -[\bar{C}, C] \end{aligned} \quad (3a)$$

$$\begin{aligned} s B_{\mu\nu} &= \partial_{[\mu} C_{\nu]} + C \cdot \partial_{[\mu} A_{\nu]} & \bar{s} B_{\mu\nu} &= \partial_{[\mu} \bar{C}_{\nu]} + \bar{C} \cdot \partial_{[\mu} A_{\nu]} \\ s C_\mu &= -\frac{1}{2} (A_\mu \cdot [C, C] - \partial_\mu M) & \bar{s} \bar{C}_\mu &= -\frac{1}{2} (A_\mu \cdot [\bar{C}, \bar{C}] - \partial_\mu \bar{M}) \\ s M &= \frac{1}{3} C \cdot [C, C] & \bar{s} \bar{M} &= \frac{1}{3} \bar{C} \cdot [\bar{C}, \bar{C}] \\ s \bar{C}_\mu + \bar{s} C_\mu &= - (A_\mu \cdot [\bar{C}, C] + \partial_\mu N) \\ 2 s N + \bar{s} M &= -\bar{C} \cdot [C, C] \\ 2 \bar{s} N + s \bar{M} &= -C \cdot [\bar{C}, \bar{C}] \end{aligned} \quad (3b)$$

Our method for building the BRS equation (3) was to require that: i) they reproduce the action of a gauge transformation when they act on the classical fields $B_{\mu\nu}$ and A_μ ; ii) the s and $\bar{s}$ operators are fully nilpotent on all fields

$$s^2 = s \bar{s} + \bar{s} s = \bar{s}^2 = 0 \quad (4)$$

In order to enforce Eq. (4), the usual composition law for the action of s and $\bar{s}$ on products of fields was imposed as

$$\begin{aligned} s(XY) &= (sX)Y \pm X sY \\ \bar{s}(XY) &= (\bar{s}X)Y \pm X \bar{s}Y \end{aligned} \quad (5)$$

where the minus sign occurs if the ghost number of X is odd [the ghost number of each field is written in parentheses in Eq. (2), and the ghost number of a product of fields is simply the sum of ghost numbers of factors]. Our construction was independent of the notion of a Lagrangian, but we had also shown that, formally, the set of fields (2) ensures the unitarity of a field theory, in Minkowski space, if its Lagrangian is BRS invariant under Eqs. (3).

The purpose of this note is to show that the BRS equations (3) possess the remarkable property of being obtainable within a formalism where all fields are considered as components of suitably defined exterior forms and, consequently, that they have an intrinsic geometrical meaning. This type of formalism was first introduced by Ne'eman and Thierry-Mieg, in the context of Yang-Mills theories⁴⁾.

The main idea is to enlarge the space-time in a space⁵⁾ with co-ordinates $(x^\mu, \theta, \bar{\theta})$, where x^μ are ordinary space-time co-ordinates, and θ and $\bar{\theta}$ are a pair of scalar Grassmanian co-ordinates. Over each point $(x^\mu, \theta, \bar{\theta})$ one considers the tangent space $\mathcal{T}$, and one defines in $\mathcal{T}$ a general Abelian exterior two-form $\mathcal{B}(x, \theta, \bar{\theta})$ and an exterior one-form $\mathcal{A}(x, \theta, \bar{\theta})$ taking its values in the Lie algebra $\mathcal{G}$.

As is now well known⁶⁾, the set of fields shown in Eq. (2) can be identified as the components of $\mathcal{B}$ and $\mathcal{A}$ when these forms are expanded on the basis of two-forms $dx^\mu \wedge dx^\nu, \dots, d\bar{\theta} \wedge d\bar{\theta}$, and of one-forms $dx^\mu, d\theta, d\bar{\theta}$. Indeed, $\mathcal{B}$ and $\mathcal{A}$ can be expanded in local co-ordinates as

$$\mathcal{A}(x, \theta, \bar{\theta}) = a_\mu dx^\mu + a_\theta d\theta + a_{\bar{\theta}} d\bar{\theta} \quad (6a)$$

$$\begin{aligned} \mathcal{B}(x, \theta, \bar{\theta}) = & b_{\mu\nu} dx^\mu \wedge dx^\nu + b_{\mu\theta} dx^\mu \wedge d\theta + b_{\mu\bar{\theta}} dx^\mu \wedge d\bar{\theta} \\ & + \frac{1}{2} b_{\theta\theta} d\theta \wedge d\theta + b_{\theta\bar{\theta}} d\theta \wedge d\bar{\theta} + \frac{1}{2} b_{\bar{\theta}\bar{\theta}} d\bar{\theta} \wedge d\bar{\theta} \end{aligned} \quad (6b)$$

and the fields in Eq. (2) are identified as

$$A_\mu = a_\mu(x, \theta, \bar{\theta}) \quad C = a_\theta(x, \theta, \bar{\theta}) d\theta \quad \bar{C} = a_{\bar{\theta}}(x, \theta, \bar{\theta}) d\bar{\theta} \quad (7a)$$

$$B_{\mu\nu} = b_{\mu\nu}(x, \theta, \bar{\theta}) \quad C_\mu = b_{\mu\theta}(x, \theta, \bar{\theta}) d\theta \quad \bar{C}_\mu = b_{\mu\bar{\theta}}(x, \theta, \bar{\theta}) d\bar{\theta}$$

$$M = b_{\theta\theta}(x, \theta, \bar{\theta}) d\theta \wedge d\theta \quad N = b_{\theta\bar{\theta}}(x, \theta, \bar{\theta}) d\theta \wedge d\bar{\theta} \quad \bar{M} = b_{\bar{\theta}\bar{\theta}}(x, \theta, \bar{\theta}) d\bar{\theta} \wedge d\bar{\theta} \quad (7b)$$

for $\theta = \bar{\theta} = 0$. Here the symbol $\wedge$ stands for the wedge product of forms.

In this identification, the ghost fields are exterior forms proportional to $d\theta$ and/or $d\bar{\theta}$. The ghost number assigned to each field in Eq. (2) can be simply obtained by counting algebraically the number of $d\theta$ and $d\bar{\theta}$ forms in the fields, with a weight +1 for $d\theta$ and -1 for $d\bar{\theta}$. Moreover, all commutation properties between fields are reproduced by the rules of exterior calculus.

In order to proceed, one introduces the exterior differential operator $\tilde{d}$ in $\mathcal{T}$

$$\hat{d} = d + s + \bar{s} \quad (8a)$$

with

$$d = dx^\mu \frac{\partial}{\partial x^\mu} \quad s = d\theta \frac{\partial}{\partial \theta} \quad \bar{s} = d\bar{\theta} \frac{\partial}{\partial \bar{\theta}} \quad (8b)$$

From this definition, the nilpotency relation $s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$ is a priori guaranteed. Then one introduces the $\mathcal{G}$ -valued two-form $\tilde{F}$ associated with $\mathcal{A}$

$$\tilde{F} = \hat{d}\mathcal{A} + \frac{1}{2} [\mathcal{A}, \mathcal{A}] \quad (9)$$

and the Abelian three-form $\tilde{G}$ constructed from $\mathcal{A}$ and $\mathcal{B}$ according to

$$\tilde{G} = \hat{d}\mathcal{B} + \tilde{F} \cdot \mathcal{A} - \frac{1}{6} \mathcal{A} \cdot [\mathcal{A}, \mathcal{A}] \quad (10)$$

Here all products are exterior products, for instance $[\mathcal{A}, \mathcal{A}]^a = f_{bc}^a \mathcal{A}^b \wedge \mathcal{A}^c$, and the $\cdot$ symbol has been defined above. $\tilde{G} - d\tilde{B}$ can be recognized as a Chern-Weil-Simon three-form^{7),8)}. The fundamental property of $\tilde{G}$ is that its exterior differential is the Abelian anomaly

$$\hat{d}\tilde{G} = (\tilde{F} - \frac{1}{2} [\mathcal{A}, \mathcal{A}])(\tilde{F} + \frac{1}{2} [\mathcal{A}, \mathcal{A}]) = \tilde{F} \cdot \tilde{F} \quad (11)$$

thanks to the Jacobi identity $[\mathcal{A}, [\mathcal{A}, \mathcal{A}]] = 0$. As a matter of fact, property (11) makes the definition (10) of $\tilde{G}$ canonical.

Our observation is that the BRS equation (3) can be obtained as a whole by postulating horizontality conditions simultaneously for $\tilde{F}$ and $\tilde{B}$, i.e.,

$$\tilde{F} \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu \quad (12a)$$

$$\tilde{G} \equiv G_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \quad (12b)$$

Indeed, once expanded on the basis of two- and three-exterior forms made with dx^μ , $d\theta$ and $d\bar{\theta}$, Eqs. (12) yield two consequences.

On the one hand, by identifying the horizontal terms (without $d\theta$ and/or $d\bar{\theta}$ dependence) on both sides of Eq. (12), the definition of $F_{\mu\nu}$ and $G_{\mu\nu\rho}$ is obtained

$$F = F_{\mu\nu} dx^\mu_\Lambda dx^\nu = (\partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]) dx^\mu_\Lambda dx^\nu \quad (13a)$$

$$G = G_{\mu\nu\rho} dx^\mu_\Lambda dx^\nu_\Lambda dx^\rho = (\partial_{[\mu} b_{\nu\rho]} - F_{[\mu\nu} \cdot a_{\rho]} - \frac{2}{3} a_{[\mu} \cdot [a_\nu, a_{\rho]}]) \quad (13b)$$

On the other hand, setting equal to zero all components in $\tilde{F}$ and $\tilde{G}$ proportional to $d\theta$ and/or $d\bar{\theta}$, the following equations are found

$$\begin{aligned} s a + D c &= 0 & \bar{s} a + D c &= 0 \\ s c + \frac{1}{2} [c, c] &= 0 & \bar{s} \bar{c} + \frac{1}{2} [\bar{c}, \bar{c}] &= 0 \\ s \bar{c} + \bar{s} c + [\bar{c}, c] &= 0. \end{aligned} \quad (14a)$$

$$\begin{aligned} s b + d \xi + c \cdot da &= 0 & \bar{s} b + d \bar{\xi} + \bar{c} \cdot da &= 0 \\ s \xi + \frac{1}{2} (d m - a \cdot [c, c]) &= 0 & \bar{s} \bar{\xi} + \frac{1}{2} (d \bar{m} - a \cdot [\bar{c}, \bar{c}]) &= 0 \\ s m - \frac{1}{3} c \cdot [c, c] &= 0 & \bar{s} \bar{m} - \frac{1}{3} \bar{c} \cdot [\bar{c}, \bar{c}] &= 0 \end{aligned}$$

$$\begin{aligned} s \bar{\xi} + \bar{s} \xi + d m + a \cdot [c, \bar{c}] &= 0 \\ 2 \bar{s} m + s \bar{m} - c \cdot [\bar{c}, \bar{c}] &= 0 \\ 2 s \bar{m} + \bar{s} m - \bar{c} \cdot [c, c] &= 0 \end{aligned} \quad (14b)$$

where $b = b_{\mu\nu}(x, \theta, \bar{\theta}) dx^\mu_\Lambda dx^\nu$, $\xi = b_\mu(x, \theta, \bar{\theta}) dx^\mu_\Lambda d\theta$, $\bar{\xi} = b_\mu(x, \theta, \bar{\theta}) dx^\mu_\Lambda d\bar{\theta}$,
 $m = b_{\theta\theta}(x, \theta, \bar{\theta}) d\theta_\Lambda d\theta$, $n = b_{\theta\bar{\theta}}(x, \theta, \bar{\theta}) d\theta_\Lambda d\bar{\theta}$, $\bar{m} = b_{\bar{\theta}\bar{\theta}}(x, \theta, \bar{\theta}) d\bar{\theta}_\Lambda d\bar{\theta}$,
 $a = a_\mu(x, \theta, \bar{\theta}) dx^\mu$, $c = a_\theta(x, \theta, \bar{\theta}) d\theta$, $\bar{c} = a_{\bar{\theta}}(x, \theta, \bar{\theta}) d\bar{\theta}$.

Setting $\theta = \bar{\theta} = 0$ in Eqs. (14), one recovers exactly the whole set of BRS equations (3). Moreover, by Taylor expanding in $\theta, \bar{\theta}$ all fields in Eq. (14), one obtains new equations from terms in $\theta, \bar{\theta}$ and $\bar{\theta}\theta$. One can easily check that these equations simply enforce the definition of s and $\bar{s}$ as graded differential exterior operators satisfying the composition rule (5), and the nilpotency relation (4).

Finally, the method appears fully consistent after checking that Eqs. (11) are their own integrability conditions: no further constraint arises if the operators $\tilde{D}$ or $\tilde{d}$ are iterated on both sides of Eqs. (1). In particular, one can verify

$$\hat{d} \hat{G} \equiv \tilde{F} \cdot \tilde{F} \equiv F \cdot F \equiv d G \quad (15)$$

This indeed follows from the covariance of G and F under BRS transformations

$$\begin{aligned} s F &= [F, c] & \bar{s} F &= [F, \bar{c}] \\ s G &= 0 & \bar{s} G &= 0 \end{aligned} \quad (16)$$

as can be seen directly from Eq. (13). One should note that Eqs. (16) make obvious the construction of the gauge invariant classical Lagrangian $\mathcal{L}_{cl} = |F|^2 + |G|^2$, which is thereby obtained as a consequence of the closure of BRS equations. The procedure for obtaining a gauge fixed and BRS invariant Lagrangian has been described in Ref. 2). One gets $\mathcal{L}_Q = |F|^2 + |G|^2 + s\bar{s}(|A|^2 + |B|^2 + |\bar{\xi}\xi| + |\bar{m}n| + |n|^2)$.

In conclusion, we find it remarkable that the geometrical method where classical and ghost fields are introduced simultaneously solves directly the problem of finding the BRS equations associated with the mixed Abelian and non-Abelian gauge transformations of Chapline and Manton. Indeed, by going into the enlarged space $\{x^\mu, \theta, \bar{\theta}\}$, we have defined generalized field strengths and, from horizontality conditions, we have found not only all relevant BRS equations, including the usual definition of gauge transformations, but also the field strengths which transform covariantly. In this approach, the construction of the full quantum Lagrangian follows trivially from the geometry. We hope that our result will motivate further studies for a deeper understanding of the structure of $\{x^\mu, \theta, \bar{\theta}\}$ space.

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