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FINITE-MODE REGULARIZATION OF THE FERMION FUNCTIONAL INTEGRAL

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A B S T R A C T

We present a regularization of the fermion functional integral in the presence of a gauge field, obtained by projecting the fermion fields on a finite subspace of the fermion wave functions. We find the basis that ensures manifest gauge covariance. We are able, in this way, to develop a non-perturbative method of calculation and test it by calculating the triangle anomalies in different models. All the calculations are carried out on a finite-size momentum space lattice. Finally we briefly discuss the zero-mode problem.

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$$[V_\mu, A_\nu] = [V_\nu, A_\mu]$$

1. INTRODUCTION

The importance of non-perturbative approaches to quantum field theory need not be stressed. They have received increasing attention in recent years, especially in the wake of the successes of the lattice approach to gauge field theories [1]. The latter has emerged as a powerful tool for both perturbative and non-perturbative (particularly numerical) methods of calculation, and therefore for descriptions of such phenomena as mass generation, phase transitions, and confinement.

The lattice version of a gauge theory is not simply a regularization scheme in the ordinary perturbative sense, in that it defines the theory in any regime [2]. In this sense, however, it is not flawless. The fermion doubling, in particular, leads to fundamental difficulties when Weyl fermions are involved [3] and requires an *ad hoc* modification in order to reproduce the axial anomalies met in the usual perturbative approach.

In the present paper we begin to explore a different regularization scheme, while keeping as a constant reference the lattice approach. Here we limit ourselves to studying the fermionic sector of a gauge field theory, although the same treatment can be applied separately to other sectors and, hopefully, to the whole theory.

Let us outline briefly our approach. The Euclidean functional integral for Dirac fermions in the presence of an external gauge field $V_\mu(x)$ is

$$Z = \int \bar{\psi} \psi \exp \left(- \int_B d^4x \bar{\psi}(x) \not{D} \psi(x) \right) \quad (1.1)$$

where $\not{D} = -i \gamma_\mu (\partial^\mu + V^\mu)$ is the Euclidean Dirac operator -- for other notations see Section 2. Formally Z is equal to $\det \not{D}$. There are several methods of regularizing this formal determinant: the ζ -function regularization [4], the point splitting method [5], and others [6], beside, of course, the lattice regularization. In order to keep as close as possible to the spirit of the functional integration, we choose a regularization based on a restriction of the functional space of the fermion wave functions. In other words, we single out in it an N -dimensional subspace and define the regularized functional integral Z_N by:

$$Z_N = \det (1 - P_N + P_N \not{D} P_N) \quad (1.2)$$

where P_N is the projector onto the chosen subspace.

If we want manifest gauge invariance to be preserved, the simplest possibility is that we choose the subspace spanned by N eigenvectors of $\not{D}$, which can be done if B , the Euclidean four-volumes, is finite. In this case Z_N reduces to

$$Z_N = \prod_{i=1}^N \lambda_i \quad (1.3)$$

where λ_i are the corresponding eigenvalues of $\not{D}$, ordered in some definite way. In this case the definitions (1.2) and (1.3) are equivalent, but using (1.2) instead of (1.3) will prove very advantageous. For we do not know how to compute λ_i , apart from some specific cases in two dimensions [6], while we are able to produce an explicit functional expression for P_N in terms of $\not{D}$ [see below eq. (2.9)], which renders eq. (1.2) a useful tool for non-perturbative calculations -- the perturbative approach in this regularization scheme will not be dealt with here.

Our attitude is to take eq. (1.2) and the variants of it as a definition of the theory, as far as fermions are concerned. As we know from the example of the lattice approach, a regularized Euclidean field theory does not necessarily give the (analytically continued) results of the Minkowski perturbative version of the theory which it is heuristically derived from. This happens, in the lattice approach, when anomalies [7] are involved. Therefore our main task in this paper will be to test our definition (1.2) for several different theories, versus the appearance and form of the triangle anomalies [8] expected on the basis of the corresponding perturbative Minkowski version of the theories. As we shall see, for some theories the right definition of the regularized functional integral is not obvious *a priori*.

The regularization (1.3) was first presented in an earlier paper [9]. However, significant changes have been made with respect to that first attempt. Not only was the case treated there a very particular one and the problems studied there are solved here, but the method of calculation is different. Throughout the paper we

will stick to a box of finite size in order to define the mathematical objects involved, which amounts to using a finite-size momentum space lattice. The calculations will be carried out without taking a continuum limit (= infinite volume limit) either before the calculations, as was done in ref. [9], or after. Removing the cutoff will be enough to provide a definite expression for the anomalies.

This paper is organized as follows. In section 2 we rederive in a more complete way the argument, first presented in ref. [9], leading to eqs. (1.2) and (1.3) and study some properties of the projection P_N . In section 3 we derive the axial anomaly for vector gauge theories. In section 4 we study the more complex case of Dirac fermions interacting with a mixed vector and axial vector potential. We discuss at length the choice of the right basis, because we think this illustrates very well the problems connected with our approach. Finally, we derive in a non-perturbative way the triangle anomaly found some years ago by Bardeen [10]. In section 5 a brief discussion is given concerning the zero modes of the Dirac operator in the context of our method.

2. THE REGULARIZATION SCHEME

The Euclidean Dirac operator in the presence of a (vector) gauge field $V_\mu(x)$, introduced in eq. (1.1), is

$$\not{D} \equiv -i \gamma_\mu (\partial^\mu + V^\mu(x)) = \not{D}^\dagger \quad (2.1)$$

where

$$V_\mu(x) = T^a V_\mu^a(x) \quad , \quad \gamma_\mu = \gamma_\mu^\dagger$$

$\{T^a | a = 1, \dots, A\}$ are the antihermitian generators, in a given representation R , of the gauge group. We shall consider the case of massive fermions, so that $\not{D}$ in eq. (1.1) will be replaced by $\not{D} - im$. The functional integral is then formally invariant under the following gauge transformations:

$$\begin{aligned}
 V_\mu &\rightarrow \Omega^{-1} V_\mu \Omega + \Omega^{-1} \partial_\mu \Omega \\
 \psi &\rightarrow \Omega^{-1} \psi \\
 \bar{\psi} &\rightarrow \bar{\psi} \Omega
 \end{aligned}
 \tag{2.2}$$

where Ω takes its values in the representation R. The latter remark assumes that we consider the measure $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ in eq. (1.1) invariant under (2.2).

We want to give eq. (1.1) a definite meaning in such a way that the invariance under (2.2) be preserved. To this end let us introduce [11] an arbitrary complete orthonormal basis $\{\psi_n(x) | n = 1, 2, \dots\}$ in the space of L_2 -integrable spinors, expand ψ and $\bar{\psi}$ into this basis, and cut the summation after the first N basis vectors. This amounts to using the approximations:

$$\begin{aligned}
 \psi(x) &\rightarrow \sum_{n=1}^N a_n \psi_n(x) \equiv \psi^{(N)}(x) \\
 \bar{\psi}(x) &\rightarrow \sum_{n=1}^N \psi_n^\dagger(x) b_n \equiv \bar{\psi}^{(N)}(x)
 \end{aligned}$$

where $\{a_n, b_k\} = \{a_n, a_k\} = \{b_n, b_k\} = 0$ for any n, k . The functional integral (1.1) becomes:

$$\begin{aligned}
 Z_N(\mathcal{D}) &\equiv \int \prod_{n=1}^N db_n da_n \exp \left(- \sum_{n,k=1}^N b_n \langle \psi_n | \mathcal{D} - i m | \psi_k \rangle a_k \right) = \\
 &= \det [(\mathcal{D} - i m)_N]
 \end{aligned}
 \tag{2.3}$$

after integration à la Berezin. In the last step we have used the new notation:

$$(\mathcal{D} - i m)_N \equiv \underline{1 - P_N} + P_N (\mathcal{D} - i m) P_N, \quad P_N \equiv \sum_{n=1}^N |\psi_n\rangle \langle \psi_n|. \tag{2.4}$$

Equation (2.3) is equal to the determinant of the finite matrix $\langle \psi_n | \mathcal{D} - i m | \psi_k \rangle$, with $1 \leq n, k \leq N$; however, keeping the notation of eqs. (2.3) and (2.4) will prove very useful^{*)}.

*) It allows us to operate directly with P_N instead of using the matrix elements, as was done to derive all the results of ref. [9]. We think eq. (2.3) is much clearer than the notation $\det (P_N \mathcal{D} P_N)$ used in that paper. We are indebted to M. Testa for suggesting it.

Consider the effect of an infinitesimal gauge transformation $\Omega(x) \approx 1 + \omega(x)$, $\omega = \omega^a T^a$, upon $\det[(\not{D} - im)_N]$. Since the latter is a functional of V_μ , we can write the variation in the following way:

$$\begin{aligned} \det[(\not{D} - im)_N] &\longrightarrow \det[(\not{D} - im)_N] + \int d^4x [\tilde{D}_\mu \omega]^a(x) \frac{\delta}{\delta V_\mu^a(x)} \det[(\not{D} - im)_N] \approx \\ &\approx \det[(\not{D} - im)_N] \cdot \left\{ 1 + \left[\text{Tr} \left(\frac{1}{(\not{D} - im)_N} P_N [\not{D}, \omega] P_N + \right. \right. \right. \\ &\quad \left. \left. + \int d^4x (\tilde{D}_\mu \omega)^a(x) \text{Tr} \left(\left(\frac{\delta P_N}{\delta V_\mu^a(x)} (\not{D} - im) P_N + P_N (\not{D} - im) \frac{\delta P_N}{\delta V_\mu^a(x)} \right) \frac{1}{(\not{D} - im)_N} \right) \right] \right\} \end{aligned} \quad (2.5)$$

where $\tilde{D}^\mu \equiv \partial^\mu + [V^\mu, \]$ and summation over repeated indices is understood. The symbol Tr stands for the trace with respect to the overall operator structure. In the course of the derivation the usual equivalence $\exp \text{Tr} \ln = \det$ has been used, together with the identity

$$P P' P = 0 \quad (2.6)$$

which holds for any projector P if P' is the derivative of P with respect to some parameter. Equation (2.6) can easily be derived by taking the derivative of the equation, $P^2 = P$.

As was pointed out in ref. [9], one can immediately recognize two simple solutions to the vanishing of the term inside the brackets in eq. (2.5). The first is $\delta P_N / \delta V_\mu^a = 0$ and $[\omega, P_N] = 0$ and leads to the lattice regularization. The second possibility is

$$[\not{D}, P_N] = 0 \quad (2.7)$$

That the aforementioned term vanishes can be verified by using the identity (2.6) and

$$(1 - P_N + P_N (\not{D} - im) P_N)^{-1} = 1 - P_N + \frac{P_N}{\not{D} - im} \quad (2.8)$$

which is valid if eq. (2.7) holds.

The condition (2.7) means that the basis vectors $\{|\psi_\mu\rangle\}$ are eigenvectors of $\mathcal{D}$:

$$\mathcal{D} |\psi_n\rangle = \lambda_n |\psi_n\rangle$$

and this requires the Euclidean space-time to be compact. From now on the regularized functional integral Z_N will be defined by eq. (2.3), where P_N is the projector onto the eigenspace corresponding to the first N eigenvalues of $\mathcal{D}$. To be precise, this definition is still short of two specifications. The first consists in excluding the presence of zero modes in the spectrum of $\mathcal{D}$ (in this respect see section 5). The second concerns the order of the eigenvalues: we choose an order of increasing absolute value, but, since the spectrum of $\mathcal{D}$ is symmetric with respect to zero, we understand N to be even^(†).

Now the functional integral defining the theory is given by the limit $\lim Z_N$, taken after all infinities have been subtracted from $\ln Z_N$. Our procedure guarantees that the results obtained are gauge covariant.

We can give an explicit functional formula for P_N . Let us consider the operator

$$P_\Lambda \equiv \theta\left(1 - \frac{\mathcal{D}^2}{\Lambda^2}\right) = \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi i(\alpha - i\varepsilon)} \exp\left[i\alpha\left(1 - \frac{\mathcal{D}^2}{\Lambda^2}\right)\right]. \quad (2.9)$$

If we apply it to the completeness $I = \sum_{n=1}^{\infty} |\psi_n\rangle \langle \psi_n|$ formed by the eigenstates of $\mathcal{D}$ and integrate over α , we get exactly eq. (2.4) provided that $|\lambda_N| < \Lambda < |\lambda_{N+2}|$. With the latter proviso P_Λ and P_N (Λ and N) will be used interchangeably in the following.

For later applications the following formula will be very useful:

$$\begin{aligned} \frac{\delta P_\Lambda(V)}{\delta V_\mu^a(x)} &= -\frac{1}{\Lambda^2} \int_{-\infty}^{+\infty} \frac{d\alpha e^{i\alpha}}{2\pi i(\alpha - i\varepsilon)} \int_0^\alpha d\tau \exp\left[-i(\alpha - \tau) \frac{\mathcal{D}^2}{\Lambda^2}\right] \\ &\quad \cdot \left\{ \mathcal{D} |x\rangle \gamma_\mu^T \langle x| + |x\rangle \gamma_\mu^T \langle x| \mathcal{D} \right\} \exp\left[-i\tau \frac{\mathcal{D}^2}{\Lambda^2}\right]. \end{aligned} \quad (2.10)$$

(†) We should also take into account the case of degenerate eigenvalues, but this will be automatically taken care of by the Λ prescription (see below).

where $|x\rangle$ is the eigenstate (properly speaking, the eigenfunctional) of the position operator corresponding to the eigenvalue x , the argument of V_μ^a . Equation (2.10) can be derived either by understanding the exponential in (2.9) as a path-ordered exponential

$$\frac{\delta}{\delta V_\mu^a(x)} \mathbb{P} \exp \left[-i \int_0^\alpha d\tau \frac{D^2}{\Lambda^2} \right] = \int_0^\alpha d\tau \mathbb{P} \exp \left[-i \int_\tau^\alpha \frac{D^2}{\Lambda^2} d\tau' \right] \cdot \left(-i \frac{\delta}{\delta V_\mu^a(x)} \frac{D^2}{\Lambda^2} \right) \cdot \mathbb{P} \exp \left[-i \int_0^\tau \frac{D^2}{\Lambda^2} d\tau' \right]$$

or by expanding the same exponential in a power series, as is done in Appendix A.

Using eqs. (2.7) and (2.10) we can easily prove that under an infinitesimal gauge transformation:

$$P_\lambda \rightarrow P_\lambda + \int d^4x \left(\tilde{D}^\mu \omega \right)_{(x)}^a \frac{\delta P_\lambda}{\delta V_\mu^a(x)} = P_\lambda + [P_\lambda, \omega]$$

This immediately implies the gauge invariance of Z_N .

3. WARD IDENTITIES IN A VECTOR GAUGE THEORY

According to the usual procedure, we could now introduce external sources for the fermion fields. However, for the Ward identities (WIs) we are going to derive this is not necessary. For example, we expect a conserved vector current to correspond to the gauge invariance of Z_N , as defined in the previous section. This is easily derived in the following way:

$$Z_N(V) = Z_N(V + \tilde{D}\omega) \simeq Z_N(V) + \int d^4x \left(\tilde{D}_\mu \omega \right)_{(x)}^a \frac{\delta Z_N(V)}{\delta V_\mu^a(x)}, \quad (3.1)$$

from which, after integration by parts, it follows that

$$\tilde{D}_\mu^{ac} \frac{\delta Z_N}{\delta V_\mu^c} \equiv \left(\partial_\mu \delta^{ac} + f^{abc} V_\mu^b \right) \frac{\delta Z_N(V)}{\delta V_\mu^c(x)} = 0 \quad (3.2)$$

Therefore

$$j_{\mu}^{a(N)}(x) \equiv -\frac{1}{Z_N} \frac{\delta Z_N}{\delta V_{\mu}^a(x)} = i \text{Tr} \left(\gamma_{\mu} T^a \langle x | \frac{1}{(\not{D} - im)_N} | x \rangle \right) \quad (3.3)$$

is the expected regularized conserved vector current. Here Tr refers to the trace with respect to the spinor and internal matrix structure. The conservation can also be proved by explicitly applying the covariant derivative $\tilde{D}_{\mu}$ to the r.h.s. of eq. (3.3) and using eq. (2.7).

The functional integral (2.1) is also formally invariant under the following chiral transformations:

$$\begin{aligned} \chi &\rightarrow \chi + \gamma_5 \tilde{\mathcal{D}} \omega \\ \psi &\rightarrow (1 + \omega \gamma_5) \psi \\ \bar{\psi} &\rightarrow \bar{\psi} (1 + \omega \gamma_5) \end{aligned} \quad (3.4)$$

Unlike the previous case, here we do not expect a conserved current but, rather, an anomaly to appear, because, as was pointed out by Fujikawa [12], the functional measure $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ is not invariant under (3.4). In our formalism this manifests itself in the following way. Under (3.4)

$$\begin{aligned} Z_N(\chi) &\rightarrow Z_N(\chi + \gamma_5 \tilde{\mathcal{D}} \omega) \simeq Z_N(\chi) + \int d^4x \left(\gamma_5 \gamma_{\mu} \right)_{ij} \left(\tilde{\mathcal{D}}^{\mu} \omega \right)^a_{(x)} \frac{\delta Z_N(\chi)}{\delta \chi_{ij}^a(x)} \simeq \\ &\simeq Z_N(\chi) - \int d^4x \omega^a(x) \tilde{\mathcal{D}}_{\mu}^{ac} \left(\left(\gamma_5 \gamma^{\mu} \right)_{ij} \frac{\delta Z_N(\chi)}{\delta \chi_{ij}^c(x)} \right). \end{aligned} \quad (3.5)$$

On the other hand,

$$Z_N(\chi + \gamma_5 \tilde{\mathcal{D}} \omega) \simeq Z_N(\chi) + \Delta Z_N(\omega, \chi). \quad (3.6)$$

Therefore, if we define

$$j_{\mu 5}^{a(N)} \equiv -\frac{1}{Z_N} \left(\gamma_5 \gamma_{\mu} \right)_{ij} \frac{\delta Z_N(\chi)}{\delta \chi_{ij}^a(x)} \quad (3.7)$$

the axial WI turns out to be

$$\left(\tilde{\mathcal{D}}^{\mu} j_{\mu 5}^{a(N)} \right)^a_{(x)} = \frac{1}{Z_N} \frac{\delta}{\delta \omega^a(x)} \Delta Z_N(\omega, \chi) \equiv \mathcal{A}^{a(N)}(x) \quad (3.8)$$

In eqs. (3.5), (3.6), and (3.7) we introduced conventions that require some comment. First of all i, j denote γ -matrix elements, and summation over repeated indices is understood. Not all the matrix elements γ_{ij}^a are linearly independent; therefore $\delta/\delta\gamma_{ij}^a(x)$ applied to a functional $F(\gamma)$ is shorthand for $\delta/\delta\bar{\gamma}_{ij}^a(x)$, where $\bar{\gamma}^a(x)$ is a generic 4×4 matrix. This derivative is meant to be applied to $\bar{F}(\bar{\gamma})$, $\bar{F}$ having the same functional dependence on $\bar{\gamma}^a$ as F on γ^a . After derivation the resulting functional is evaluated at $\bar{\gamma}^a = \gamma^a$:

$$\left. \frac{\delta F(\gamma)}{\delta \gamma_{ij}^a(x)} \right|_{\bar{\gamma}^a = \gamma^a} = \left. \frac{\delta \bar{F}(\bar{\gamma})}{\delta \bar{\gamma}_{ij}^a(x)} \right|_{\bar{\gamma}^a = \gamma^a} \quad (3.9)$$

Next, we have to evaluate $\Delta Z_N(\omega, \gamma)$. We can do it easily working out explicitly the integral in eq. (3.5). Then

$$\begin{aligned} \mathcal{A}^{a(N)}(x) &= \mathcal{B}^{a(N)}(x) + \mathcal{A}_1^{a(N)}(x) + \mathcal{A}_2^{a(N)}(x) \\ \mathcal{B}^{a(N)}(x) &= -2im \text{Tr} \left(\gamma_5 T^a \langle x | \frac{1}{(D-im)_N} | x \rangle \right) \equiv 2m j_5^{(N)} \\ \mathcal{A}_1^{a(N)}(x) &= -2 \text{Tr} \left(\gamma_5 T^a \langle x | P_N | x \rangle \right) \\ \mathcal{A}_2^{a(N)}(x) &= 2 \tilde{D}_\gamma^{ac} (\gamma_5 \gamma^\mu)_{ij} \text{Tr} \left(P_N \frac{\delta P_N}{\delta \gamma_{ij}^c} P_N - P_N \frac{\delta P_N}{\delta \gamma_{ij}^c} \frac{P_N}{D-im} \right) \end{aligned} \quad (3.10)$$

It would seem that $\mathcal{A}_2^{a(N)}$ should vanish, owing to eq. (2.6). However, one should remember that $\bar{P}_N(\bar{\gamma})$ is not a projector, in general. Therefore we are not allowed to draw the above conclusion. However, we can prove explicitly that $\mathcal{A}_2^a \equiv \lim_{N \rightarrow \infty} \mathcal{A}_2^{a(N)} = 0$ (see Appendix C). The $\mathcal{B}^{a(N)}$ term is the usual mass-term contribution to $\tilde{D}_{\mu 5}^{ja}$. So we are left with $\mathcal{A}_1^a(x)$, which is in fact different from zero. It is worth while noticing that the anomaly appears when, in order to comply with the regularization prescription, instead of taking the operator limit $P_N \xrightarrow{N \rightarrow \infty} I \equiv \text{identity}$ (which would give a vanishing anomaly), we take the weak limit $\lim_{N \rightarrow \infty} \langle x | P_N | x \rangle$.

In order to calculate explicitly such an expression as $\langle x | P_N | x \rangle$ we must expand it using a complete set of eigenstates of the momentum operator:

$$\langle x | P_N | x \rangle = \sum_{\{k_n\}} \langle x | P_N | k_n \rangle \langle k_n | x \rangle \quad (3.11)$$

We easily find such a set if the theory is defined in a box of size $2L$ symmetric with respect to the origin of the coordinates -- the four sizes could be different from one another, but, for the sake of simplicity, we take them equal. Requiring antiperiodic boundary conditions for the eigenfunctions of the momentum operator $-i \partial_\mu$ (in order to avoid infrared problems), we find the eigenvalues

$$k_n = \left(n_n + \frac{1}{2}\right) \frac{\pi}{L}, \quad n_n = 0, \pm 1, \pm 2, \dots$$

and the corresponding eigenfunctions

$$\frac{1}{4L^2} e^{i k_n x} \quad (3.12)$$

Inserting (2.9) into $\langle x | P_N | x \rangle$, then expanding it as in eq. (3.11) and using the basis (3.12), we find

$$\mathcal{A}_1^{a(N)}(x) = \frac{2}{(4L^2)^2} \sum_{\{k_n\}} \int \frac{d\alpha e^{i\alpha}}{2\pi i(\alpha - i\varepsilon)} \text{Tr} \left(\gamma_5 T^a \exp \left[-i\alpha \left(\frac{\not{x} + \not{\mathcal{D}}_x}{\Lambda} \right)^2 \right] \right) \quad (3.13)$$

where $\not{\mathcal{D}}_x$ is the operator $\not{D}$ acting on the space of square-integrable spinors ("evaluated" at the point x , so to speak). Next, we expand the exponential in eq. (3.13) into an asymptotic series in $1/\Lambda^2$. In order to emphasize the power counting in $1/\Lambda$ we also introduce $\tilde{L} = L\Lambda$. Using the properties of γ -traces we easily find:

$$\mathcal{A}_1^{a(N)}(x) = \frac{2\Lambda^4}{16\tilde{L}^4} \int \frac{d\alpha e^{i\alpha}}{2\pi i(\alpha - i\varepsilon)} \sum_{\{\tilde{k}_n\}} e^{-i\alpha \tilde{k}^2} \left(\frac{\alpha^2}{2\Lambda^4} \right) \text{Tr} \left(T^a \gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho V^{\mu\nu} V^{\lambda\rho} \right). \quad (3.14)$$

where $V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu]$, and the following expansion has been used:

$$\left(\tilde{k} + \frac{\partial}{\Lambda}\right)^2 = \tilde{k}^2 - 2i \tilde{k}_\mu \frac{D^\mu}{\Lambda} - \frac{D_\mu D^\mu}{\Lambda^2} - \frac{1}{2} \gamma_\mu \gamma_\nu \frac{V^{\mu\nu}}{\Lambda^2} \quad (3.15)$$

where $\tilde{k} = (n_\mu + \frac{1}{2}) \pi/\tilde{L}$, $D_\mu = \partial_\mu + V_\mu$.

The regularization prescription implies that we first carry out the sum over k_μ and then integrate over α . The summation over k_μ involves such series as

$$\sum_{n=-\infty}^{+\infty} \exp\left[-(n+\frac{1}{2})^2 z\right] \equiv \theta_2(0, z). \quad (3.16)$$

This is the well-known 2nd theta function [13]. From its asymptotic behaviour for small z (see Appendix B) we learn that

$$\prod_{\mu=1}^4 \sum_{n_\mu=-\infty}^{+\infty} \exp\left[-i\alpha \left(n_\mu + \frac{1}{2}\right)^2 \frac{\pi^2}{\tilde{L}^2}\right] \approx -\frac{\tilde{L}^4}{\alpha^2 \pi^2} + \mathcal{O}(\tilde{L}^{-\beta}) \quad (3.17)$$

for arbitrary positive β . One can convince oneself that this is the only relevant singularity in the limit $\Lambda \rightarrow \infty$. Thus, from eq. (3.14), one gets:

$$\mathcal{A}_1^{a(1)}(x) = \frac{1}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{tr} \left(T^a V^{\mu\nu} V^{\lambda\rho} \right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right), \quad (3.18)$$

where tr refers only to the internal degrees of freedom. A few details of the above calculations are reported in Appendix C, while Appendix B is devoted to the theta functions.

A comment is in order before ending this section. All the calculations in this and the following sections are carried out using a finite-size momentum space lattice to define the objects involved. There is no need to take a continuum limit in order to find definite results, since removing the cut-off is enough for this purpose. From a practical point of view this means that taking the infinite volume limit (as in ref. [9]) is an admissible short cut. On the other hand, it means that passing through such a limit may not be completely necessary.

4. (V,A) THEORIES

Let us consider the functional integral for Euclidean Dirac fermions, coupled to an external vector field $V_\mu = V_\mu^a T^a$ and axial vector field $A_\mu = A_\mu^a T^a$,

$$Z(V, A) = \int \bar{\psi} \psi \exp \left[- \int \mathcal{L} d^4x \right], \quad \mathcal{L} = \bar{\psi} (\not{D} - im) \psi \quad (4.1)$$

$$\not{D} = -i (\not{\partial} + \not{V} + \gamma_5 \not{A}). \quad (4.2)$$

This type of theory was studied in connection with anomalies especially by Bardeen [10], see also ref. [14]. The approach was perturbative. Recently there has been a renewed interest in this kind of models from a non-perturbative point of view [15,16]. Applying our method to them will prove to be enlightening for the method itself.

$Z(V,A)$ is invariant under the following two groups of gauge transformations, expressed here in infinitesimal form

$$a) \begin{cases} V_\mu \rightarrow V_\mu + \tilde{D}_{V_\mu} \omega \\ A_\mu \rightarrow A_\mu + [A_\mu, \omega] \\ \psi \rightarrow (1 - \omega) \psi \\ \bar{\psi} \rightarrow \bar{\psi} (1 + \omega) \end{cases} \quad b) \begin{cases} V_\mu \rightarrow V_\mu + [A_\mu, \lambda] \\ A_\mu \rightarrow A_\mu + \tilde{D}_{V_\mu} \lambda \\ \psi \rightarrow (1 + \gamma_5 \lambda) \psi \\ \bar{\psi} \rightarrow \bar{\psi} (1 + \gamma_5 \lambda) \end{cases} \quad (4.3)$$

where $\omega = \omega^a(x) T^a$ and $\lambda = \lambda^a(x) T^a$ and $\tilde{D}_{V_\mu} = \partial_\mu + [V_\mu, \]$. Once again this invariance is simply formal, unless we provide an unambiguous definition for $Z(V,A)$. Following the method of section 2 we look for a suitable basis on which we can define our regularization. On the basis of the perturbative results we expect a conserved WI to correspond to the symmetry (4.3a) and an anomalous WI to (4.3b). Two bases seem the most natural choices to meet the first requirement: the basis of eigenvectors of $\not{D}$ and the basis of eigenvectors of $\not{D}_V = -i (\not{\partial} + \not{V})$.

Let us consider the first case. $\mathcal{D}$ is not Hermitian, therefore we have to continue analytically A_μ to $i A_\mu$. At the end of the calculations we will continue the results back. We introduce P_N as in eq. (2.4) and recall that $[P_N, \mathcal{D}] = 0$. The regularized functional integral is

$$Z_N(V, A) = \det (\mathcal{D} - i m)_N, \quad (\mathcal{D} - i m)_N = 1 - P_N + P_N (\mathcal{D} - i m) P_N \quad (4.4)$$

Let us first derive the conserved vector WI. To this end consider the effect of the gauge transformation (4.3a) on $Z_N(V, A)$:

$$\begin{aligned} Z_N(V, \tilde{\mathcal{D}}_V \omega, A + [A, \omega]) &\simeq Z_N(V, A) + \Delta Z_N(V, A, \omega) \simeq \\ &\simeq Z_N(V, A) - \int d^4x \, \omega^a(x) \tilde{\mathcal{D}}_{V\mu}^{ac} \frac{\delta Z_N(V, A)}{\delta V_\mu^c(x)} - \int d^4x \, \omega^a(x) f^{abc} A_\mu^b(x) \frac{\delta Z_N(V, A)}{\delta A_\mu^c(x)} \end{aligned} \quad (4.5)$$

where $D_{V\mu}^{ac} = \partial_\mu \delta^{ac} + f^{abc} V_\mu^b(x)$. Using $[P_N, \mathcal{D}] = 0$ it is straightforward to prove that $\Delta Z_N = 0$, from which

$$\tilde{\mathcal{D}}_V^\mu j_{\mu 5}^{(N)} + [A^\mu, j_{\mu 5}^{(N)}] = 0 \quad (4.6)$$

follows, provided that we define

$$j_{\mu 5}^{a(N)} = -\frac{1}{Z_N} \frac{\delta Z_N}{\delta V_\mu^a}, \quad j_{\mu 5}^{a(N)} = -\frac{1}{Z_N} \frac{\delta Z_N}{\delta A_\mu^a} \quad (4.7)$$

As expected, the WI corresponding to (4.3b) exhibits an anomaly. Operating as in (4.5) we get

$$\left(\tilde{\mathcal{D}}_V^\mu j_{\mu 5}^{(N)} + [A^\mu, j_{\mu 5}^{(N)}] \right)^a = -\frac{1}{Z_N} \frac{\delta}{\delta \lambda^a} \Delta Z_N(V, A, \lambda) \equiv \mathcal{B}^{a(N)} + \mathcal{A}^{a(N)} \quad (4.8)$$

where

$$\mathcal{B}^{a(N)}(x) = -2 i m \text{Tr} \left(\gamma_5 T^a \langle x | \frac{P_N}{(\mathcal{D} - i m)_N} | x \rangle \right) \equiv 2 m j_5^{(N)} \quad (4.9)$$

and

$$\begin{aligned}
 \mathcal{A}^{a(N)} &= -2 \operatorname{tr} \left(\gamma_5 T^a \langle x | P_N | x \rangle \right) \equiv \mathcal{A}^a(\Lambda^2) + \mathcal{A}_1^{a(N)} + \mathcal{A}_2^{a(N)} \\
 \mathcal{A}^a(\Lambda^2) &= \frac{4}{\pi^2} \Lambda^2 \operatorname{tr} \left(T^a \tilde{D}_V^\mu A_\mu \right) \\
 \mathcal{A}_1^{a(N)} &= \frac{1}{4\pi^2} \operatorname{tr} \left(T^a \left(\frac{2}{3} [\tilde{D}_V^\mu V_{\mu\lambda}, A^\lambda] - \frac{1}{3} [\tilde{D}_V^\mu A^\lambda, V_{\mu\lambda}] + \right. \right. \\
 &\quad \left. \left. + \frac{1}{3} \tilde{D}_V^\mu \tilde{D}_{V\mu} \tilde{D}_V^\lambda A_\lambda + \frac{2}{3} \left\{ \left\{ \tilde{D}_V^\mu A^\lambda, A_\lambda \right\}, A_\mu \right\} + \frac{1}{3} \left\{ \tilde{D}_V^\mu A_\mu, A_\lambda A^\lambda \right\} + \right. \right. \\
 &\quad \left. \left. + \frac{2}{3} A_\mu \tilde{D}_V^\lambda A_\lambda A^\mu + \frac{1}{6} [[A^\mu, A^\lambda], A_{\mu\lambda}] \right) \right) + O\left(\frac{1}{\Lambda^2}\right) \\
 \mathcal{A}_2^{a(N)} &= \frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} \operatorname{tr} \left(T^a \left(\frac{1}{4} V^{\mu\nu} V^{\lambda\rho} + \frac{1}{12} A^{\mu\nu} A^{\lambda\rho} \right. \right. \\
 &\quad \left. \left. - \frac{1}{6} V^{\mu\nu} A^\lambda A^\rho - \frac{1}{6} A^\mu A^\nu V^{\lambda\rho} - \frac{2}{3} A^\mu V^{\nu\lambda} A^\rho + \right. \right. \\
 &\quad \left. \left. - \frac{1}{3} A^\mu A^\nu A^\lambda A^\rho \right) \right) + O\left(\frac{1}{\Lambda^2}\right),
 \end{aligned} \tag{4.10}$$

with the following definitions: $V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu]$, $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] - [V^\nu, A^\mu]$ and $\tilde{D}_V^\mu = \partial^\mu + [V^\mu, \cdot]$. $\mathcal{B}^{a(N)}$ is the expected contribution proportional to the mass. $\mathcal{A}^0(\Lambda^2)$ and $\mathcal{A}_1^a = \lim_{N \rightarrow \infty} \mathcal{A}_1^{a(N)}$ can be subtracted away by adding to $\ln Z_N$ the term, invariant under (4.3a),

$$\begin{aligned}
 &-2 \frac{\Lambda^2}{\pi^2} \int d^4x \operatorname{tr} (A_\mu A^\mu) - \frac{1}{12\pi^2} \int d^4x \operatorname{tr} \left(-\frac{1}{4} V_{\mu\lambda} V^{\mu\lambda} + A_\mu A_\lambda V^{\mu\lambda} + \right. \\
 &\quad \left. + \frac{1}{2} \tilde{D}_V^\mu A_\mu \tilde{D}_V^\lambda A_\lambda + \frac{1}{2} A_\mu A_\lambda A^\mu A^\lambda - \frac{3}{2} (A_\mu A^\mu)^2 \right).
 \end{aligned} \tag{4.11}$$

Some details of the calculations are reported in Appendix D.

Finally $\mathcal{A}_2^a = \lim_{N \rightarrow \infty} \mathcal{A}_2^{a(N)}$ satisfies the Wess-Zumino consistency conditions [17] (see below). Indeed, when expressed in terms of $\tilde{V}_{\mu\nu} = V_{\mu\nu} + [A_\mu, A_\nu]$, it is exactly the result found long ago by Bardeen [10] by means of a perturbative calculation. The same expression was obtained in ref. [15] by means of a ζ -regularization, though the way it is derived still deserves some clarification, as the authors recognize.

In the $\mathcal{D}_V$ -basis the regularized functional integral is

$$Z_N^V(V, A) = \det[(\mathcal{D} - im)_N^V], \quad (\mathcal{D} - im)_N^V = 1 - P_N^V + P_N^V (\mathcal{D} - im)_N^V P_N^V \quad (4.12)$$

where P_N^V is defined as in eq. (2.4), $[P_N^V, \mathcal{D}_V] = 0$. Proceeding as in eqs. (4.4) and (4.5), and defining

$$\begin{aligned} j_r^{a(N)} &\equiv - \frac{1}{Z_N^V} \frac{\delta Z_N^V(V, A)}{\delta V_r^a(x)} = \\ &= - \mathcal{Z}_2 \left[\left(\frac{\delta P_N^V}{\delta V_r^a(x)} \mathcal{D} P_N^V + P_N^V \frac{\delta \mathcal{D}}{\delta V_r^a(x)} P_N^V + P_N^V \mathcal{D} \frac{\delta P_N^V}{\delta V_r^a(x)} \right) \frac{1}{(\mathcal{D} - im)_N^V} \right], \end{aligned} \quad (4.13a)$$

$$j_{r5}^{a(N)} \equiv - \frac{1}{Z_N^V} \frac{\delta Z_N^V(V, A)}{\delta A_r^a(x)} = - \mathcal{Z}_2 \left(P_N^V \frac{\delta \mathcal{D}}{\delta A_r^a(x)} P_N^V \frac{1}{(\mathcal{D} - im)_N^V} \right), \quad (4.13b)$$

we find

$$\widehat{\mathcal{D}}_V^r j_r^{(N)} + [A^r, j_{r5}^{(N)}] = 0 \quad (4.14a)$$

$$\left(\widehat{\mathcal{D}}_V^r j_{r5}^{(N)} + [A^r, j_r^{(N)}] \right)^a = - \frac{1}{Z_N^V} \frac{\delta}{\delta \lambda^a} \Delta Z_N^V(V, A, \lambda) \equiv \mathcal{B}^{a(N)} + \mathcal{A}^{a(N)} \quad (4.14b)$$

where

$$\begin{aligned} \mathcal{B}^{a(N)}(x) + \mathcal{A}^{a(N)}(x) &= - \text{Tr} \left(\gamma_5 T^a \langle x | \left\{ \mathcal{D}, \frac{P_N^V}{(\mathcal{D} - im)_N^V} \right\} | x \rangle - \right. \\ &\quad \left. - f^{abc} A_r^b(x) \mathcal{Z}_2 \left(\left(\frac{\delta P_N^V}{\delta V_r^c(x)} \mathcal{D} P_N^V + P_N^V \mathcal{D} \frac{\delta P_N^V}{\delta V_r^c(x)} \right) \frac{1}{(\mathcal{D} - im)_N^V} \right) \right). \end{aligned} \quad (4.15)$$

Equation (4.14a) can also be proved directly (see Appendix D). The explicit evaluation of the anomaly gives (for details see Appendix D):

$$\mathcal{B}^{a(N)} = -2im \text{Tr} \left(\gamma_5 T^a \langle x | \frac{P_N^V}{(\mathcal{D} - im)_N^V} | x \rangle \right) \equiv 2m j_5^{(N)}$$

$$\begin{aligned}
 \mathcal{A}^{a(N)} &= \mathcal{A}^e(\Lambda^2) + \mathcal{A}_1^{a(N)} + \mathcal{A}_2^{a(N)} \\
 \mathcal{A}^e(\Lambda^2) &= \frac{\Lambda^2}{8\pi^2} \text{tr} \left(T^a \tilde{D}_V^\mu A_\mu \right) \\
 \mathcal{A}_1^{a(N)} &= \frac{1}{16\pi^2} \text{tr} \left(T^a \left(\frac{11}{9} [\tilde{D}_V^\mu V_{\mu\lambda}, A^\lambda] + \frac{1}{3} [\tilde{D}_V^\mu A^\lambda, V_{\mu\lambda}] - \right. \right. \\
 &\quad \left. \left. - \frac{2}{3} \tilde{D}_{V_\mu} \tilde{D}_V^\mu \tilde{D}_V^\lambda A_\lambda - \frac{1}{6} \{ \tilde{D}_V^\mu A_\mu, A_\lambda A^\lambda \} + \right. \right. \\
 &\quad \left. \left. - \frac{1}{6} \{ \tilde{D}_V^\mu A^\lambda, \{ A_\mu, A_\lambda \} \} \right) \right) + O\left(\frac{1}{\Lambda^2}\right)
 \end{aligned} \tag{4.16}$$

$$\begin{aligned}
 \mathcal{A}_2^{a(N)} &= \frac{1}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} \text{tr} \left(T^a \left(V^{\mu\nu} V^{\lambda\rho} + \frac{1}{3} A^{\mu\nu} A^{\lambda\rho} + \right. \right. \\
 &\quad \left. \left. + \frac{1}{3} \left(V^{\mu\nu} A^\lambda A^\rho - 2 A^\mu V^{\nu\lambda} A^\rho + A^\mu A^\nu V^{\lambda\rho} \right) \right) \right)
 \end{aligned}$$

$\mathcal{B}^{a(N)}$ is as expected and the quadratic divergence $\mathcal{A}^a(\Lambda^2)$ can be cancelled as in the previous case, but the overall expression for the anomaly is unacceptable.

The problem with eqs. (4.16), especially with $\mathcal{A}_2^{a(N)}$, is not as much that it does not coincide with eqs. (4.10), but that it does not satisfy the Wess-Zumino consistency conditions [17]. Let us briefly recall what the latter are. Defining

$$\begin{aligned}
 X_x^a &= \partial_\mu^x \frac{\delta}{\delta V_\mu^a(x)} + f^{abc} \left(V_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)} + A_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)} \right) \\
 Y_x^a &= \partial_\mu^x \frac{\delta}{\delta A_\mu^a(x)} + f^{abc} \left(V_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)} + A_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)} \right)
 \end{aligned} \tag{4.17}$$

eqs. (4.14) can be rewritten as

$$X_x^a \ln Z_N^V = 0 \tag{4.18a}$$

$$Y_x^a \ln Z_N^V = \mathcal{A}_x^{a(N)} + \mathcal{B}_x^{a(N)} \tag{4.18b}$$

Using the commutation relation satisfied by X^a and Y^a , we can derive for $\mathcal{A}^{a(N)}(x)$:

$$X_y^b \mathcal{A}^{a(N)}(x) = -f^{abc} \delta(x-y) \mathcal{A}^{c(N)}(x) \quad (4.19a)$$

$$Y_x^a \mathcal{A}^{b(N)}(y) - Y_y^b \mathcal{A}^{a(N)}(x) = 0 \quad (4.19b)$$

$\mathcal{A}_2^{a(N)}$ satisfies the first set of conditions [covariance under (4.3a)] but not the second, as can easily be proved from the absence of a quartic term in A_μ , $F^a \sim \epsilon_{\mu\nu\lambda\rho} \text{tr} (T^a A^\mu A^\nu A^\lambda A^\rho)$ [compare with eqs. (4.10)].

We can ask ourselves whether it is possible to modify $\mathcal{A}_2^{a(N)}$ by adding suitable terms to $\ln Z_N$, in such a way as to satisfy (4.13b). If this is possible, it is only at the price of destroying the conserved WI (4.14a). Indeed $\mathcal{A}_2^{a(N)}$ should at least contain a term proportional to F^a . This could be generated [through (4.3b)] only by a term in $\ln Z_N$ such as $\mathcal{F} \sim \int d^4x \epsilon_{\mu\nu\lambda\rho} \text{tr} (V^\mu A^\nu A^\lambda A^\rho)$. However, $\mathcal{F}$ generates a non-vanishing contribution to the WI (4.14a) or (4.18a). Indeed $X_x^a \mathcal{F}$ contains a term $\sim \epsilon_{\mu\nu\lambda\rho} \text{tr} [T^a \partial^\mu (A^\nu A^\lambda A^\rho)]$. It is very easy to see that there exists no counter-term which can cancel this unwanted anomaly.

There is also another problem, presumably connected with the previous one. There exists no combination of counterterms, invariant under (4.3a), which, added to $\ln Z_N$, can cancel $\mathcal{A}_1^a = \lim_{N \rightarrow \infty} \mathcal{A}_1^{a(N)} (\neq)$. These two facts force us to conclude that the definition (4.4) is inconsistent.

We must add that, in carrying out the calculations to derive the explicit form of $\mathcal{A}_1^{a(N)}$ (and only in this case!) we met a few ambiguities, mentioned in Appendix D. We resolved them using consistency conditions based on such identities as $P^2 = P$, etc. As a matter of fact, they could be a hint of some inadequacy of our procedure when several P_V -factors are involved. However, in this case the corresponding ambiguous terms would not affect our final conclusions. We are developing a different calculation method in order to bypass these ambiguities. We hope it will

#) Such a combination exists when the gauge group is Abelian. In this case $\mathcal{A}_2^{(N)}$ corresponds to the right expression for the anomaly.

prove useful not only in this case, but in any complicated calculation where more than one P_N -factor is involved. A puzzling aspect of the results of this section (and of similar examples not reported here) is that they are basis-dependent. But, however surprising, it is to be expected. For one observes that the operation of passing from one basis to another in eq. (2.3) or (4.4) is not, in general, a unitary transformation, since the summation over n [in eq. (2.4)] is truncated^{**}). It is only in the limit $N \rightarrow \infty$ that different orthonormal bases become unitarily equivalent. However, owing to the regularization prescription, this limit is the last operation to be performed. This produces expressions for the anomaly different according to the different bases we start with. The differences have the same general features as the anomaly itself and, in general, we must expect it to be non-vanishing when we take the limit $N \rightarrow \infty$.

5. THE ZERO MODES

In sections 2 and 3 we have considered only trivial V_μ configurations, that is only connections defined on the trivial bundle above a box with (anti-)periodic boundary conditions (a torus). Therefore we have avoided any problem with zero modes of the Dirac operator in a theory of massless fermions. If we take into account non-trivial V_μ configurations then we are bound to find some zero eigenvalues. In this case the definition (2.3) with $[P_N, \emptyset] = 0$ gives 0. In order to find a sensible definition of the functional integral we must subtract the zero-mode contribution.

Let us call

$$P_0 \equiv \sum_{j=0}^{N_0} |\psi_j^0\rangle \langle \psi_j^0| \quad (5.1)$$

where $|\psi_j^0\rangle$ are zero eigenvectors of $\emptyset$, and define

$$\overline{P}_N \equiv P_N - P_0 \quad (5.2)$$

^{**}) Of course if we perform a unitary transformation *only* in the subspace projected out by P_N , everything is left unchanged.

Then a good definition for the fermion functional integral is:

$$\bar{Z}_N = \det(\bar{D}_N) \quad , \quad \bar{D}_N = 1 - \bar{P}_N + \bar{P}_N \not{D} \bar{P}_N \quad (5.3)$$

As a simple application of this definition we are going to recover the index theorem for the Dirac operator [18].

Indeed, let us consider the chiral current:

$$j_{\gamma_5}^{(N)} = i \operatorname{Tr} \left(\gamma_5 \gamma_\mu \langle x | \frac{\bar{P}_N}{\bar{D}_N} | x \rangle \right) \quad (5.4)$$

and calculate its divergence. Explicitly applying ∂^μ we get

$$\partial^\mu j_{\gamma_5}^{(N)}(x) = -2i \operatorname{Tr} \left(\gamma_5 \langle x | \bar{P}_N | x \rangle \right) \quad (5.5)$$

This formula could be obtained also as a WI in an analogous way to the WI derivation of the previous sections. One should implement the variation of $\bar{Z}_N$ under the local chiral transformation:

$$\psi \rightarrow \psi + \gamma_5 \not{D} \omega \quad (5.6)$$

and then proceed exactly as in eqs. (3.5), (3.6), and (3.8), defining $j_{\mu 5}^{(N)}$ through eq. (5.4). In this case $\mathcal{B}^{(N)} = 0$ and again $\mathcal{A}_2^{(N)} = 0$, which one can prove as in Appendix C using the following representation of P_0 :

$$P_0 = \lim_{\varepsilon \rightarrow 0} \theta(\varepsilon - \not{D}^2) \quad (5.7)$$

Finally one recovers eq. (5.5). Integrating it over the box we find:

$$0 = -2i \int_B d^4x \operatorname{Tr} \left(\gamma_5 \langle x | P_N | x \rangle \right) + 2i \int_B d^4x \operatorname{Tr} \left(\gamma_5 \langle x | P_0 | x \rangle \right) \quad (5.8)$$

The first integrand is equal to eq. (3.18) with T^a replaced by the unit matrix. If we choose the eigenstates $|\psi_j^0\rangle$ to be eigenstates of γ_5 , i.e. $\gamma_5 |\psi_k^0(\pm)\rangle = \pm |\psi_k^0(\pm)\rangle$, $k = 1, \dots, N_0/2$,

$$\int_B d^4x \text{Tr} \left(\gamma_5 \langle x | P_0 | x \rangle \right) = \int_B d^4x \sum_{k=1}^{N_0/2} \left(\psi_k^{o(+)*}(x) \psi_k^{o(+)}(x) - \psi_k^{o(-)*}(x) \psi_k^{o(-)}(x) \right) = n_+ - n_-.$$

Therefore

$$n_+ - n_- = - \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\rho} \text{tr} \left(F^{\mu\nu} F^{\lambda\rho} \right).$$

By the way, we know from a theorem of Schwartz [19] that, on a torus, $n_- = 0$.

6. CONCLUSION

We have seen in this paper that finite-mode regularization (or finite-size momentum lattice regularization) can preserve manifest gauge covariance. Coupled to the functional formula for P_N , eq. (2.9), it provides a viable non-perturbative method of calculation, which we have tested by computing the triangle anomalies in several theories.

A rather surprising feature is that the calculations can be entirely carried out in a discrete formalism, without passing through the infinite volume limit.

The finite-mode regularization provides us with a definition of the fermion functional integral in a gauge theory, valid for any regime, either perturbative or non-perturbative. As we have seen, the anomalies are sensitive to the basis chosen in order to regularize the functional integral. To avoid inconsistencies we must choose the basis carefully. Inferring from the examples of this paper and from others, we conclude that a safe basis is always that of eigenvectors of the (complete) Dirac operator involved.

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Derivation of formula (2.10). Consider an operator A depending on a parameter λ . The expression $\exp(\alpha A)$, where α is a number, is defined by the corresponding formal power series. Then, denoting $A' = dA/d\lambda$,

$$\begin{aligned}
 \frac{d}{d\lambda} \exp(\alpha A) &= \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \sum_{k=0}^{n-1} A^k A' A^{n-k-1} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\alpha^{n+k+1}}{(n+k+1)!} A^n A' A^k = \\
 &= \sum_{k,n=0}^{\infty} \int_0^\alpha dt_0 \int_0^{t_0} dt_1 \dots \int_0^{t_{n+k-1}} dt_{n+k} A^n A' A^k = \\
 &= \sum_{n=0}^{\infty} \int_0^\alpha dt_0 \int_0^{t_0} dt_1 \dots \int_0^{t_{n-1}} dt_n A^n A' \sum_{k=0}^{\infty} \frac{A^k}{k!} t_n^k = \\
 &= \sum_{n=0}^{\infty} \int_0^\alpha dt_0 \int_0^{t_0} dt_1 \dots \int_0^{t_{n-1}} dt_n A^n A' \exp(t_n A) = \\
 &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\alpha dt (\alpha-t)^n A^n A' \exp(tA) = \\
 &= \int_0^\alpha dt \exp[(\alpha-t)A] A' \exp(tA), \tag{A.1}
 \end{aligned}$$

where the Cauchy formula

$$\int_0^\alpha dt \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n f(t_n) = \frac{1}{n!} \int_0^\alpha f(t) (\alpha-t)^n dt \tag{A.2}$$

for an integrable function $f(t)$, has been used. Formula (2.10) is recovered if we put $A = -i\not{D}^2$, $\lambda = V_\mu^a(x)$ and insert eq. (A.1) in eq. (2.9).

APPENDIX B

The θ_2 function. The calculations of this paper involve series such as the one in eq. (3.14):

$$\sum_{n=-\infty}^{+\infty} \exp \left[- \left(n + \frac{1}{2} \right)^2 z \right] \equiv \theta_2 (0, z)$$

This is the second Jacobi θ -function with zero argument. It is analytic for $\text{Re } z > 0$. Its Mellin transform is given by [20]:

$$\mathcal{M} \left(\theta_2 (0, z) \right) = 2 \left(2^{2s} - 1 \right) \Gamma(s) \zeta(2s) \quad (\text{B.1})$$

and ζ is the well-known Riemann ζ -function. The poles of the Γ -function are cancelled by corresponding zeros, so that on the r.h.s. of eq. (B.1), there is only one pole at $s = \frac{1}{2}$ with residue $\sqrt{\pi}$. Therefore the asymptotic expansion of $\theta_2(0, z)$ for small z is [20]:

$$\theta_2 (0, z) \approx \sqrt{\pi} z^{-\frac{1}{2}} + O(|z|^\beta) \quad \text{for arbitrary large } \beta. \quad (\text{B.2})$$

Using the theorem on the product of asymptotic expansions [21], from (B.2) we easily find eq. (3.17).

With the same method we can find the asymptotic behaviour of

$$\sum_{n=-\infty}^{+\infty} \left(n + \frac{1}{2} \right)^{2k} \exp \left[- \left(n + \frac{1}{2} \right)^2 z \right] = (-1)^k \frac{d^k}{dz^k} \theta_2 (0, z) \approx \Gamma \left(k + \frac{1}{2} \right) z^{-(k+\frac{1}{2})} + O(|z|^\beta) \quad (\text{B.3})$$

with $k = 1, 2, 3, \dots$.

Using (B.3), $k_\mu = (n_\mu + \frac{1}{2}) \pi/L$ and $k^2 = \sum_\mu k_\mu^2$, we get the formulae

$$\sum_{\{k\}} k_\mu k_\nu e^{-i\alpha k^2} \approx - \frac{L^4}{2\pi^2 i \alpha^3} \delta_{\mu\nu} \quad (\text{B.4a})$$

$$\sum_{\{k\}} k_\mu k_\nu k_\lambda k_\rho e^{-i\alpha k^2} \approx \frac{L^4}{4\alpha^4 \pi^2} \left(\delta_{\mu\nu} \delta_{\lambda\rho} + \delta_{\mu\lambda} \delta_{\nu\rho} + \delta_{\mu\rho} \delta_{\nu\lambda} \right) \quad (\text{B.4b})$$

from which all the other formulae needed in this paper can be derived through integration [21] of the series expansions in α . For example:

$$\sum_{\{k\}} \frac{k_\mu k_\nu}{k^2} e^{-i\alpha k^2} \approx \frac{L^4}{4\alpha^2\pi^2} \delta_{\mu\nu} \quad (\text{B.5})$$

$$\sum_{\{k\}} \frac{k_\mu k_\nu}{(k^2)^2} e^{-i\alpha k^2} \approx -\frac{iL^4}{4\alpha^2\pi^2} \delta_{\mu\nu}$$

APPENDIX C

Evaluation of $\mathcal{A}_1^{\alpha(N)}$, eq. (3.10). The only step missing between eq. (3.11) and eq. (3.12) is the expansion

$$\begin{aligned} \langle x | P_\Lambda | x \rangle &= \frac{\Lambda^4}{16 L^4} \sum_{\{\vec{k}_r\}} \int \frac{d\alpha e^{i\alpha \vec{k}^2}}{2\pi i (\alpha - i\epsilon)} \left[1 - 2\alpha \vec{k} \cdot \vec{D} + i\alpha \frac{D^2}{\Lambda^2} + i\alpha \gamma_\mu \gamma_\nu \frac{V^{\mu\nu}}{\Lambda^2} + \right. \\ &+ 2\alpha^2 \frac{(\vec{k} \cdot \vec{D})^2}{\Lambda^2} - i\frac{\alpha^2}{\Lambda^3} \{ \vec{k} \cdot \vec{D}, D^2 \} - i\frac{\alpha^2}{2\Lambda^3} \{ \vec{k} \cdot \vec{D}, D^2 \} - i\frac{\alpha^2}{2\Lambda^3} \{ \vec{k} \cdot \vec{D}, \gamma_\mu \gamma_\nu V^{\mu\nu} \} + \\ &- \frac{4\alpha^3}{3\Lambda^3} (\vec{k} \cdot \vec{D})^3 - \frac{\alpha^2}{2\Lambda^4} (D^2)^2 - \frac{\alpha^2}{8\Lambda^4} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho V^{\mu\nu} V^{\lambda\rho} - \frac{\alpha^2}{4\Lambda^4} \{ D^2, \gamma_\mu \gamma_\nu V^{\mu\nu} \} + \\ &+ \frac{2i\alpha^3}{3\Lambda^4} \left((\vec{k} \cdot \vec{D})^2 D^2 + (\vec{k} \cdot \vec{D}) D^2 (\vec{k} \cdot \vec{D}) + D^2 (\vec{k} \cdot \vec{D})^2 \right) + \\ &+ \frac{i\alpha^3}{3\Lambda^4} \gamma_\mu \gamma_\nu \left((\vec{k} \cdot \vec{D})^2 V^{\mu\nu} + (\vec{k} \cdot \vec{D}) V^{\mu\nu} (\vec{k} \cdot \vec{D}) + V^{\mu\nu} (\vec{k} \cdot \vec{D})^2 \right) + \\ &\left. + \frac{16\alpha^4}{4!\Lambda^4} (\vec{k} \cdot \vec{D})^4 + O(\Lambda^{-5}) \right], \end{aligned}$$

(C.1)

obtained using eq. (3.13).

Proof that $\mathcal{A}_2^a$ in eq. (3.10) vanishes. Let us put $f(\alpha) = \exp[-i\alpha(\vec{D}^2/\Lambda^2)]$ and use eq. (2.10). Then

$$\begin{aligned} 2 \tilde{D}^\mu \left((\gamma_5 \gamma_\mu)_{ij} \tau_2 \left(P_N \frac{dP_N}{d\alpha_{ij}^a} P_N \right) \right) &= \frac{2i}{\Lambda^2} \int \frac{d\alpha e^{i\alpha}}{2\pi i (\alpha - i\epsilon)} \int_0^\alpha d\tau \tau \left(P_\Lambda f(\alpha - \tau) \left[\gamma_5 |x\rangle T_\gamma^a \langle x| \gamma_5 - \right. \right. \\ &- \gamma_5 |x\rangle T^a \langle x| \gamma_5 \gamma_5 - \gamma_5 |x\rangle T^a \langle x| \gamma_5 \left. \right] f(\tau) P_\Lambda \Big) = \\ &= \frac{8i}{\Lambda^2} \int \frac{d\alpha e^{i\alpha}}{2\pi i (\alpha - i\epsilon)} \int_0^\alpha d\tau \text{Tr} \left(T^a \gamma_5 \langle x| f(\tau) P_\Lambda f(\alpha - \tau) \gamma_5^2 |x\rangle \right) = \\ &= (\text{integrating over } \tau \text{ and to the order } \Lambda^0) \sim \\ &\sim \int \frac{d\alpha_1 d\alpha_2 e^{i(\alpha_1 + \alpha_2)}}{(2\pi i)^2 (\alpha_1 - i\epsilon)(\alpha_2 - i\epsilon)} \sum_{\{k\}} e^{-i(\alpha_1 + \alpha_2)k^2} \alpha_1 \cdot \\ &\cdot \text{Tr} \left(\varepsilon_{\mu\nu\lambda\rho} V^{\mu\nu} V^{\lambda\rho} \right) \left(-i \frac{(\alpha_1 + \alpha_2)}{4} - k^2 \frac{(\alpha_1 + \alpha_2)^2}{8} \right) = 0 \end{aligned} \quad (C.2)$$

In a similar way we can prove that the second term in $\mathcal{A}_2^{a(N)}$ also gives a vanishing contribution.

Proof of eq. (4.14a). From eq. (4.13a,b) we get:

$$\begin{aligned}
 & \tilde{D}_{V_N}^{ac} \mathcal{T}_2 \left(P_N^\nu \frac{1}{\delta V_N^c} P_N^\nu \frac{1}{(B-im)_N^\nu} \right) + f^{abc} V_N^b \mathcal{T}_2 \left(P_N^\nu \frac{1}{\delta A_N^c} P_N^\nu \frac{1}{(B-im)_N^\nu} \right) = \\
 & = \mathcal{T}_2 \left(P_N^\nu \left(|x\rangle T^a \langle x| B - B |x\rangle T^a \langle x| \right) P_N^\nu \frac{1}{(B-im)_N^\nu} \right) = \\
 & = -i \text{Tr} \left(T^a \langle x| \left\{ [\gamma_5 A, P_N^\nu], \frac{P_N^\nu}{(B-im)_N^\nu} \right\} |x\rangle \right). \tag{D.1}
 \end{aligned}$$

On the other hand, using eq. (2.10) and setting $f(\tau) = \exp[-i \tau (B_V^2/\Lambda^2)]$:

$$\begin{aligned}
 & \tilde{D}_{V_N}^{ac} \mathcal{T}_2 \left[\left(\frac{1}{\delta V_N^c} P_N^\nu B P_N^\nu + P_N^\nu B \frac{1}{\delta V_N^c} P_N^\nu \right) \frac{1}{(B-im)_N^\nu} \right] = \\
 & = \tilde{D}_{V_N}^{ac} \mathcal{T}_2 \left[-\frac{1}{\Lambda^2} \int \frac{d\alpha e^{i\alpha}}{2\pi i (\alpha-i\epsilon)} \int_0^\alpha d\tau \left(f(\alpha-\tau) (B_V |x\rangle \gamma_\mu T^c \langle x| + |x\rangle \gamma_\mu T^c \langle x| B_V) \right. \right. \\
 & \quad \left. \left. \cdot f(\tau) B P_N^\nu \frac{1}{(B-im)_N^\nu} + \frac{1}{(B-im)_N^\nu} P_N^\nu B f(\alpha-\tau) (B_V |x\rangle \gamma_\mu T^c \langle x| + |x\rangle \gamma_\mu T^c \langle x| B_V) f(\tau) \right) \right] = \\
 & = \mathcal{T}_2 \left[-\frac{1}{\Lambda^2} \int \frac{d\alpha e^{i\alpha}}{2\pi i (\alpha-i\epsilon)} \int_0^\alpha d\tau \left(f(\alpha-\tau) \left[|x\rangle T^a \langle x|, B_V^2 \right] f(\tau) B \frac{P_N^\nu}{(B-im)_N^\nu} + \right. \right. \\
 & \quad \left. \left. + \frac{1}{(B-im)_N^\nu} P_N^\nu B f(\alpha-\tau) \left[|x\rangle T^a \langle x|, B_V^2 \right] f(\tau) \right) \right] = \\
 & = (\text{integrating over } \tau) = \\
 & = \mathcal{T}_2 \left(\left[|x\rangle T^a \langle x|, P_N^\nu \right] B P_N^\nu \frac{1}{(B-im)_N^\nu} + \frac{P_N^\nu}{(B-im)_N^\nu} B \left[|x\rangle T^a \langle x|, P_N^\nu \right] \right) = \\
 & = i \text{Tr} \left(T^a \langle x| \left\{ [\gamma_5 A, P_N^\nu], \frac{P_N^\nu}{(B-im)_N^\nu} \right\} |x\rangle \right). \tag{D.2}
 \end{aligned}$$

In the last step we have used the obvious relation

$$P_N^\nu [P_N^\nu, \gamma_5 A] P_N^\nu = 0$$

When integrating over τ we have made use of the fact that the commutator of P_N^V with a given operator has an expression similar to eq. (2.10). Finally (D.1) and (D.2) give opposite contributions to the vector WI, so that eq. (4.14a) is verified.

Derivation of eq. (4.16). This is too long to be reported here. We give the starting expression. With manipulations similar to the ones just used $\mathcal{A}^{a(N)}$ can be cast into the form:

$$\begin{aligned} & -\text{Tr} \left(T^a \left(2\gamma_5 \langle x | P_N^V | x \rangle + \gamma_5 \langle x | \left[\left[P_N^V, i\gamma_5 A \right], \frac{P_N^V}{(B-im)_N^V} \right] | x \rangle - \right. \right. \\ & \left. \left. - \frac{1}{\Lambda^2} \int \frac{d\alpha e^{i\alpha}}{2\pi i (\alpha - i\varepsilon)} \int_0^\alpha d\tau \langle x | \left[A, \left\{ B_V, \not{f}(\tau) \left[\left[P_N^V, i\gamma_5 A \right], \frac{P_N^V}{(B-im)_N^V} \right] \not{f}(\alpha - \tau) \right\} \right] | x \rangle \right) \right) \end{aligned} \quad (D.3)$$

Then we use the expansion

$$\frac{P_N^V}{(B-im)_N^V} = \frac{P_N^V}{B_V-im} + \frac{P_N^V}{B_V-im} i\gamma_5 A \frac{P_N^V}{B_V-im} + \frac{P_N^V}{B_V-im} i\gamma_5 A \frac{P_N^V}{B_V-im} i\gamma_5 A \frac{P_N^V}{B_V-im} + \dots$$

The rest is straightforward though lengthy algebra.

In the calculation of $\mathcal{A}_1^{a(N)}$ in eqs. (4.16) we are faced with the evaluation of ambiguous integrals such as

$$\int \frac{d\alpha_1 d\alpha_2 e^{i(\alpha_1+\alpha_2)}}{(2\pi i)^2 (\alpha_1 - i\varepsilon)(\alpha_2 - i\varepsilon)} \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2}$$

Its value ($= 0$) is determined by using consistency relations such as $P^2 = P$, etc.

As for eqs. (4.10) we first use

$$\left(\not{k} + \frac{\not{B}}{\Lambda} \right)^2 = k^2 - 2i \not{k} \cdot \frac{D_V}{\Lambda} + i\gamma_5 \left[\not{k}, \frac{A}{\Lambda} \right] - \frac{D_V^2}{\Lambda^2} - \frac{1}{2} \gamma_\mu \gamma_\nu \frac{V^{\mu\nu}}{\Lambda^2} + \frac{i}{\Lambda^2} \gamma_5 \left[\not{B}_V, A \right] + \frac{AA}{\Lambda^2}$$

where $D_V^\mu = \partial^\mu + V^\mu$, and then expand P_N as we have done in eq. (C.1). The rest is straightforward algebra. No ambiguities are met during this calculation.

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