Ref.TH.3546-CERN

ON A CLASSICAL UNIFIED THEORY FOR INTERACTION AND MATTER FIELDS

N. A. Batakis ^{*)}
CERN - Geneva

A B S T R A C T

A classical unified theory is proposed for the gravitational and electromagnetic fields, as well as a possibly charged matter field incorporated into the spacetime geometry as, respectively, the symmetric-metric part of the connection, the metricity tensor and the contorsion tensor. The construction seems to admit a maximal extension, incorporating the electroweak interactions.

^{*)} On leave from the Department of Physics, University of Ioannina, Greece.

The relevance of gauge theories in physics, and in particular their fundamental role in the unification of the electromagnetic and weak interactions, has motivated analogous considerations for general relativity as a step towards the eventual unification of all interactions and matter fields. Such an ultimate theory will presumably admit a classical interpretation, and it seems quite appealing, at least among relativists, that there may also exist a geometric framework, generalizing that of Riemannian geometry, which would fully accommodate that classical limit. Unfortunately, in spite of enormous efforts during several decades, we do not yet have a definite scheme unifying even just the e.m. field with gravity. There *has been*, however, one interesting development in that direction, involving not another interaction but what is considered the classical analogue of spin. The relevant construction is known as the Einstein-Cartan theory, in which spin is given a geometric interpretation in terms of torsion in a Riemann-Cartan spacetime, and gravity is thereby identified as the gauge theory of the Poincaré group [1].

In a recent paper we proposed what *then* appeared as a novel interpretation of torsion, which seemed to provide a unification scheme for the gravitational and e.m. fields [2]. However, this result can be modified, so that the original interpretation of torsion is essentially restored in a more satisfying unification scheme, involving not only the gravitational and e.m. fields but also the (classical analogue of) matter fields as well. We will now present *ab initio* this construction. The possibility of its generalization for the case of a non-Abelian classical Yang-Mills field will be discussed briefly at the end. In the following, greek indices take the values 0, 1, 2, 3, the metric is $g_{\mu\nu} = \text{diag} (-1, 1, 1, 1)$ in an orthonormal frame, and $\overset{\Gamma}{\nabla}$ refers to a general connection Γ , with the semi-colon reserved for $\overset{\circ}{\nabla}$ (see eq. (1)).

At the limit of a weak gravitational field, the Newtonian potential is given by $1/2 (g_{00} + 1)$, implying the standard association of the metric tensor with the gravitational potential, while the field itself is associated with the symmetric *metric* part of the connection ("Christoffel symbols"):

$$\overset{\circ}{\Gamma}^{\alpha}_{\beta\gamma} = \{\alpha_{\beta\gamma}\} = \frac{1}{2} g^{\alpha\mu} (g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}). \quad (1)$$

One should be able to transform this field to zero at one point, in fact along any given curve (because of the principle of equivalence), which is elegantly afforded by a corresponding property of the connection (1). On the other hand, some analogous possibility should *also* exist for the e.m. field, and, if the latter is to be incorporated into the geometry at the level of the connection^{*)}, it has to enter its *symmetric part*. To be more precise, consider the motion of a small charged particle in a given electric field. One can balance the electric force on the particle by a proper gravitational force, that is by a proper general coordinate transformation. Its motion will then be that of a "free" particle along *some* geodesic -- not, of course, with respect to the connection (1), but to some other connection incorporating the e.m. field as well. Our claim follows if we recall that only the symmetric part of a connection can be transformed to zero along a curve [4]. A torsion $T^{\alpha}_{\beta\gamma}$ does give a symmetric contribution, if it is employed in terms of the contorsion tensor:

$$K_{\alpha\beta\gamma} = \frac{1}{2} (T_{\gamma\alpha\beta} + T_{\beta\alpha\gamma} - T_{\alpha\beta\gamma}). \quad (2)$$

However, there will always remain the irreducible antisymmetric part

$$T_{\alpha\beta\gamma} = K_{\alpha\gamma\beta} - K_{\alpha\beta\gamma},$$

making torsion better suited for the interpretation of the matter fields. We are thus led to consider a more general connection, which we write as:

$$\Gamma^{\alpha}_{\beta\gamma} = \overset{\circ}{\Gamma}^{\alpha}_{\beta\gamma} + Q^{\alpha}_{\beta\gamma} + K^{\alpha}_{\beta\gamma}. \quad (3)$$

*) This view is supported by the analogy with the gravitational field. There is an interesting relevant comment by C.P. Johnson Jr., criticizing Einstein's incorporation of the e.m. field at the level of the metric [3]. Our choice here in no way upsets the standard interpretation of gauge potentials, as the latter refers to a connection in a different principal bundle. Instead, we simply provide a linkage between the two hitherto unrelated connections (see also the discussion at the end).

In the Einstein-Cartan theory (and similarly in general relativity), compatibility between the metric $g_{\mu\nu}$ and the connection (3) is guaranteed by the *sufficient* condition that the symmetric tensor

$$Q^{\alpha}{}_{\beta\gamma} = Q^{\alpha}{}_{\gamma\beta} \quad (4)$$

vanishes. There exists, however, a *weaker* sufficient and necessary condition for compatibility [4], namely:

$$Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} + Q_{\gamma\alpha\beta} = 0, \quad (5)$$

in which case $Q_{\alpha\beta\gamma}$ can be identified with the metricity tensor:

$$\nabla_{\gamma}^{\Gamma} g_{\alpha\beta} = Q_{\alpha\beta\gamma}. \quad (6)$$

If we further require invariance of the volume element:

$$\nabla_{\mu}^{\Gamma} \varepsilon_{\alpha\beta\gamma\delta} = 0, \quad (7)$$

we must also have

$$Q^{\alpha}{}_{\alpha\beta} = 0 \quad (8)$$

We will postulate that *such* a metricity tensor may be given a physical interpretation in terms of the e.m. field $F_{\mu\nu}$. The extra index will be provided by the gradient of a scalar field $\phi_{\alpha} = \phi_{;\alpha}$. Such a field has also been employed in our previous considerations [2] but we have since provided an independent motivation for its introduction, in connection with Mach's Principle [5]. We will briefly outline that result and then return to the present development.

The Lagrangian (with $g = -\det g_{\mu\nu}$)

$$\overset{\circ}{\mathcal{L}} = (16\pi G)^{-1} \sqrt{g} \overset{\circ}{R} \quad (9)$$

has been shown [5] to incorporate Mach's Principle into general relativity, if the gravitational coupling is related to some cosmic scalar field ϕ by

$$G = -(16\pi)^{-1} \varphi_{\alpha} \varphi^{\alpha}. \quad (10)$$

No additional terms are required in (9), and the resulting theory virtually coincides with general relativity for the important cases of i) an empty, static, and spherically symmetric space-time (reproducing the Schwarzschild solution), and ii) a dust-filled, homogeneous, and isotropic space-time (reproducing the Friedmann models with dust). In both cases G reduces to Newton's constant G_N , but its relation to ϕ through (10) allows a non-trivial interpretation of Mach's Principle.

To proceed with the construction of $Q_{\alpha\beta\gamma}$, one can show that, in view of (4), (5), (8), there exists only one combination, linear in both $F_{\alpha\beta}$ and ϕ_γ :

$$Q_{\alpha\beta\gamma} = F_{\alpha(\beta} \phi_{\gamma)} + \frac{1}{3} \phi^\lambda \left[F_{\lambda\alpha} g_{\beta\gamma} + g_{\alpha(\beta} F_{\gamma)\lambda} \right]. \quad (11)$$

$F_{\alpha\beta}$ is, for the moment, a shorthand notation for

$$F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}, \quad (12)$$

employed to ensure gauge invariance. For an analogous construction of the contorsion tensor, we immediately realize that since $K^\alpha_{\beta\gamma}$ will accommodate the matter fields, its *explicit* form must depend on an underlying phenomenology or quantum theory of matter. We will assume that the latter can be expressed in terms of an antisymmetric tensor field [1]

$$S_{\alpha\beta} = -S_{\beta\alpha}, \quad (13)$$

such that the e.m. current will be given by

$$j^\alpha = e \sqrt{g} S^{\alpha\beta}{}_{;\beta}, \quad (14)$$

with e a unit of charge. For the case of neutral matter, $S_{\alpha\beta}$ must obviously be divergenceless. In analogy with the previous case, a *traceless* contorsion linear in $S_{\alpha\beta}$ and ϕ_γ is given by

$$K_{\alpha\beta\gamma} = \frac{2}{3} e \left(\phi_{[\alpha} S_{\beta]\gamma} - S_{\alpha\beta} \phi_\gamma - \phi^\lambda S_{\lambda[\alpha} g_{\beta]\gamma} \right). \quad (15)$$

The curvature scalar corresponding to the connection (3) with its assumed symmetries, is given by:

$$R = \overset{\circ}{R} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + Q_{\alpha\beta\gamma} K^{\alpha\beta\gamma} + K_{\alpha\beta\gamma} K^{\alpha\beta\gamma}. \quad (16)$$

It would obviously be satisfying if

$$\mathcal{L} = (16\pi G)^{-1} \sqrt{g} R \quad (17)$$

provided the *total* Lagrangian for the theory. Indeed, using (10), (13-17), and performing a partial integration for the QK term, we can establish the result:

$$\mathcal{L} = (16\pi G)^{-1} \sqrt{g} \mathring{R} - \frac{1}{4} \sqrt{g} F^2 + A_\alpha j^\alpha - \frac{1}{3} e^2 \sqrt{g} S^2, \quad (18)$$

where $F^2 = F_{\alpha\beta} F^{\alpha\beta}$ and $S^2 = S_{\alpha\beta} S^{\alpha\beta}$. The dynamical quantities involved in (18) are ϕ (see eq. (10)), A_α , and $g_{\alpha\beta}$; variation of $\mathcal{L}$ with respect to these will provide all the field equations. If we assume for simplicity that ϕ does not enter the construction of $S_{\alpha\beta}$, these equations are, respectively:

$$\Phi^\alpha{}_{;\alpha} = 0 \quad (19)$$

$$\sqrt{g} F^{\alpha\beta}{}_{;\beta} = j^\alpha \quad (20)$$

$$\mathring{R}_{\alpha\beta} - \frac{1}{2} \mathring{R} g_{\alpha\beta} = 8\pi G \left(T^{(\varphi)}_{\alpha\beta} + T^{(EM)}_{\alpha\beta} + T^{(matter)}_{\alpha\beta} \right) \quad (21)$$

where

$$\Phi^\alpha = G^{-2} \mathring{R} \varphi^\alpha \quad (22)$$

$$T^{(\varphi)}_{\alpha\beta} = 2 \left(G^{-1}{}_{;\mu;\nu} g_{\alpha\beta} - G^{-1}{}_{;\alpha;\beta} - \Phi_\alpha \varphi_\beta \right) \quad (23)$$

$$T^{(EM)}_{\alpha\beta} = F_\alpha{}^\mu F_{\beta\mu} - \frac{1}{4} F^2 g_{\alpha\beta} \quad (24)$$

$$\sqrt{g} T^{(matter)}_{\alpha\beta} = \frac{2}{3} e^2 \frac{\delta}{\delta g^{\alpha\beta}} (\sqrt{g} S^2). \quad (25)$$

$F_{\mu\nu}$ can now be identified with the e.m. field through (12), and the conservation law $j^\alpha{}_{;\alpha} = 0$ following from (20) is in agreement with the definition (14).

Finally, taking the divergence of (21) and using the rest of the field equations plus the contracted Bianchi identity for the Riemannian connection, we establish the important result:

$$\sqrt{g} T^{\text{(matter)} \beta}_{\alpha\beta} = F_{\alpha\beta} j^\beta. \quad (26)$$

When the e.m. field is switched off, the theory reduces to the scalar-tensor theory of ref. [5], the results of which we have already outlined. A similar limit can be obtained, giving us Maxwell's theory in flat spacetime.

An important development of the above construction would be its generalization for the case of a non-Abelian Yang-Mills field [6]. To outline such a program, it should first of all be clear that the geometric interpretation of gauge potentials as the components of a connection in a principal bundle over a curved spacetime does not automatically provide a geometric unification of the corresponding Y-M field with gravity. Such structures are simply generalizations of the Einstein-Maxwell system. On the other hand, for a U(1) gauge group, the present theory does provide a linkage between fibres and the base space, in terms of the connection (3). For the case of an N-dimensional non-Abelian group, the potentials, fields, etc., will carry, as usual, an additional internal index $i = 0, 1, \dots, (N-1)$. Since $T^\alpha_{\beta\gamma}$ should remain gauge invariant, ϕ must be also generalized to a vector ϕ^i , contracted in (11) and (15) with $F^i_{\alpha\beta}$ and $S^i_{\alpha\beta}$, respectively, by means of the group metric (killing form) g^{ij} , while (10) must be generalized to

$$g^{\alpha\beta} \varphi^i_\alpha \varphi^j_\beta = -16\pi G g^{ij}. \quad (27)$$

The requirement of non-degeneracy for g^{ij} in (27) shows that our formalism would at most allow $N = 4$. If the four spacetime vectors ϕ^i_α are chosen mutually orthogonal, with for example ϕ^0_α timelike and $\phi^{1,2,3}_\alpha$ spacelike (each with absolute square-length $16\pi G$), the group metric is $\text{diag}(1, -1, -1, -1)$, suggesting $U(1) \times SU(2)$ as the *maximal* relevant group. The construction would conceivably

then provide a classical unification for the gravitational and electroweak interactions, with the ϕ^i related to some Higgs fields (ϕ^0 is here a singlet and (ϕ^1, ϕ^2, ϕ^3) a vector in the adjoint representation of $SU(2)$) and with everything else incorporated into the torsion. In that sense, the present theory seems to imply that strong interactions, being responsible for the deeper structure of matter, cannot be transformed to zero even at one point, like spin. Classically, then, they would not appear as forces at all, at least not on the same footing as the gravitational and electroweak ones.

We conclude with a discussion of possible difficulties with the present theory, as far as we can see: i) the expression (10) for G can be viewed as an eikonal equation, which could develop caustics in extreme situations. However, as torsion has been shown to possibly avert the occurrence of singularities [1], this problem may well not arise. ii) The role of the geodesics of (3) is not quite clear. Of course, the equations of motion for charged particles, with the correct Lorentz-force law, cannot be postulated separately, as they should follow from the field equations. Further, in non-metric theories there may be no *canonical* way to lift the connection in a spinor bundle, which could cause difficulties with the incorporation of spinors into the theory [7]. It should be recalled, however, that, although the connection (3) is not metric, it is nevertheless *compatible* with the metric [4]; see also refs. [1] and [8]. iii) The Lagrangian (18) contains no kinetic term for the scalar field, which furthermore may not even have positive energy in (23). These aspects, together with the $(\phi_\alpha \phi^\alpha)^{-1}$ term, will make problematic a standard quantization program. We may remark however that, for example for the case of a nearly homogeneous and isotropic universe, up to first order in the infinitesimal quantity $\epsilon = 1 + (16\pi G_N)^{-1} \phi_\alpha \phi^\alpha$, we find $G^{-1} = G_N^{-1} (1 + \epsilon)$ from (10), whereby a rescaling of ϕ , combined with some sort of Higgs mechanism, could possibly provide a resolution. Finally, it is not at all obvious if the eventually resulting quantum theory will succeed in lifting the non-renormalizability of general relativity [9].

The author is thankful to S. Ferrara, M. Jacob, Ch. Mukku, D. Nanopoulos, T. Regge, Th. Schücker and other members of the Theory Division at CERN for their hospitality and for useful discussions.

REFERENCES

- [1] F.W. Hehl, P. Heyde, G.D. Kerlick and J.M. Nester, Rev. Mod. Phys. 48 (1976) 393, and references cited therein.
- [2] N.A. Batakis, Phys. Lett. 90A (1982) 115.
- [3] C.P. Johnson Jr., Phys. Rev. 87 (1953) 320.
- [4] E. Schrödinger, Space-Time Structure (Cambridge Univ. Press, London, 1954).
- [5] N.A. Batakis, submitted to Physics Letters A.
- [6] For standard results see, for example,
T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213, and
references cited therein.
- [7] A. Trautman, *in* "Symposia Mathematica" Vol. XII (1973) 139.
- [8] A.A. Coley, Phys. Rev. D 27 (1983) 728.
- [9] H.T. Nieh, Phys. Lett. 88A (1982) 388.