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CONFORMAL ANOMALY AND COHOMOLOGY

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A B S T R A C T

We show that the trace anomaly is a non-trivial element of a suitable cohomology group. So we extend the geometrical interpretation of anomalies previously considered in the Adler-Bell-Jackiw case.

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It is by now understood that the Adler-Bell-Jackiw anomaly ¹⁾ (ABJ) in gauge theories has a cohomological origin. This was first pointed out by Biedenharn ²⁾ ; Becchi, Rouet and Stora ³⁾ (BRS) showed, by introducing the BRS operator (which is a coboundary operator), that the integrated ABJ anomaly is a BRS invariant quantity, but it is not the BRS transform of any integrated local quantity. Accordingly this anomaly is a non-trivial element of some cohomology group. It was later shown ⁴⁾ that this cohomology arises from the action of the group of (local) gauge transformations on the space of vector potentials. Indeed the action of any Lie group on a manifold naturally induces a cohomology [see Refs. 4) and 5)].

In this letter we suggest that another anomaly appearing in field theory, namely the trace or conformal anomaly ^{6),7)}, is as well a non-trivial element of a suitable cohomology group. This cohomology arises from the action of the conformal (Weyl) group on the space $\mathcal{M}$ of all metric tensors and on the space $\mathcal{F}$ of matter fields. To avoid any misunderstanding, we underline that we do not interpret the conformal symmetry as a gauge symmetry (so we do not introduce gauge breaking terms in the Lagrangian, etc.) and $g_{\mu\nu}$ will always be an external field.

To be definite, we will consider a self-interacting scalar field ϕ coupled in a conformally invariant way to an external Riemannian metric $g_{\mu\nu}$. The classical action reads

$$S(\phi; g) = \int_M d^4x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{6} R \phi^2 + \frac{\lambda}{4} \phi^4 \right) \quad (1)$$

where R is the scalar curvature of $g_{\mu\nu}$ and M is assumed to be a four-dimensional compact manifold without boundary. S is invariant under conformal transformations $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$, $\phi \rightarrow \phi/\Omega$, where Ω is any positive function on M .

One possible way to represent the action of our coboundary operator is to introduce an anticommuting scalar field $\xi = \xi(x)$ and to consider $\Omega(x) = \exp(\epsilon \xi)$, where ϵ is an anticommuting infinitesimal parameter. Then the infinitesimal action of the Weyl group on $\mathcal{M} \times \mathcal{F}$ can be represented by $\delta g_{\mu\nu} = 2\epsilon \xi g_{\mu\nu}$, $\delta \phi = -\epsilon \xi \phi$. We can now write the classical Ward identity

$$\Xi \Gamma_\epsilon = 0 \quad (2)$$

where

$$\begin{aligned}\Xi &= \Xi_g + \Xi_\phi \\ \Xi_g &= \int d^4x \, 2 \xi \, g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \\ \Xi_\phi &= - \int d^4x \, \xi \, \phi \frac{\delta}{\delta \phi}\end{aligned}\tag{3}$$

Here Γ_{cl} has the same functional expression as S , provided that the quantum field, still denoted by ϕ , be replaced by its classical counterpart. In other words, we introduce an external current coupled to ϕ , we define a generating functional and perform a Legendre transformation in order to obtain the generating functional Γ of the 1PI Green functions. The 0-th order term in the $\hbar$ expansion of Γ is Γ_{cl} .

Thanks to the anticommutativity of ξ , it is easy to check that Ξ is a nilpotent operator, i.e., $\Xi^2 = 0$. As we shall see at the end of this paper, introducing the nilpotent operator Ξ is tantamount to dealing with the cohomology of the Lie algebra of the Weyl group with coefficients in the space of functionals on $\mathcal{M} \times \mathcal{F}$. Of course, we are only interested in the subspace of local functionals.

Let us consider now a quantized version of Eq. (2). Even though the existence of a quantum theory for the action (1) is an open question, we suppose here that the quantum action principle⁸⁾ holds. In the simpler case we will discuss at the end of the paper, this assumption is unnecessary. In the following, we will refer to a BPHZ renormalization scheme ; nevertheless we will not write normal product symbols, since in this paper only algebraic relations are relevant. On this basis, the quantum analogue Γ of the action should obey an equation of the form

$$\Xi \Gamma = \Delta \Gamma, \quad \Delta \Gamma = \hbar \Delta + \mathcal{O}(\hbar^2)\tag{4}$$

$\Delta \Gamma$ is of order $\hbar$ and contains all the possible insertions of dimension 4 ($\dim \phi = 1$, $\dim g_{\mu\nu} = 0$, $\dim \xi = 0$) and with "ghost number" 1 (that is, containing ξ only once). Since $\Xi^2 = 0$, from Eq. (4) we have that Δ must satisfy the following consistency condition

$$\Xi \Delta = 0\tag{5}$$

This implies that Δ has the form $\Delta = \Xi \gamma + \bar{\Delta}$, with $\Xi \bar{\Delta} = 0$, but there exists no $\bar{\gamma}$ such that $\bar{\Delta} = \Xi \bar{\gamma}$. The functional $\bar{\Delta}$ is an anomaly.

Now we write Δ as a linear combination

$$\Delta = \sum_i a_i \Delta_i \quad (6)$$

of terms Δ_i forming a complete basis. Such Δ_i are listed in Table 1, together with their E transforms $E\Delta_i$. Any other invariant of dimension 4 and "ghost number" 1 can be reduced to a linear combination of those listed in Table 1 by integrating by parts. Note also that to derive this Table we only used integration by parts and the contracted Bianchi identity, namely $\nabla_\lambda R = 2g^{\rho\nu} \nabla_\rho R_{\mu\lambda}$, where ∇_λ is the covariant derivative.

We next require that Δ in Eq. (6) satisfies the condition (5), yielding the following relations among the coefficients a_i

$$\begin{aligned} a_1 + a_2 + 3a_3 &= 0 \\ 6a_5 + a_6 &= 0 \end{aligned} \quad (7)$$

while a_4, a_7 and a_8 are arbitrary. The first of the relations (7) was obtained by Duff⁷⁾, who used a more heuristic argument based on dimensional regularization. It is clear that the true reason for the existence of such relations is of cohomological nature.

We want now to select among the possible Δ 's given by Eq. (6) with the conditions (7) those which cannot be expressed in the form EQ , for some local functional Q . Since the operator E preserves dimensions, Q should be a linear combination of the functionals γ_i of dimension 4 and "ghost number" 0 listed in Table 2. From this Table we see that $\Delta_4 = \frac{1}{6} E \gamma_1$, $\Delta_8 = \frac{1}{6} E \gamma_6$, so these two terms are cohomologically trivial. Any linear combination of Δ_4 and Δ_8 can be written as $E\gamma$, and a modified action $\Gamma' = \Gamma + \gamma$ can be defined such that $E\Gamma' = \bar{\Delta}$, where, because of Eq. (7), $\bar{\Delta}$ is given by

$$\bar{\Delta} = \int dV \left\{ (a F + b G + c K + d \phi^4) \right\} \quad (8)$$

with

$$\begin{aligned} F &= R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \\ G &= R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \\ K &= \phi \square \phi + \frac{1}{6} R \phi^2 \end{aligned}$$

The coefficients a, b, c and d in Eq. (8) are left undetermined by Eq. (7), and to evaluate them one has to use some supplementary tool. It is interesting to notice that F is the square of the Weyl tensor and G is the Euler density. We have excluded from the beginning a possible contribution to the anomaly from the Pontryagin density, because of parity.

Let us come to the trace of the energy momentum tensor. This is classically defined by

$$(\theta_r^r)_c \equiv \frac{\delta}{\delta \xi} \Xi_g \Gamma_{cl} = - \frac{\delta}{\delta \xi} \Xi_\phi \Gamma_{cl} ,$$

and of course vanishes on-shell. We now define

$$\theta_r^r = \frac{\delta}{\delta \xi} \Xi_g \Gamma' = \frac{\delta}{\delta \xi} (\bar{\Delta} - \Xi_\phi \Gamma')$$

The term $\Xi_\phi \Gamma$ is a contact term. Moreover, using the equations of motion, we can eliminate the term $R\phi^2$. Therefore the trace anomaly (on-shell) becomes

$$\theta_r^r = a F + b G + d' \phi^4 + c' \phi \square \phi .$$

Note that when g is flat, only the last two terms survive : this corresponds to the results obtained by Lowenstein⁹⁾ and Schroer¹⁰⁾.

Usually the trace anomaly is derived for much simpler theories, i.e., free matter fields interacting with an external metric. In this case we can give a more direct treatment. Let

$$W(g) = \ln \int \mathcal{D}\phi \exp[-S(\phi; g)]$$

be the effective action for a theory in which the action S is given by (1) with $\lambda = 0$. Performing a conformal transformation we get

$$\begin{aligned} W(\Omega^2 g) &\simeq W(g) + \varepsilon \Delta(g, \xi) \\ &\simeq W(g) + \varepsilon \Xi_g W(g) . \end{aligned}$$

Therefore

$$\Xi_g W(g) = \Delta(g, \xi)$$

when $\Delta(g, \xi)$ is a priori a combination of all terms of dimension 4 and linear in ξ . Accordingly, $\Delta(g, \xi) = \sum_{i=1}^4 a_i \Delta_i$. Since, when operating on terms of this form, $\Xi_g = \Xi$, the above analysis still holds, and the trace anomaly reads

$$\bar{\Delta}(g, \xi) = \int dV \xi (a F + b G),$$

provided that the effective action is redefined by adding a term proportional to R^2 .

We have not considered so far soft breaking terms of the conformal symmetry, but they can be easily incorporated in the above analysis. It is more interesting to extend the same analysis to other matter fields. For spin $\frac{1}{2}$ fields, the results only change by contact terms. A vector field A_μ would introduce a new anomaly, in that $\int dV F_{\mu\nu} F^{\mu\nu}$ is cohomologically non-trivial. A hardly unexpected result ¹¹⁾.

We would like now to say a few words about the mathematical meaning of the operator Ξ and the field ξ , stressing that we are in fact dealing with the cohomology of the Lie algebra of the Weyl group. Let $\mathcal{M} \times \mathcal{F}$ be the product manifold of the manifold of Riemannian metrics $\mathcal{M}$ with the manifold of scalar fields $\mathcal{F}$ [see, e.g., Ref. 12)]. Let G be the group of maps $\lambda : M \rightarrow \mathbb{R}^+$, with pointwise product. Its (Abelian) Lie algebra $\mathcal{G}$ is the linear space of maps $\lambda : M \rightarrow \mathbb{R}$. Any element $\lambda \in \mathcal{G}$ induces a fundamental vector field Z_λ on $\mathcal{M} \times \mathcal{F}$. We consider the space $\Omega^0(\mathcal{M} \times \mathcal{F})$ of all functionals on $\mathcal{M} \times \mathcal{F}$ and the spaces C_k of skew symmetric multilinear functions of $\mathcal{G} \times \dots \times \mathcal{G}$ (k factors) to $\Omega^0(\mathcal{M} \times \mathcal{F})$. Then we consider the operators $\delta_k : C_k \rightarrow C_{k+1}$ given by

$$\delta_k \psi(\lambda_1, \dots, \lambda_{k+1}) = \sum_i (-)^{i+1} \mathcal{L}_{Z_{\lambda_i}} \psi(\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{k+1}).$$

where $\mathcal{L}_{Z_{\lambda_i}}$ is the Lie derivative along Z_{λ_i} and $\hat{}$ denotes omission. It is easy to see that for any k , $\delta_{k+1} \circ \delta_k = 0$. Accordingly the k -th cohomology group is defined as

$$H^k(\mathcal{G}, \Omega^0(\mathcal{M} \times \mathcal{F})) = \text{Ker } \delta_k / \text{Im } \delta_{k-1}$$

Actually, one should restrict oneself to considering functionals in $\Omega^0(\mathcal{M} \times \mathcal{F})$ that are integrals of the fields ϕ , $g_{\mu\nu}$ and their derivatives, in complete analogy to what has been done in Ref. 4).

To see that the operator Ξ considered in this paper reproduces the cohomology $H^k(\mathfrak{g}, \Omega^0(\mathcal{M} \times \mathcal{F}))$, we shall limit ourselves to check a simple example. Let $\psi \in C_1$, and $\psi(\lambda) = \int f(\lambda) d^4x$, where $f(\lambda)$ is a function over M depending on $(g, \phi) \in \mathcal{M} \times \mathcal{F}$. Now

$$\begin{aligned} \delta \psi(\lambda_1, \lambda_2) &= \mathcal{L}_{Z_{\lambda_1}} \psi(\lambda_2) - \mathcal{L}_{Z_{\lambda_2}} \psi(\lambda_1) = \\ &= \int_M d^4x \left\{ 2\lambda_1 g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \lambda_1 \phi \frac{\delta}{\delta \phi} \right\} f(\lambda_2) - \\ &\quad - \int_M d^4x \left\{ 2\lambda_2 g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \lambda_2 \phi \frac{\delta}{\delta \phi} \right\} f(\lambda_1). \end{aligned}$$

By means of the following "formal" transformation $\lambda_i \rightarrow \xi$, and for any function h , $h(\lambda_1, \lambda_2) \rightarrow h(\lambda_2, \lambda_1) \rightarrow h(\xi, \xi)$, we see that $\delta \psi(\lambda_1, \lambda_2)$ becomes $\Xi \psi(\xi) = \int d^4x \xi (2g_{\mu\nu} (\delta/\delta g_{\mu\nu}) - \phi (\delta/\delta \phi)) f(\xi)$.

In conclusion, the cohomological method discussed here reproduces all the known results about the trace anomalies. Although it is not a complete method (the coefficients of non-trivial elements of the relevant cohomology group must be calculated in any given model), it has two important advantages: it exposes the mathematical origin of the anomalies and it is very simple. Indeed, it seems to us an indispensable tool when more complex theories (with torsion or non-metric connections) are studied.

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i	Δ_i	$\Xi \Delta_i$
1	$\int dV \xi R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$	$4 \int dV R \xi \square \xi$
2	$\int dV \xi R_{\mu\nu} R^{\mu\nu}$	$4 \int dV R \xi \square \xi$
3	$\int dV \xi R^2$	$12 \int dV R \xi \square \xi$
4	$\int dV \xi \square R$	0
5	$\int dV \xi \phi \square \phi$	$-\int dV \phi^2 \xi \square \xi$
6	$\int dV \xi R \phi^2$	$6 \int dV \phi^2 \xi \square \xi$
7	$\int dV \xi \phi^4$	0
8	$\int dV \xi \square \phi^2$	0

Table 1 - Insertions of dimension 4 and "ghost number" 1.
We denote by $dV = \sqrt{g} d^4x$ the volume element,
 $R_{\mu\nu\lambda\rho}$, $R_{\mu\nu}$ and R the Riemann tensor, the Ricci
tensor and the scalar curvature of $g_{\mu\nu}$, res-
pectively. $\square$ is the covariant Laplacian.

j	γ_j	$\Xi \gamma_j$
1	$\int dV R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}$	$4 \int dV \xi \square R$
2	$\int dV R_{\mu\nu} R^{\mu\nu}$	$4 \int dV \xi \square R$
3	$\int dV R^2$	$12 \int dV \xi \square R$
4	$\int dV \square R$	$-2 \int dV \xi \square R$
5	$\int dV \phi \square \phi$	$-\int dV \xi \square \phi^2$
6	$\int dV \phi^2 R$	$6 \int dV \xi \square \phi^2$
7	$\int dV \phi^4$	0

Table 2 - Functionals of dimension 4 and "ghost number" 0.

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