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ASYMPTOTIC EXPANSIONS FOR THE LANDAU DENSITY
AND DISTRIBUTION FUNCTIONS

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ABSTRACT

Asymptotic expansions are given for the Landau function

$$\phi(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{u \ln u + \lambda u} du$$

and for its integral, as $\lambda \rightarrow \pm\infty$. Further, it is shown that $\phi(\lambda)$ is indeed a probability density function, as is always assumed in applications.

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1. INTRODUCTION

In 1944 Lev Landau [10] derived an integro-differential equation for the probability density function (*pdf*) of the energy loss of heavy particles traversing a slab of material. This energy loss is the result of collisions with atomic electrons. The resulting *pdf*,

$$\phi(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{u \ln u + \lambda u} du, \quad (1)$$

and its generalization, the so-called Vavilov distribution [16], play an important role in high energy physics (see, e.g. [2], [4], [6], [14], [15]). It is not, however, immediately clear from Landau's analysis (which contains some approximations difficult to justify from a mathematical point of view) that his result is a *pdf*. In particular, it would seem to be very difficult to prove its non-negativity for all λ and its normalization from the given representations of $\phi(\lambda)$ in form of parameter integrals. This point has apparently not yet been treated in the relevant literature. Also, confusion has arisen from the fact that Moyal [12] has introduced another *pdf*,

$$\chi(\lambda) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\lambda + \exp(-\lambda) \right)^2 \right\}, \quad (2)$$

which he calls an "explicit expression of Landau's distribution". This function is much easier to compute and has sometimes been used instead of Landau's original function (1) (e.g., in [13]).

We do not wish to comment on the physical implications of using (2) instead of (1), but would like to draw attention to the fact that the two functions are quite different, in particular as concerns their maximum value and their asymptotic behaviour as $\lambda \rightarrow \infty$ (Fig. 1).

In this note we show that Landau's function (1) is indeed a *pdf*. We also discuss the asymptotic behaviour of $\phi(\lambda)$ as $\lambda \rightarrow \pm\infty$, extending the investigations of Landau [10] and Börsch-Supan [3]. We also give asymptotic formulae, as $\lambda \rightarrow \pm\infty$, for the corresponding Landau distribution function

$$\Phi(\lambda) = \int_{-\infty}^{\lambda} \phi(\lambda) d\lambda . \quad (3)$$

2. LANDAU'S MOMENT GENERATING FUNCTION

Let $f(\varepsilon, x)$ be the *pdf* of the energy loss, where ε denotes the energy and x the thickness of the material. The moment generating function of $f(\varepsilon, x)$ may then be defined by the Laplace transforms

$$F(s, x) = \int_0^{\infty} e^{-s\varepsilon} f(\varepsilon, x) d\varepsilon . \quad (4)$$

[In the literature on probability theory, the function $F(s, x)$, with $s = -it$, is called the characteristic function of $f(\varepsilon, x)$.] By solving a certain integro-differential equation formally by the Laplace transform, Landau obtained the following representation for $F(s, x)$:

$$F(s, x) = \exp \left\{ -x \int_0^{\infty} w(\varepsilon) (1 - e^{-s\varepsilon}) d\varepsilon \right\} , \quad (5)$$

where $w(\varepsilon)$ is the *pdf* of the energy loss per unit length. Using a mixture of physical and mathematical arguments to obtain an expression for the function $w(\varepsilon)$ and to evaluate certain definite integrals, Landau arrives at the following approximation for $F(s, x)$:

$$\tilde{F}(s, x) = e^{\psi(s, x)} , \quad (6)$$

with

$$\psi(s, x) = xs(\ln s\alpha + \gamma - 1) , \quad (7)$$

where $\alpha > 0$ is a constant and $\gamma = 0.57721 \dots$ is Euler's constant. Applying the inversion formula for the Laplace transform to (4), and using (6), one obtains

$$\tilde{f}(\varepsilon, x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\psi(s, x) + s\varepsilon} ds \quad (\sigma \geq 0) . \quad (8)$$

Setting $s = u/x$ and defining

$$\lambda = \varepsilon/x + \ln(\alpha/x) + \gamma - 1 ,$$

finally yields

$$\tilde{f}(\varepsilon, x) = \frac{1}{x} \phi(\lambda) ,$$

where

$$\phi(\lambda) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{u \ln u + \lambda u} du \quad (9)$$

is the $p\hat{d}f$ of Landau's distribution.

We now show that $\tilde{F}(s, x)$ is a moment generating function of a $p\hat{d}f$. According to Loève [11], a function $\exp[\psi(s)]$ is a moment generating function of a $p\hat{d}f$ if $\psi(s)$ can be written in the form

$$\psi(s) = as + \int_{-\infty}^{\infty} \left(e^{-s\xi} - 1 + \frac{s\xi}{1 + \xi^2} \right) \frac{1 + \xi^2}{\xi^2} d\Psi(\xi) , \quad (10)$$

where $a \in \mathbb{R}$ and $\Psi(\xi)$ is a distribution function (up to a multiplicative constant), with $\Psi(-\infty) = 0$. We therefore wish to show that $\psi(s, x)$ as defined by (7) can be written in the form (10). To do this, we consider the function

$$\begin{aligned} I(A) &= \int_0^A \left(e^{-s\xi} - 1 + \frac{s\xi}{1 + \xi^2} \right) \frac{d\xi}{\xi^2} \\ &= \int_0^A \left(e^{-s\xi} - 1 + s\xi \right) \frac{d\xi}{\xi^2} - s \int_0^A \frac{\xi dx}{1 + \xi^2} \\ &= -s + \frac{1 - e^{-sA}}{A} + s \int_0^{sA} \frac{1 - e^{-t}}{t} dt - s \ln \sqrt{1 + A^2} , \end{aligned}$$

with $A > 0$. Using [5, No. 8.2111 and 8.2121],

$$\int_0^z \frac{1 - e^{-t}}{t} dt = \int_z^\infty e^{-t} dt/t + \gamma + \ln z \quad (\text{Re } z > 0) \quad (11)$$

we obtain

$$I(A) = -s(1 - \gamma - \ln s) + \frac{1 - e^{-sA}}{A} + s \int_{sA}^\infty e^{-t} dt/t - s \ln \frac{\sqrt{1 + A^2}}{A} ,$$

and hence, for $\text{Re } s > 0$,

$$I(\infty) = \int_0^{\infty} \left(e^{-s\xi} - 1 + \frac{s\xi}{1 + \xi^2} \right) \frac{d\xi}{\xi^2} = s(\ln s + \gamma - 1). \quad (12)$$

Therefore, from (7):

$$\begin{aligned} \psi(s, x) &= xI(\infty) + xs \ln \alpha \\ &= (x \ln \alpha)s + \int_{-\infty}^{\infty} \left(e^{-s\xi} - 1 + \frac{s\xi}{1 + \xi^2} \right) \frac{1 + \xi^2}{\xi^2} d\Psi(\xi), \end{aligned} \quad (13)$$

where

$$\Psi(\xi) = \begin{cases} x \arctan \xi & (\xi \geq 0) \\ 0 & (\xi < 0) \end{cases},$$

which is of the form (10). This result, proved under the assumption $\operatorname{Re} s > 0$, holds for all s by analytic continuation. The function $\exp[\psi(s, x)]$ is therefore a moment generating function of a pdf.

3. REPRESENTATIONS OF $\phi(\lambda)$

Starting from the integral (1), Börsch-Supan [3] has derived an integral representation of $\phi(\lambda)$ which is complicated but nevertheless suitable for numerical evaluation. He chooses σ in such a way that the path of integration passes through the saddle point of the integrand in (1). Thus $\sigma = u_0$, where $u_0 = 1/\exp(\lambda + 1)$ is the solution of

$$\frac{d}{du} e^u \ln u + \lambda u = 0.$$

Setting $u = \sigma + iy$ in (1) gives

$$\phi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(\sigma+iy)\ln(\sigma+iy) + \lambda(\sigma+iy)} dy.$$

Using $\lambda = -1 - \ln \sigma$, and combining the complex conjugate values of the integrand, one obtains

$$\begin{aligned} \phi(\lambda) &= \frac{1}{\pi} e^{-\sigma} \int_0^{\infty} \exp \left\{ \frac{\sigma}{2} \ln \left(1 + \frac{y^2}{\sigma^2} \right) - y \arctan \frac{y}{\sigma} \right\} \\ &\quad \cos \left\{ \frac{1}{2} y \ln \left(1 + \frac{y^2}{\sigma^2} \right) - y + \sigma \arctan \frac{y}{\sigma} \right\} dy, \end{aligned} \quad (14)$$

where

$$\sigma = 1/\exp(\lambda+1) . \quad (15)$$

This expression is particularly suitable for negative λ . For positive λ , Landau [10] chooses the path of integration to lie along the branch cut from $-\infty$ to 0, and obtains

$$\begin{aligned} \phi(\lambda) &= \frac{1}{2\pi i} \int_{-\infty}^0 \left(e^{-i\pi u} - e^{i\pi u} \right) e^u \ln|u| + \lambda u \, du \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-u} \ln u - \lambda u \sin \pi u \, du . \end{aligned} \quad (16)$$

By setting $\sigma = 0$ in the integral (1), we obtain

$$\begin{aligned} \phi(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it} \ln it + \lambda it \, dt \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-(\pi/2)t} \cos (t \ln t + \lambda t) dt . \end{aligned} \quad (17)$$

as another integral representation. Seltzer and Berger [14] have derived this formula for $\phi(\lambda)$ as a limiting case of the Vavilov density function [16].

It turns out that the integrals (14) and (16) provide a tool for computing $\phi(\lambda)$ on almost the whole of the λ -axis. By using an accurate numerical integration procedure, they can be used to compute values needed for the construction of suitable approximations by series of Chebyshev polynomials or by rational functions. It is clear that these approximations will then allow a much faster computation of $\phi(\lambda)$. Once it is possible to compute $\phi(\lambda)$ fast, an additional numerical integration will allow the construction of similar approximations for the distribution function $\Phi(\lambda)$.

The functions $\phi(\lambda)$ and $\Phi(\lambda)$ are plotted in Fig. 2. We mention that $\phi(\lambda)$ has its maximum at $\lambda^* = -0.22278 \, 29812 \, 5640$, with $\phi(\lambda^*) = 0.18065 \, 56338 \, 2055^*$.

In addition to their numerical usefulness, the formulae (14) and (16) also serve as starting points for the derivation of asymptotic expressions for $\phi(\lambda)$ as $\lambda \rightarrow \pm\infty$.

) Fano [4] finds $\lambda^ = 0.225$, but this is probably a misprint. Landau [10] gives the value $\lambda^* = -0.05$.

4. ASYMPTOTIC EXPANSIONS FOR THE DENSITY FUNCTION $\phi(\lambda)$

4.1 $\lambda \rightarrow -\infty$

In order to find an asymptotic expansion for $\phi(\lambda)$, as $\lambda \rightarrow -\infty$, we follow Börsch-Supan [3] and write (14) as

$$\phi(\lambda) = \frac{1}{\pi} e^{-\sigma} \int_0^{\infty} \exp\left(-\frac{y^2}{2\sigma}\right) \operatorname{Re} \exp\left\{\sigma\left[f\left(\frac{y}{\sigma}\right) + ig\left(\frac{y}{\sigma}\right)\right]\right\} dy, \quad (18)$$

where

$$f(t) = \frac{1}{2} \ln(1+t^2) - t \arctan t + \frac{1}{2} t^2, \quad (19)$$

$$g(t) = \frac{1}{2} t \ln(1+t^2) + \arctan t - t.$$

Writing

$$h(t) = f(t) + ig(t) = \sum_{n=3}^{\infty} b_n t^n \quad (20)$$

we find the coefficients $c_{2n}(\sigma)$ of the power series

$$\begin{aligned} \operatorname{Re} \exp\left\{\sigma h\left(\frac{y}{\sigma}\right)\right\} &= \exp\left[\sigma f\left(\frac{y}{\sigma}\right)\right] \cos\left[\sigma g\left(\frac{y}{\sigma}\right)\right] \\ &= \sum_{n=0}^{\infty} c_{2n}(\sigma) \left(\frac{y}{\sigma}\right)^{2n} \end{aligned} \quad (21)$$

by using the recurrence relation for the coefficients of the power series representing the exponential function of an arbitrary power series (see, e.g. Henrici [8, § 1.6]):

$$c_n = \frac{1}{n} \sum_{k=1}^n k b_k c_{n-k}; \quad c_0 = 1.$$

It turns out that the coefficients $c_{2n}(\sigma)$ are polynomials in σ of degree $[(2n)/3]$, where $[\xi]$ denotes the integer part of ξ . We list them up to $n = 6$:

$$\begin{aligned} c_0(\sigma) &= 1 \\ c_2(\sigma) &= 0 \\ c_4(\sigma) &= \frac{1}{12} \sigma \\ c_6(\sigma) &= -\frac{1}{72} \sigma^2 - \frac{1}{30} \sigma \\ c_8(\sigma) &= \frac{17}{1440} \sigma^2 + \frac{1}{56} \sigma \end{aligned} \quad (22)$$

$$c_{10}(\sigma) = -\frac{1}{864} \sigma^3 - \frac{403}{50400} \sigma^2 - \frac{1}{90} \sigma$$

$$c_{12}(\sigma) = \frac{1}{31104} \sigma^4 + \frac{13}{10368} \sigma^3 + \frac{839}{151200} \sigma^2 + \frac{1}{132} \sigma .$$

Substituting (21) in (18) and using the formula [5, No. 3.4612],

$$\int_0^{\infty} y^{2n} e^{-y^2/(2\sigma)} dy = \sqrt{\frac{\pi}{2}} (2n-1)!! \sigma^{n+\frac{1}{2}} \quad (23)$$

we obtain an asymptotic series of the form

$$\phi(\lambda) = \frac{e^{-\sigma/\sqrt{\sigma}}}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \bar{a}_n \sigma^{-n} \quad (\lambda \rightarrow -\infty, \sigma \rightarrow \infty) . \quad (24)$$

The actual computation has been carried out by REDUCE [7] and lead to the coefficients

$$\begin{aligned} \bar{a}_0 &= 1 \\ \bar{a}_1 &= 1/24 \\ \bar{a}_2 &= -23/1152 \\ \bar{a}_3 &= 11237/414720 \\ \bar{a}_4 &= -2482411/39813120 \\ \bar{a}_5 &= 272785979/1337720832 \\ \bar{a}_6 &= -4175309343349/4815794995200 \\ \bar{a}_7 &= 525035501918789/115579079884800 \end{aligned} \quad (25)$$

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Although this computation is in principle straightforward, it turns out that the power series (20) has to be computed up to the 42nd power in order to get $\bar{a}_7$ correctly. It is clear from this that the computation would have been difficult without the use of a formula manipulation system.

From the coefficients (25), it seems that as $n \rightarrow \infty$,

$$\left| \frac{\bar{a}_{n+1}}{\bar{a}_n} \right| = O(n) . \quad (26)$$

Only the zeroth coefficient in (24) is given in [3] and [10], and Landau [10] writes

$$\phi(\lambda) \sim \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left[(\lambda+1) + 2 \exp(-\lambda-1) \right] \right\} \quad (\lambda \rightarrow \infty) . \quad (27)$$

In addition, Börsch-Supan [3] derives some bounds on $\phi(\lambda)$, as $\lambda \rightarrow -\infty$.

4.2 $\lambda \rightarrow \infty$

This case is somewhat more complicated. Börsch-Supan [3] introduces the substitutions

$$\lambda = \omega + \ln \omega + B, \quad u = t/\omega \quad (28)$$

in the integral (16), giving

$$\phi(\lambda) = \frac{1}{\pi\omega} \int_0^\infty e^{-t} e^{-t(\ln t + B)/\omega} \sin \frac{\pi t}{\omega} dt . \quad (29)$$

The second exponential factor in this integral is then expanded as a power series, leading to an asymptotic formula of the form

$$\phi(\lambda) = \frac{1}{\omega^2 + \pi^2} + \frac{\alpha}{\omega^3} + O(\omega^{-4}) . \quad (30)$$

In order to get a better error term, the parameter B is chosen in [3] such as to give $\alpha = 0$ in (30). Since here we are interested in a general asymptotic expansion in terms of $1/\omega$ we set $B = 0$ and expand both $\exp[-t \ln t/\omega]$ and $\sin(\pi t/\omega)$ as power series. Multiplying the two series leads to

$$\begin{aligned} \phi(\lambda) &= \sum_{n=2}^{\infty} \left(\sum_{j,k} c_{j,k} \int_0^\infty e^{-t} t^j \ln^k t dt \right) \omega^{-n} \\ &= \int_0^\infty e^{-t} \left(\frac{t}{\omega^2} - \frac{t^2 \ln t}{\omega^3} + \frac{1}{6} \frac{t^3 (3 \ln^2 t - \pi^2)}{\omega^4} + \dots \right) dt . \end{aligned} \quad (31)$$

A general closed expression for the integral

$$R_{j,k} = \int_0^\infty e^{-\mu t} t^j \ln^k t dt , \quad (32)$$

together with its explicit form for $0 \leq j \leq 5$, $0 \leq k \leq 5$, is given in [9]. For $\mu = 1$, this expression reads

$$R_{j,k} = (-1)^j k! \sum_{m=0}^{\min(j,k)} (-1)^m S_{j+1}^{(m+1)} b_{k-m}, \quad (33)$$

where $S_{j+1}^{(m+1)}$ are the Stirling numbers of the first kind [1, No. 24.1.3]. The constants b_n are defined by the power series expansion of the gamma function [5, No. 8.3211]

$$\Gamma(1+x) = \sum_{n=0}^{\infty} b_n x^n \quad (|x| \leq 1) \quad (34)$$

i.e. $b_0 = 1$,

$$b_n = \frac{1}{n} \sum_{k=1}^n (-1)^k \tilde{\zeta}(k) b_{n-k}, \quad (35)$$

where $\tilde{\zeta}(1) = \gamma$, $\tilde{\zeta}(k) = \zeta(k)$ for $k > 1$, γ is again Euler's constant, and $\zeta(k)$ is the Riemann zeta function for integer argument. Substituting (33) in (31) and using the expression [5, No. 9.5421]

$$\zeta(2k) = \frac{2^{2k-1} \pi^{2k} |B_{2k}|}{(2k)!} \quad (36)$$

where the B_{2k} are the Bernoulli numbers, we find with the help of REDUCE the asymptotic series

$$\phi(\lambda) = \frac{1}{\omega^2} \sum_{n=0}^{\infty} a_n^+ \omega^{-n} \quad (\lambda \rightarrow \infty, \omega \rightarrow \infty), \quad (37)$$

where $\omega(\lambda)$ is the (unique) solution of the equation

$$\omega + \ln \omega - \lambda = 0 \quad (38)$$

and

$$a_0^+ = 1$$

$$a_1^+ = 2\gamma - 3$$

$$a_2^+ = 3\gamma^2 - 11\gamma - \frac{\pi^2}{2} + 6$$

$$\begin{aligned}
 a_3^+ &= 4\gamma^3 - 25\gamma^2 - (2\pi^2 - 35)\gamma + \frac{25}{6}\pi^2 - 10 \\
 a_4^+ &= 5\gamma^4 - \frac{137}{3}\gamma^3 - 5\left(\pi^2 - \frac{45}{2}\right)\gamma^2 + \left(\frac{137}{6}\pi^2 - 85\right)\gamma \\
 &\quad + \frac{\pi^4}{12} - \frac{75}{4}\pi^2 + 15 \\
 a_5^+ &= 6\gamma^5 - \frac{147}{2}\gamma^4 - 2\left(5\pi^2 - \frac{406}{3}\right)\gamma^3 + \frac{147}{2}(\pi^2 - 5)\gamma^2 \\
 &\quad + \left(\frac{\pi^4}{2} - \frac{406}{3}\pi^2 + 175\right)\gamma - \frac{49}{40}\pi^4 + \frac{245}{4}\pi^2 - 21 \\
 a_6^+ &= 7\gamma^6 - \frac{1089}{10}\gamma^5 - \frac{7}{2}\left(5\pi^2 - \frac{469}{3}\right)\gamma^4 + \frac{1}{2}\left(363\pi^2 - \frac{6769}{3}\right)\gamma^3 \\
 &\quad + 7\left(\frac{\pi^4}{4} - \frac{469}{6}\pi^2 + 140\right)\gamma^2 - \left(\frac{363}{40}\pi^4 - \frac{6769}{12}\pi^2 + 322\right)\gamma \\
 &\quad - \frac{5}{24}\pi^6 + \frac{3283}{360}\pi^4 - \frac{490}{3}\pi^2 + 28 .
 \end{aligned} \tag{39}$$

Equation (38) can be solved straightforwardly by Newton iteration, using the starting value ω_0 given in [3]:

$$\begin{aligned}
 \omega_{k+1} &= \omega_k - \omega_k \frac{\omega_k + \ln \omega_k - \lambda}{\omega_k + 1} \\
 \omega_0 &= \lambda - \frac{\lambda \ln \lambda}{\lambda + 1} .
 \end{aligned} \tag{40}$$

The function $\omega(\lambda)/\lambda$ is plotted in Fig. 3 and decimal values of a_n^- and a_n^+ are given in Table 1.

5. ASYMPTOTIC EXPANSIONS FOR THE DISTRIBUTION FUNCTION $\phi(\lambda)$

5.1 $\lambda \rightarrow -\infty$

From the expansion (24) for $\phi(\lambda)$, we obtain by integration

$$\phi(\lambda) = \int_{-\infty}^{\lambda} \phi(\lambda) d\lambda = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n^- \int_{-\infty}^{\lambda} e^{-\sigma} \sigma^{n+\frac{1}{2}} d\lambda . \tag{41}$$

Since $\lambda = -1 - \ln \sigma$, we have $d\lambda = -d\sigma/\sigma$, and hence

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n^- \int_{\sigma}^{\infty} e^{-\sigma} \sigma^{-n-\frac{1}{2}} d\sigma = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n^- \Gamma(-n+\frac{1}{2}, \sigma) , \tag{42}$$

where $\Gamma(\alpha, x)$ is the complementary incomplete gamma function, whose asymptotic expansion [1, No. 6.5.32]

$$\Gamma(\alpha, x) = x^{\alpha-1} e^{-x} \left[1 + \frac{\alpha-1}{x} + \frac{(\alpha-1)(\alpha-2)}{x^2} + \dots \right] \quad (x \rightarrow \infty) \quad (43)$$

enables us to determine the coefficients in

$$\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\sigma}}{\sqrt{\sigma}} \sum_{n=0}^{\infty} A_n^- \sigma^{-n} \quad (\lambda \rightarrow -\infty, \sigma \rightarrow \infty) . \quad (44)$$

Again with REDUCE, we find from (42) and (43)

$$\begin{aligned} A_0^- &= 1 \\ A_1^- &= -11/24 \\ A_2^- &= 769/1152 \\ A_3^- &= -680863/414720 \\ A_4^- &= 226287557/39813120 \\ A_5^- &= -169709463197/6688604160 \\ A_6^- &= 667874164916771/4815794995200 \\ A_7^- &= -103663334225097487/115579079884800 \end{aligned} \quad (45)$$

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For $n \rightarrow \infty$, it seems that

$$\frac{A_{n+1}^-}{A_n^-} = O(n) . \quad (46)$$

5.2 $\lambda \rightarrow \infty$

In order to find an expansion for $\Phi(\lambda)$ as $\lambda \rightarrow \infty$, we write

$$\Phi(\lambda) = \int_{-\infty}^{\infty} \phi(\lambda) d\lambda - \int_{\lambda}^{\infty} \phi(\lambda) d\lambda = 1 - \int_{\lambda}^{\infty} \phi(\lambda) d\lambda .^{*}) \quad (47)$$

*) This follows from the fact that $\phi(\lambda)$ is a pdf.

Using (37) and $d\lambda = (1 + 1/\omega) d\omega$, we see that

$$\begin{aligned}\Phi(\lambda) &= 1 - \int_{\omega}^{\infty} \frac{1}{\omega^2} \left(1 + \frac{1}{\omega}\right) \left(\sum_{n=0}^{\infty} a_n^+ \omega^{-n}\right) d\omega \\ &= 1 - \sum_{n=1}^{\infty} A_n^+ \omega^{-n} \quad (\lambda \rightarrow \infty, \omega \rightarrow \infty),\end{aligned}\tag{48}$$

where

$$\begin{aligned}A_1^+ &= 1 \\ A_2^+ &= \gamma - 1 \\ A_3^+ &= \gamma^2 - 3\gamma - \frac{\pi^2}{6} + 1 \\ A_4^+ &= \gamma^3 - \frac{11}{2} \gamma^2 - \left(\frac{\pi^2}{2} - 6\right)\gamma + \frac{11}{12} \pi^2 - 1 \\ A_5^+ &= \gamma^4 - \frac{25}{3} \gamma^3 - \left(\pi^2 - \frac{35}{2}\right)\gamma^2 + 5\left(\frac{5}{6} \pi^2 - 2\right)\gamma + \frac{\pi^4}{60} - \frac{35}{12} \pi^2 + 1 \\ A_6^+ &= \gamma^5 - \frac{137}{12} \gamma^4 - 5\left(\frac{\pi^2}{3} - \frac{15}{2}\right)\gamma^3 + \frac{1}{2}\left(\frac{137}{6} \pi^2 - 85\right)\gamma^2 \\ &\quad + \left(\frac{\pi^4}{12} - \frac{75}{4} \pi^2 + 15\right)\gamma - \frac{137}{720} \pi^4 + \frac{85}{12} \pi^2 - 1 \\ &\dots\end{aligned}\tag{49}$$

Landau [10] considers

$$1 - \Phi(\lambda) = \int_{\lambda}^{\infty} \Phi(\lambda) d\lambda = \int_0^{\infty} e^{-u \ln u - \lambda u} \frac{\sin \pi u}{\pi u} du.\tag{50}$$

By using the substitution (28) and choosing $B = \gamma - 1$, he obtains the coefficient A_1^+ in (48). Decimal values of A_n^- and A_n^+ are given in Table 2.

Figure 4 shows the function

$$-\log_{10} |f(\lambda) - f_N(\lambda)|$$

for $f = \omega^2 \phi$ and $f = \Phi$ ($\lambda > 10$), where N is the number of terms retained in the asymptotic series (37) and (48), respectively. This function is approximately equal to the number of correct digits in the approximation.

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Table 1

n	a_n^-			a_n^+		
0	1.00000	00000	00000E+00	1.00000	00000	00000E+00
1	4.16666	66666	66667E-02	-1.84556	86701	96934E+00
2	-1.99652	77777	77778E-02	-4.28464	07430	38385E+00
3	2.70953	89660	49383E-02	3.23719	33381	47130E+01
4	-6.23515	81589	18467E-02	-6.81085	21405	83044E+01
5	2.03918	46525	41914E-01	-3.31185	46603	71593E+01
6	-8.67003	13188	38562E-01	3.67825	81601	72458E+02
7	4.54265	16843	88753E+00			

Table 2

n	A_n^-			A_n^+		
0	1.00000	00000	00000E+00			
1	-4.58333	33333	33333E-01	1.00000	00000	00000E+00
2	6.67534	72222	22222E-01	-4.22784	33509	84671E-01
3	-1.64174	14158	95062E+00	-2.04340	31377	45106E+00
4	5.68374	33740	43531E+00	7.02182	31596	08230E+00
5	-2.53729	26717	94170E+01	-7.14731	76048	71827E+00
6	1.38684	09381	67955E+02	-1.68711	78001	59106E+01
7	-8.96903	95812	47819E+02

Fig. 1

Fig. 2

Fig. 3

Fig. 4