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INVARIANCE OF ACTIONS, RHEONOMY AND THE NEW MINIMAL
 $N = 1$ SUPERGRAVITY IN THE GROUP MANIFOLD APPROACH

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A B S T R A C T

A new definition of rheonomy is proposed based on Bianchi identities instead of field equations. For theories with auxiliary fields, the transformation rules are obtained in a completely geometrical way and invariance of the action is equivalent to $d\mathcal{L} = 0$, which means surface-independence of the action integral. For theories without auxiliary fields, the transformation rules are found by requiring that the action be invariant, just as in the component approach. Previous methods of obtaining the transformation rules which start from rheonomy of field equations and use certain recipes to find the off-shell extensions of the rules are abandoned. New minimal supergravity is worked out in detail; it is the gauge theory based on a free differential algebra which includes the auxiliary fields.

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1. - INTRODUCTION

In this article we analyze various concepts which play a rôle in the group manifold approach to supergravity^{1),2)} and their relation to the x space component approach. To illustrate the general discussion we consider a number of case studies. One of them is new: the formulation of $N = 1$ new minimal supergravity in terms of the group manifold approach. In fact, it was this concrete case study which led to our sharpened understanding of general concepts³⁾.

The concepts which play a central rôle in the group manifold approach, and whose relation with x space concepts we are interested in, are the following

- i) Rheonomy¹⁾. In the geometric framework popularly called group manifold, the variational field equations obtained from the Lagrangian are entirely written in terms of exterior differential forms with the exclusion of the Hodge duality operation. Therefore, they can be implemented either on the x space manifold (in this way obtaining the same result as in the usual approach) or on any larger manifold containing x space. In particular, if they are implemented on the full group manifold, one obtains algebraic relations between curvature components in x space (inner directions) and curvature components in directions orthogonal to x space (outer directions). When it happens that the former completely determine the latter, a solution of the field equations on the x space submanifold can be uniquely extended to a solution on the whole group manifold. Mathematically, this lifting corresponds to a solution of the Cauchy problem in the outer directions, for which the x space fields form a complete set of initial data. The possibility of this lifting and its uniqueness is what has been called rheonomy.

This rheonomic lifting can also be viewed as an x space transformation of the fields, which maps solutions of the x space field equations into new solutions. From this point of view, it is nothing other than the on-shell supersymmetry transformation.

- ii) Bianchi identities⁴⁾. The fields entering the geometric Lagrangian are one-form potentials of some super Lie algebra $\mathfrak{G}$. They can be considered either as one-forms living on the x space manifold, or as one-forms living on the whole group manifold. In both cases the Bianchi identities, which are exterior form equations, must hold on the corresponding space. The algebraic relations between inner and outer components of the curvatures which we obtain from the field equations (hereafter named rheonomic conditions) are differential constraints on the one-form potentials when they are considered as forms on the whole group manifold. These constraints are a system of differential equations involving both inner and outer derivatives. In general,

the integrability conditions of this system are a set of purely x space differential equations which we can identify with the x space field equations. This can be proved by inserting the explicit rheonomic parametrization of the outer curvatures into the Bianchi identities: one finds that they are satisfied if and only if the inner field equations hold²⁾.

This state of affairs means that, in general, the rheonomic lifting can be performed only on solutions of x space field equations and not on arbitrary field configurations. This is the counterpart of the familiar fact that the supersymmetry algebra without auxiliary fields is closed only on-shell. Thus, one can perform rheonomic lifting on those fields which satisfy field equations. If one desires a closed off-shell algebra or, equivalently, the possibility of lifting an arbitrary field configuration instead of a classical solution, one has to modify the rheonomic differential constraints, namely, to modify the relations between inner and outer curvature components. This is done by the addition of new fields (auxiliary fields) in the parametrization of outer curvatures. The new objects are determined by the requirement that the Bianchi identities should hold without the use of inner field equations. Moreover, their outer derivatives, namely their supersymmetry transformation rules, will be determined by Bianchi identities themselves.

- iii) Invariance of the action. In the group manifold approach the action functional of a supergravity theory is uniquely fixed by a set of general principles which make no reference to the x space concept of supersymmetry invariance. However, if the result is to be identical to the Noether one, it is evident that these principles should be equivalent to the requirement of off-shell invariance. The problem is thus how to obtain the correct transformation rules which leave the action invariant.

To formulate the problem, let us first recall the aforementioned principles.

- a) The action is written entirely in the language of differential forms, so that the equations of motion are also form equations. This is necessary if one wants to extend the x space field equations to the entire group manifold.
- b) The group-flat space (= vacuum), corresponding to all curvatures R^A equal to zero should be a solution.
- c) Non-trivial solutions ($R^A \neq 0$) on the whole group manifold must also exist.
- d) The non-trivial solutions must be rheonomic in the sense previously discussed. (Outer curvature components are uniquely determined by inner ones.)

This set of principles guarantees that the manifold of classical solutions of the x space field equations is invariant under a closed algebra of supersymmetry transformations. This statement is explained in the following way. First consider a classical x space solution. Because of rheonomy, it can be uniquely extended to a solution on the whole group manifold. This solution can now be mapped into a new solution on the same space by means of a diffeomorphism. The restriction to x space of this new group manifold solution is a new x space solution. By means of this detour in the group manifold, one defines a mapping of x space solutions into x space solutions. Its infinitesimal form gives the algebra of on-shell supersymmetry transformation rules. The form of the on-shell transformation rules is just a Lie derivative

$$\begin{aligned} \delta_\epsilon \mu^A &= \mathcal{L}_\epsilon \mu^A \equiv \underline{\epsilon} \rfloor d\mu^A + d(\underline{\epsilon} \rfloor \mu^A) = \\ &= \nabla \epsilon^A + \underline{\epsilon} \rfloor R^A = \nabla \epsilon^A + 2 E^F \mu^G R^A_{FG} \end{aligned} \quad (1.1)$$

where ∇ is the group covariant derivative and R^A_{FG} the curvature components. If one remembers that all outer components (R^A_{IO}, R^A_{OO}) are, for a solution, functions of the inner ones, then from (1.1) one obtains a formula which gives the variation of x space objects in terms of x space objects only. This is the on-shell supersymmetry transformation rule. These transformations form an on-shell closed algebra. Indeed, the Lie derivatives always form an algebra¹⁾

$$[\mathcal{L}_{\epsilon_1}, \mathcal{L}_{\epsilon_2}] = \mathcal{L}_{[\epsilon_1, \epsilon_2]} \quad (1.2)$$

provided they are consistently defined, namely, provided the operator used in their definitions is a true exterior derivative ($d^2 = 0$). If one uses the equivalent form $\mathcal{L}_\epsilon = \nabla \epsilon + \underline{\epsilon} \rfloor R$, checking that $d^2 = 0$ amounts to checking that the Bianchi identities are satisfied by R . This is the case for the on-shell rheonomic curvatures. Thus, the rheonomy principle used in the group manifold construction of the action guarantees the existence of a closed on-shell supersymmetry algebra.

A symmetry of the equations of motion is not necessarily a symmetry of the action. Since, however, we know that the same action is determined in the Noether approach by the very requirement of invariance, it makes sense to inquire whether the rheonomic transformation can be promoted to an off-

shell transformation leaving the action invariant. For this purpose, it is crucial to note that while going from solutions to configurations, formula (1.1) becomes ambiguous. Indeed, the replacement of R_{I0}^A and R_{00}^A by their expression in terms of inner curvatures R_{II}^A is defined, on-shell, only modulo any addition of inner-field equations. Going off-shell, this ambiguity has to be fixed in order for the transformation to be a symmetry of the action. One way of fixing it is the ordinary requirement of invariance as in the x space approach. Explicit computation, however, shows that the result one gets in this way is the same as one would get by demanding that the curvatures are such as to make $d\mathcal{L} = 0$ on the whole group manifold. By $d\mathcal{L}$ we mean the exterior derivative of the Lagrangian. This result has a clear-cut interpretation in the case of models with a closed off-shell algebra. In these cases, $d\mathcal{L} = 0$ is the condition for the action to be independent of the choice of integration surface in the group manifold. Therefore, $d\mathcal{L} = 0$ is indeed the invariance condition for the action, since supersymmetry transformations are nothing other than shifts of the integration surface in the group manifold.

As we already announced, we will exemplify the above discussion by detailed analysis of some cases. They are the following:

- i) Simple ($N = 1$) supergravity in $D = 4$ dimensions without auxiliary fields, both in first and second order formalism. Note that the transition from first to second order formalism is obtained by equating to zero the space-time components R_{mn}^a of the torsion R^a , rather than the full two-form R^a .
- ii) Simple $d = 4$ supergravity with the new minimal auxiliary fields A_μ and $t_{\mu\nu}$ ⁵⁾. The interest in this model lies in the fact that these x space fields can be interpreted as one- and two-forms on the group manifold and already appear at the level of the Cartan integrable system which corresponds to this model. [Cartan integrable systems^{6),7)} are known in mathematics as free graded differential algebras⁸⁾. For the extensions to superalgebras we refer to Ref. 9).] This model exhibits many analogies with $d = 11$ supergravity.

Both models are based on the more general concept of "Cartan Integrable Systems" [free differential algebra], rather than on the concept of (super) Lie algebra. A question which naturally arises in these cases is whether the free differential algebra is equivalent to an underlying Lie algebra (or more than one) and whether, correspondingly, the dynamical theory written in terms of higher forms is equivalent to a theory written in terms of one-forms (ordinary gauge potentials).

As far as the algebraic equivalence is concerned, an answer was already given in Ref. 6) for the case of $d = 11$ supergravity. Here the problem will be reconsidered at the dynamical level also. In particular, using as a test the simpler case of the Sohnius-West model, we shall point out challenging unsolved problems occurring in the $d = 11$ case. Sections 2 and 3 are devoted to the case studies. Section 4 deals with the problem of underlying (super) Lie algebras for Cartan integrable system theories. Metric and γ matrices conventions in four dimensions are given in Appendix A of Ref. 4); for $d = 11$ supergravity conventions are given in the Appendix of Ref. 6).

2. - SIMPLE $d = 4$ POINCARÉ SUPERGRAVITY WITHOUT AUXILIARY FIELDS

The action of this model is uniquely determined by the principles of the geometric approach reviewed in the introduction. It reads

$$I = \int_{M_4} \mathcal{L} \quad (2.1a)$$

$$\mathcal{L} = R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4 \bar{\Psi} \wedge \gamma_5 \gamma_a \rho \wedge V^a \quad (2.1b)$$

The curvatures defining the super Poincaré algebra in $d = 4$ and entering Eq. (2.1b) are the following:

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \quad (2.2a)$$

$$R^a = DV^a - \frac{i}{2} \bar{\Psi} \wedge \gamma^a \Psi \equiv dV^a - \omega^{ab} \wedge V^b - \frac{i}{2} \bar{\Psi} \wedge \gamma^a \Psi \quad (2.2b)$$

$$\rho = D\Psi \equiv d\Psi + \frac{i}{2} \omega^{ab} \wedge \Sigma_{ab} \Psi \quad (2.2c)$$

The field equations in the outer directions (= orthogonal to x space) determine the outer components of the curvatures completely: rheonomy holds.

The field equations read

$$\mathcal{L} \epsilon_{ijkp} R^k \wedge V^l = 0 \quad (2.3a)$$

$$\mathcal{L} R^{mn} \wedge V^k \epsilon_{mnkl} + 4 \bar{\Psi} \wedge \gamma_5 \gamma_l \mathcal{S} = 0 \quad (2.3b)$$

$$8 \gamma_5 \gamma_m \mathcal{S} \wedge V^m - 4 \gamma_5 \gamma_m \Psi \wedge R^m = 0 \quad (2.3c)$$

and the solution of the outer equations reads

$$R^a = R^a_{mn} V^m \wedge V^n \quad (2.3a)'$$

$$\mathcal{S} = \mathcal{S}_{mn} V^m \wedge V^n - \left(\frac{1}{2} \gamma_m \gamma_k R^k_{mn} - \frac{i}{6} \gamma_m \tilde{\epsilon}_{rs} \gamma_k R^k_{rs} \right) \Psi \wedge V^n \quad (2.3b)'$$

$$\begin{aligned} R^{ab} = R^{ab}_{mn} V^m \wedge V^n - \epsilon^{abrs} \bar{\Psi} \wedge \gamma_5 \gamma_m \mathcal{S}_{rs} V^m \\ - \epsilon^{trrs} [a \bar{\Psi} \wedge \gamma_5 \gamma_t \mathcal{S}_{rs} V^b] \end{aligned} \quad (2.3c)'$$

The field equations in inner directions (= x space field equations) read

$$R^k_{mn} = 0 \quad (2.4a)$$

$$R^k_{lm} - \frac{1}{2} \delta^k_l R^m_{mn} = 0 \quad (2.4b)$$

$$\gamma^m \mathcal{S}_{mn} = 0 \quad (2.4c)$$

The inner components of the curvatures are what, in the x space approach, are called supercovariant objects¹⁰⁾.

The curvatures in (2.3)' satisfy the Bianchi identities if and only if all x space field equations in (4) are satisfied^{4),2)}. Turning to the transformation rules, we recall from the introduction that via rheonomy, the general diffeomorphisms

$$\delta \mu^A = (\nabla \epsilon)^A + \epsilon^B R_{BC}^A \mu^C \quad (2.5)$$

become x space transformations mapping solutions of (2.4) into solutions of (2.4). The transformation laws in (2.5) form a closed x space algebra if and only if the curvatures in (2.3)' satisfy the Bianchi identities on the group manifold.

We now turn to the question of the off-shell extension of (2.5). We will discuss separately first and second order formalism, i.e., with and without imposing (2.4a). In both cases, we must determine the off-shell form of R_{BC}^A in (2.5). There is an infinity of such extensions, all differing by terms proportional to the left-hand side of (2.4). For instance, on-shell, we could have written instead of (2.3c)':

$$\begin{aligned} R^{kl} = R_{mn}^{kl} V^m \wedge V^n + i \bar{g}^{kl} \gamma^m \psi \wedge V_m \\ + 2i \bar{g}^m [k \gamma^l] \psi \wedge V_m \end{aligned} \quad (2.3c)''$$

This form is actually found if one solves the Bianchi identities for (2.3), imposing only (2.4a)³⁾. In going off-shell, one could take (2.3c) or (2.3c)' or any other expression as long as they are the same when (2.4) holds.

Our procedure will be to choose one particular expression for the R_{BC}^A , denoted by $R_{BC}^{(0)A}$, and to add to it arbitrary terms proportional to the left-hand side of (2.4). We will choose for $R_{BC}^{(0)A}$ the curvatures obtained from the outer field equations; thus, in first order formalism we choose (2.3c) and not (2.3c)'. The transformation rules we will consider then have the form

$$\delta \mu^A = (\nabla \epsilon)^A + 2 \epsilon^B \left(\overset{\circ}{R}{}^A{}_{\cdot B \zeta} + \Delta R^A{}_{\cdot B \zeta} \right) \mu^\zeta \quad (2.5)'$$

where ΔR^A_{BC} are proportional to (2.4) field equations.

To investigate the invariance of the action, we could perform the usual x space manipulations: writing the transformation laws as follows

$$\delta \omega^{ab} = \underline{\epsilon} \rfloor R^{ab} \quad (2.6a)$$

$$\delta \psi = \nabla \epsilon + \underline{\epsilon} \rfloor \not{S} = \not{D} \epsilon + \underline{\epsilon} \rfloor \not{S} \quad (2.6b)$$

$$\delta V^a = i \bar{\epsilon} \gamma^a \psi + \underline{\epsilon} \rfloor R^a \quad (2.6c)$$

the symbols $\underline{\epsilon} \rfloor R^{ab}$, $\underline{\epsilon} \rfloor \not{S}$, $\underline{\epsilon} \rfloor R^a$ are to be determined in such a way as to leave the action invariant. At this point, they do not yet have a particular geometrical meaning. For the corresponding variation of the action, one would find straightforwardly

$$\begin{aligned} \delta I = \int_{M_4} & \left\{ 2 \underline{\epsilon} \rfloor R^{ab} \wedge R^c \wedge V^d \epsilon_{abcd} + 8 (\underline{\epsilon} \rfloor \bar{\not{S}}) \not{\gamma}_5 \not{\gamma}_a \not{S} \wedge V^a \right. \\ & - 4 (\underline{\epsilon} \rfloor \bar{\not{S}}) \not{\gamma}_5 \not{\gamma}_a \psi \wedge R^a - 4 \bar{\not{S}} \wedge \not{\gamma}_5 \not{\gamma}_a \epsilon \wedge R^a \\ & \left. - (\underline{\epsilon} \rfloor R^d) (2 R^{ab} \wedge V^c \epsilon_{abcd} + 4 \bar{\psi} \wedge \not{\gamma}_5 \not{\gamma}_d \not{S}) \right\} \end{aligned} \quad (2.7)$$

In first order formalism, one finds that $\delta I = 0$ is achieved with:

$$\underline{\epsilon} \rfloor \not{S} = 0 \quad (2.8a)$$

$$\underline{\epsilon} \rfloor R^a = 0 \quad (2.8b)$$

$$\underline{\epsilon} \rfloor R^{ab} = -\epsilon^{abrs} \bar{\epsilon} \gamma_5 \gamma_m g_{rs} V^m - \epsilon^{trrs} [a \bar{\epsilon} \gamma_5 \gamma_t g_{rs} V^b] \quad (2.8c)$$

As far as the x space approach is concerned, Eqs. (2.8) are the end of the story: one has determined the supersymmetry transformation laws which leave the action invariant yet have no geometrical interpretation. If one remembers, however, that the off-shell supersymmetry transformations should be the extension to configurations of the rheonomic symmetry of the solution manifold (2.5), then one is led to interpret $\underline{\epsilon} \rfloor \rho$, $\underline{\epsilon} \rfloor R^a$, $\underline{\epsilon} \rfloor R^{ab}$ as the contractions along $\vec{\epsilon} = \epsilon^{\alpha\dot{\alpha}}_{\alpha}$ of the corresponding curvature two-forms ρ , R^a , and R^{ab} . From this point of view, Eqs. (2.8) are constraints on the curvature components in superspace; indeed, Eqs. (2.8) imply:

$$R^a = R^a_{mn} V^m \wedge V^n \quad (2.9a)$$

$$g = g_{mn} V^m \wedge V^n \quad (2.9b)$$

$$R^{ab} = R^{ab}_{mn} V^m \wedge V^n - \epsilon^{abrs} \bar{\psi} \wedge \gamma_5 \gamma_m g_{rs} V^m - \epsilon^{trrs} [a \bar{\epsilon} \gamma_5 \gamma_t g_{rs} V^b] \quad (2.9c)$$

Comparing Eqs. (2.9) with (2.3)', we see that they differ by terms proportional to the left-hand side of the inner equations (2.4). This shows explicitly that the on-shell rheonomic symmetry is promoted to an off-shell invariance of the action by resolving the ambiguity previously discussed. In particular, with reference to Eq. (2.5)', we see that if $R_{BC}^{(0)A}$ is defined via Eq. (2.3)' then

$$\Delta R^a = 0 \quad (2.10a)$$

$$\Delta \mathcal{S} = \left\{ \frac{1}{2} \gamma_m \gamma_k R_{mn}^k - \frac{i}{6} \gamma_m \vec{\Sigma}_{rs} \gamma_k R_{rs}^k \right\} \psi \wedge V^n \quad (2.10b)$$

$$\Delta R^{ab} = 0 \quad (2.10c)$$

We point out that in any case, both (2.9) and (2.3)' satisfy Bianchi identities on the group manifold only via implementation of the inner equations (2.4): as extensively discussed in the introduction, this means that the algebra of the corresponding transformations closes only on-shell.

One defines second order formalism by setting $R_{mn}^a = 0$. In that case, the equation $\delta I = 0$ implies

$$\underline{\epsilon} \rfloor R^a = 0 \quad (2.11a)$$

$$\underline{\epsilon} \rfloor \mathcal{S} = 0 \quad (2.11b)$$

$$\underline{\epsilon} \rfloor R^{ab} = \text{chain rule} \quad (2.11c)$$

Namely, in second order formalism, we have $\Delta \rho = 0$, $\Delta R^a = 0$.

Considering now Eq. (2.7), it is easy to verify that if $\underline{\epsilon} \rfloor \rho$, $\underline{\epsilon} \rfloor R^a$ and $\underline{\epsilon} \rfloor R^{ab}$ are interpreted as the contractions of the corresponding curvatures, then the entire integrand of δI in (2.7) can be interpreted as the contraction $\underline{\epsilon} \rfloor$ of the exterior derivative of the Lagrangian:

$$\delta I = \int_{M_H} \underline{\epsilon} \rfloor d\mathcal{L} \quad (2.12)$$

Indeed, from (2.1b), one obtains

$$d\mathcal{L} = 2 R^{ab} \wedge R^c \wedge V^d \epsilon_{abcd} + 4 \bar{\psi} \wedge \gamma_5 \gamma_m \psi \wedge V^m - 4 \bar{\psi} \wedge \gamma_5 \gamma_m \psi \wedge R^m \quad (2.13)$$

and the contraction operator applied to (2.13) yields the integrand of (2.7). This is no accident. Indeed, under a diffeomorphism:

$$\delta \mu^A = \epsilon \rfloor d\mu^A + d(\epsilon \rfloor \mu^A) = \mathcal{L}_\epsilon \mu^A \quad (2.14)$$

the Lagrangian $\mathcal{L}$ would transform into

$$\delta \mathcal{L} = \epsilon \rfloor d\mathcal{L} + d(\epsilon \rfloor \mathcal{L}) = \mathcal{L}_\epsilon \mathcal{L} \quad (2.15)$$

so that the variation of the action is given by (2.12).

If $d\mathcal{L} = 0$, then clearly the action would not depend on the surface M^4 chosen. This is just Stokes' theorem

$$I(M^4 + \delta M^4) - I(M^4) = \int_{M^4} d\mathcal{L} \quad (2.16)$$

In general, $d\mathcal{L}$ does not vanish, and $d\mathcal{L} = 0$ is an equation which implies relations between the components R_{BC}^A of the curvatures $R^A = d\mu^A + 1/2 f_{BC}^A \mu^B \wedge \mu^C$ as we have discussed in our case study. If the Bianchi identities for the solutions $R^A = R_{BC}^A \mu^B \wedge \mu^C$ of $d\mathcal{L} = 0$ hold, then these objects $R_{BC}^A \mu^B \wedge \mu^C$ are true curvatures, namely $R^A = d\tilde{\mu}^A + 1/2 f_{BC}^A \tilde{\mu}^B \wedge \tilde{\mu}^C$ for certain $\tilde{\mu}^A$. These particular $\tilde{\mu}^A$ for which $d\mathcal{L} = 0$ is zero, span a manifold of off-shell fields on which the action functional is defined. For all fields in this manifold, the action is surface independent, and hence supersymmetric invariant. The algebra of these

supersymmetry transformations is off-shell closed because it is generated by genuine Lie derivatives.

In the next case study, we shall encounter such a situation; in fact, in that example the outer components of R_{BC}^A are entirely determined in terms of the inner components by the two criteria of $d\mathfrak{f} = 0$ and fulfilment of Bianchi identities. Moreover, in this case the inner components are entirely free. Geometrically, this means that an arbitrary configuration of the fields in x space has a unique extension to the group manifold which preserves the value of the action. In other words, we again have a Cauchy problem; however, the initial data are now completely off-shell. When the solution of the equation $d\mathfrak{f} = 0$ does not satisfy the Bianchi identities for off-shell initial data, the geometrical interpretation of off-shell transformations is ruled out while it remains for the on-shell ones. However, we just remind the reader that this is of no consequence for the group manifold approach, because, as pointed out in the introduction, the Lagrangian is uniquely determined precisely by the requirement of equation of motion invariance, not action invariance.

3. - SIMPLE $d = 4$ SUPERGRAVITY WITH THE NEW MINIMAL AUXILIARY FIELDS

In the group manifold approach, scalars and spin $1/2$ fields have no natural geometrical interpretation, although they may be added as zero-forms. In particular, auxiliary fields generally do not arise automatically within this approach, but must be found by the same techniques based on Bianchi identities as used in the superspace approach. The model presented now is occasioned by the observation that there is an exception to this state of affairs: there exists a set of auxiliary fields for $N = 1$ supergravity for which all fields are gauge fields. This is the "new minimal" off-shell formulation by Sohnius and West⁵⁾. The auxiliary fields are an axial vector gauge field A_μ and a gauge antisymmetric tensor $T_{\mu\nu}$.

The complete set of fields including auxiliaries can be associated with the following "Cartan integrable system" (differential free algebra):

$$\begin{aligned} R^{mn} &= d\omega^{mn} - \omega^m{}_k \wedge \omega^{kn} \\ R^m &= DV^m - \frac{i}{2} \bar{\psi} \wedge \gamma^m \psi \\ S &= D\psi - \frac{i}{2} \gamma_5 \psi \wedge A \quad ; \quad R^\square = dA \\ R^\otimes &= dT - \frac{i}{2} \bar{\psi} \wedge \gamma_m \psi \wedge T^m \end{aligned} \tag{3.1}$$

The first three lines define a superalgebra, namely the super Poincaré algebra plus a chiral external charge, while the last line defines the curvature of the two-form T . This system is consistent due to the following two identities:

$$\bar{\Psi} \wedge \gamma_5 \gamma_m \Psi = 0 \quad ; \quad \bar{\Psi} \gamma_m \Psi \wedge \bar{\Psi} \wedge \gamma^m \Psi = 0 \quad (3.2)$$

Consistency (= closure) of this system implies the following Bianchi identities:

$$\begin{aligned} \mathcal{D} R^{mu} &= 0 \\ \mathcal{D} R^m + R^{mu} \wedge V_m - i \bar{\Psi} \wedge \gamma^m \not{S} &= 0 \\ \mathcal{D} \not{S} - \frac{i}{2} \gamma_5 \not{S} \wedge A - \frac{i}{2} R^{mu} \wedge \not{Z}_{mu} \Psi - i \gamma_5 \Psi \wedge R^\square &= 0 \\ dR^\square &= 0 \\ dR^\otimes - i \bar{\Psi} \wedge \gamma_m \not{S} \wedge V^m + \frac{i}{2} \bar{\Psi} \wedge \gamma_m \Psi \wedge R^m &= 0 \end{aligned} \quad (3.3)$$

The action can be obtained by the standard geometric approach using the principles of vacuum existence, non-triviality and rheonomy. It reads:

$$\begin{aligned} I = \int_{M_4} \mathcal{L} \quad ; \quad \mathcal{L} = & R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4 \bar{\Psi} \wedge \gamma_5 \gamma_a \not{S} \wedge V^a \\ & - 8 R^\square \wedge T + \alpha \left(\oint_m R^\otimes \wedge V^m + \frac{1}{8} \oint_m \oint^n V^i \wedge V^j \wedge V^k \wedge V^l \epsilon_{ijkl} \right) \end{aligned} \quad (3.4)$$

The first part is the cohomology part, while the terms proportional to α containing the zero-form f_m have been added in order to obtain a kinetic term for T , since it appears in the x space formulation of Ref. 5). One could add more general terms, for example terms with R^m or extra zero-forms f_{mn} , which would also give A a kinetic term, but just for simplicity we will restrict our attention to (3.4).

To verify whether rheonomy holds, we write down the field equations. They read

$$\begin{aligned}
 \delta \omega^{mn} &: \epsilon_{mni} R^i \wedge V^j = 0 \\
 \delta f^m &: R^\otimes = f_i V_j \wedge V_k \wedge V_l \in ijkl \\
 \delta T &: -8R^\square + \alpha V^m \wedge \mathcal{D}f_m - \frac{i\alpha}{2} f_m \bar{\Psi} \wedge \gamma^m \Psi - \alpha f_m R^m = 0 \quad (3.5) \\
 \delta A &: R^\otimes = 0 \\
 \delta \Psi &: 8\gamma_5 \gamma_m \rho \wedge V^m - i\alpha \gamma_m \Psi \wedge V^m f_m - 4\gamma_5 \gamma_m \Psi \wedge R^m = 0 \\
 \delta V^m &: 2R^{ij} \wedge V^k \in mijk - 4\bar{\Psi} \gamma_5 \gamma_m \rho - \alpha f_m R^\otimes + \frac{\alpha}{2} f_\mu f^\mu \epsilon_{mijk} V^i \wedge V^j \wedge V^k = 0
 \end{aligned}$$

From now on, we restrict ourselves to second order formalism, which means that we assume that the VV and VVV parts of the $\delta \omega^{mn}$ and δf_m equations are satisfied: $R_{mn}^k = 0$ and $R_{mnp}^\otimes = f^q \epsilon_{qmpn}$.

We first look at the outer field equations. In that case, we find

$$\begin{aligned}
 R^m &= 0 ; R^\otimes = f_i V_j \wedge V_k \wedge V_l \in ijkl \\
 \rho &= S_{mn} V^m \wedge V^n + \frac{i\alpha}{8} \gamma_5 \Psi \wedge f_m V^m \\
 R^\square &= R_{mn}^\square V^m \wedge V^n - \frac{\alpha}{8} \bar{\Psi} \Psi \wedge V^m - \frac{i\alpha}{16} f_m \bar{\Psi} \wedge \gamma^m \Psi \\
 R^{ab} &= \text{as in eq (2.3b)}'
 \end{aligned} \quad (3.6)$$

where $\bar{\Psi} \Psi$ is the four-part of $\mathcal{D}f_m$. If one imposes all field equations, $f^k = 0$, so that $\varphi_m = 0$, rheonomy holds.

To obtain the transformation rules, we must fix the beforementioned ambiguity. As explained in the previous sections, the outer field equations cannot be used to yield the correct laws. To see this, it is sufficient to note the presence of the extraneous object φ_m in $R^\square$ in (3.6).

To fix the ambiguity correctly, we will discuss two methods. The first method is what one does in x space: find ΔR_{BC}^A (and thus $\delta \mu^A$) by requiring invariance of the action (Noether method). This will fix not only ΔR_{BC}^A (in particular φ_m) but also the free parameters α in the action, and the end

result would be Ref. 5). Afterwards, one could study the algebra and discover that it closes. This closure means that $R^A + \Delta R^A$ satisfy the Bianchi identities, with $D\varphi_m$ being determined by the Bianchi identities themselves. In other words, the transformation rules $\delta\mu^A = (D\varepsilon)^A + \varepsilon^B (R_{BC}^A + \Delta R_{BC}^A)\mu^C$ are diffeomorphisms in the group manifold. This suggests our second method. We assume from the start that the x space transformation rules are group manifold diffeomorphisms, hence that the Bianchi identities are satisfied by $R^A + \Delta R^A$ where R_{BC}^A is given by the outer field equations, and ΔR^A is proportional to the x space field equations. Moreover, we will require that the x space action be invariant. The result will be a stronger property: the action is independent of the choice of M_4 in the group manifold.

The most general form of the $R^A + \Delta R^A$ curvatures discussed above is the following one:

$$R^a = 0 \quad (3.7a)$$

$$R^{kl} = R^{kl}_{mn} V^m \wedge V^n + ic \bar{\psi} \gamma^{[k} \rho^{l]} \wedge V_m + id \bar{\psi} \gamma^{mkl} \psi \wedge V_m + ie \epsilon^{klmn} \bar{\psi} \gamma_m \psi f^n \quad (3.7b)$$

$$R^\square = R^\square_{mn} V^m \wedge V^n + \bar{\psi} \chi_m \wedge V^m + ig \bar{\psi} \gamma^m \psi f_m \quad (3.7c)$$

$$\rho = \rho_{mn} V^m \wedge V^n + ai \gamma_5 \psi f_m \wedge V^m + ib \gamma_5 \gamma_{mn} \psi \wedge V^m \wedge V^n \quad (3.7d)$$

$$R^{\star} = f^i V^j \wedge V^k \wedge V^l \epsilon_{ijkl} \quad (3.7e)$$

where χ_m is a spinor made out of the left-hand side of the spinorial equation of motion: $\gamma^m \rho_{mn}$. That the above is the most general ansatz follows from a scaling argument. Note that the Bianchi identities (3.3) are invariant under the following scale transformation:

$$\begin{array}{ll}
 \omega^{mn} \rightarrow \omega^{mn} & R^{mn} \rightarrow R^{mn} \\
 V^m \rightarrow \lambda V^m & R^m \rightarrow \lambda R^m \\
 A \rightarrow A & R^\square \rightarrow R^\square \\
 \psi \rightarrow \lambda^{1/2} \psi & \rho \rightarrow \lambda^{1/2} \rho \\
 T \rightarrow \lambda^2 T & R^\otimes \rightarrow \lambda R^\otimes
 \end{array} \tag{3.8}$$

Hence the solutions of Bianchi identities must be in agreement with the scaling assignments (3.8). From (3.8) we can also work out the scaling assignments of the inner components R_{mn}^{kl} , $R_{mn}^\square$, ρ_{mn} , f_m which are given below:

$$\begin{array}{l}
 R_{mn}^{kl} \rightarrow \lambda^{-2} R_{mn}^{kl}, \quad R_{mn}^\square \rightarrow \lambda^{-2} R_{mn}^\square \\
 \rho_{mn} \rightarrow \lambda^{-3/2} \rho_{mn}, \quad f_m \rightarrow \lambda^{-1} f_m
 \end{array} \tag{3.9}$$

Using (3.8) and (3.9), it is easy to check that (3.7) is indeed the most general ansatz for curvatures whose outer components are parametrized by inner ones. Now inserting (3.7) into the Bianchi identities (3.3) and using the algebraic techniques described in Ref. 4), one obtains the solution for the parameters c , d , e , g , a , b and for the expression χ_m . It is the following:

$$\begin{array}{ll}
 c = -2 & d = 1 \\
 e = -3/2 & a = a \\
 b = -3/2 & g = \frac{3-a}{2}
 \end{array} \tag{3.10}$$

$$\chi_m = \left(\frac{ia}{3!} \epsilon_{mijk} \gamma^i \rho^{jk} - \gamma_5 \gamma^n \rho_{nm} \right)$$

Equation (3.10) shows that Bianchi identities fix everything except the free parameter α . It follows that for every value of α one has a different closed algebra of off-shell transformations. Only one of these families of algebra is, however, going to leave the action invariant, as we shall see in a moment.

Let us consider the Lagrangian (3.4) and take its exterior derivative. If $R^a = 0$, we find:

$$\begin{aligned} d\mathcal{L} = & 4\bar{\psi} \gamma_5 \gamma_a \psi \wedge V^a - 8R^a \wedge R^a + \alpha \mathcal{F}^a \wedge R^a \wedge V_a \\ & + i\alpha \mathcal{F}^a \wedge \gamma^b \psi \wedge V_b \wedge V_a - \frac{i\alpha}{2} \mathcal{F}^a R^a \wedge \bar{\psi} \wedge \gamma_a \psi \\ & + \frac{\alpha}{4} \mathcal{F}^a \mathcal{F}^b \epsilon_{ijkl} V^i \wedge V^j \wedge V^k \wedge V^l + \frac{i\alpha}{4} \mathcal{F}^a \mathcal{F}^b \epsilon_{ijkl} V^i \wedge V^j \wedge V^k \wedge V^l \end{aligned} \quad (3.11)$$

As we have already pointed out, the action $I = \int_{M_4} \mathcal{L}$ is independent of the choice of M_4 in the group manifold, if and only if $d\mathcal{L} = 0$ and the curvatures satisfy Bianchi identities. When this happens, the action I is invariant under the infinitesimal diffeomorphisms:

$$\begin{aligned} \delta_\epsilon V^a &= \mathcal{L}_\epsilon V^a = \epsilon^\mu dV^\mu + d\epsilon^\mu V^\mu = (\nabla \epsilon)^a + \epsilon^\mu R^\mu_a \\ \delta_\epsilon \psi^\alpha &= \mathcal{L}_\epsilon \psi^\alpha = \epsilon^\mu d\psi^\mu + d\epsilon^\mu \psi^\mu = (\nabla \epsilon)^\alpha + \epsilon^\mu \rho^\mu_\alpha \\ &\dots\dots\dots \\ &\dots\dots\dots \end{aligned} \quad (3.12)$$

If we insert Eq. (3.7) into (3.11) and use the result (3.10) from Bianchi identities, we find that $d\mathcal{L} = 0$ if and only if

$$\begin{aligned} \alpha &= 3 \\ \alpha &= 36 \end{aligned} \quad (3.13)$$

Hence the requirement of invariance ($d\mathcal{L} = 0$) both fixes the Lagrangian ($\alpha = 36$) and picks up one out of the one-parameter infinity of closed off-shell algebras ($a = 3$). Our results are summarized in the table).

4. - UNDERLYING (SUPER) LIE ALGEBRA AND HIDDEN POTENTIALS FOR "GAUGE" THEORIES OF CARTAN INTEGRABLE SYSTEMS

A) $d = 11$ Supergravity

The model we have discussed in the previous section exhibits an important analogy with $d = 11$ supergravity. The analogy lies in the fact that both theories are based on Cartan integrable systems rather than ordinary (super) Lie algebras. This means that among our physical potentials there are forms of degree higher than one. The Sohnius-West model has the two-form $T_{\mu\nu}$ while $d = 11$ supergravity has the three-form $A_{\mu\nu\rho}$ (11). An important difference between the two cases is that $T_{\mu\nu}$ does not propagate, while $A_{\mu\nu\rho}$ does. This, however, is of no relevance for the algebraic structure of the theory: in both cases, a higher degree form cannot be considered as the potential of any group generator, and therefore one cannot say which symmetry it is gauging.

In the case of $d = 11$ supergravity, the following was shown⁶⁾: the generalized Maurer-Cartan equations which one obtains by equating to zero the curvatures, namely:

$$\begin{aligned} R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega_c^b = 0 \\ R^a &= dV^a - \omega^{ab} \wedge V_b - \frac{i}{2} \bar{\psi} \Gamma^a \psi = 0 \\ \rho &= d\psi - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi = 0 \\ R^\square &= dA - \frac{1}{2} \bar{\psi} \Gamma_{ab} \psi \wedge V^a \wedge V^b = 0 \end{aligned} \tag{4.1}$$

are equivalent to a set of ordinary Maurer-Cartan equations on ordinary one-forms. Generally speaking, the problem is that of writing for A (or any other p -form in the free differential algebra) a suitable representation as a polynomial (with constant coefficients) in terms of newly-introduced one-forms, (together with those already appearing in the differential algebra), in such a way that the generalized Maurer-Cartan equations [(4.1) in this case] are a consequence of ordinary Maurer-Cartan equations obeyed by the new and old one-forms. In the specific case of (4.1), one finds a minimal dichotomic solution, that is, there are two minimal sets of one-forms which represent (4.1) in the previous sense. They are respectively given by:

$$\begin{aligned} 1^{st} \text{ set} &= \{ \omega^{ab}, V^a, \psi, B^{ab}, B^{a_1 \dots a_5}, \eta_{(1)} \} \\ 2^{nd} \text{ set} &= \{ \omega^{ab}, V^a, \psi, B^{ab}, B^{a_1 \dots a_5}, \eta_{(2)} \} \end{aligned} \quad (4.2)$$

where ω^{ab}, V^a, ψ are the old one-forms (describing super Poincaré algebra in $d = 11$), $B^{ab}, B^{a_1 \dots a_5}$ are antisymmetric one-forms with two- and five-vector indices respectively, and $\eta_{(1)}$ and $\eta_{(2)}$ are new 32-component Majorana spinor one-forms. They obey the following ordinary Maurer-Cartan equations:

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^b = 0 \quad (4.3a)$$

$$R^a = dV^a - \omega^{ab} \wedge V_b - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi = 0 \quad (4.3b)$$

$$\rho = d\psi - \frac{1}{4} \omega^{ab} \wedge \Gamma_{ab} \psi = 0 \quad (4.3c)$$

$$R^{a_1 \dots a_5} = dB^{a_1 \dots a_5} - \frac{i}{2} \bar{\psi} \wedge \Gamma^{a_1 \dots a_5} \psi = 0 \quad (4.3d)$$

$$\begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix} = d \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \frac{1}{4} \Gamma_{ab} \omega^{ab} \wedge \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad (4.3e)$$

$$\begin{aligned} &+ i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Gamma_a \psi \wedge V^a + \begin{pmatrix} 1/5 \\ -1/2 \end{pmatrix} \Gamma_{ab} \psi \wedge B^{ab} + \\ &+ i \begin{pmatrix} 1/240 \\ -1/144 \end{pmatrix} \Gamma_{a_1 \dots a_5} \psi \wedge B^{a_1 \dots a_5} = 0 \end{aligned}$$

$$(4.3f)$$

The three-form A is given by

$$\begin{aligned}
 A = & B_{ab} \wedge V^a \wedge V^b + \bar{\psi} \wedge \Gamma^a \begin{pmatrix} 0 \\ \gamma_2 \end{pmatrix} \wedge V^a + \bar{\psi} \Gamma^{ab} \begin{pmatrix} \frac{1}{2} \gamma_1 \\ \frac{1}{5} \gamma_2 \end{pmatrix} \wedge B^{ab} \\
 & + i \bar{\psi} \wedge \Gamma^{a_1 \dots a_5} \begin{pmatrix} \frac{1}{144} \gamma_1 \\ \frac{1}{240} \gamma_2 \end{pmatrix} \wedge B^{a_1 \dots a_5} \\
 & + \left(\frac{4}{15} \right) B^{a_1 a_2} \wedge B^{a_2 a_3} \wedge B^{a_3 a_4} + \left(-\frac{5}{144} \right) B^{a_1 \dots a_4} \wedge B^{a_4 a_5} \wedge B^{a_5 a_1} \\
 & + \left(\frac{1}{4! 6!} \right) \epsilon_{b_1 \dots b_5 a_1 \dots a_5} B^{b_1 \dots b_5} \wedge B^{a_1 \dots a_5} \wedge V^m + \\
 & \left(\frac{1}{2 (\gamma_2)^2} \right) \epsilon_{a_1 \dots a_6 b_1 \dots b_5} B^{a_1 \dots a_3 p_1 p_2} \wedge B^{a_4 \dots a_6 p_1 p_2} \wedge B^{b_1 \dots b_5} \quad (4.4) \\
 & - \left(\frac{1}{2 (\gamma_2)^2} \right) \epsilon_{a_1 \dots a_6 b_1 \dots b_5} B^{a_1 \dots a_3 p_1 p_2} \wedge B^{a_4 \dots a_6 p_1 p_2} \wedge B^{b_1 \dots b_5}
 \end{aligned}$$

This minimal solution is a representation of the vacuum structure of the theory, which is described by the generalized Maurer-Cartan equation (4.1) (zero curvature solution of the theory).

The question now is whether the entire theory can be rewritten in terms of the underlying potentials (4.2) discovered, analyzing the structure of the free differential algebra at zero curvature. Physically, this programme is analogous to the conversion from metric formalism to vierbein formalism in general relativity. Eq. (4.4) is the analogue of

$$g_{\mu\nu}(x) = V^a_\mu(x) V^b_\nu(x) \gamma_{ab}$$

In both cases, an object carrying several world indices but no "flat index", on which a group action could be defined, is replaced by a product of objects with only one world index (one-form), but one or several flat ones belonging to group representations. With this procedure, new symmetries are introduced or, rather, made explicit. In the case of gravity, the new symmetry is, of course, Lorentz invariance. In our case it is given by the superalgebra (4.3). We can therefore try to express our theory in terms of either one of the two minimal sets (4.2). However, as soon as one inserts Eq. (4.4) into the following field equation of $d = 11$ supergravity⁶⁾

$$R^{\square} = dA - \frac{1}{2} \bar{\psi} \Gamma^{ab} \psi \wedge V_a \wedge V_b = F_{abcd} V^a \wedge V^b \wedge V^c \wedge V^d \quad (4.5)$$

one immediately finds that, in order for (4.5) to be satisfied, F_{abcd} has to be zero, namely we cannot go out of the vacuum. Hence the minimal parametrization (4.4) is insufficient to describe the excited states of the theory. The problem is overcome by taking for A a new parametrization in terms of V^a , ψ , ω^{ab} , B^{ab} , $B^{a_1 \dots a_5}$ and both η_1 and η_2 , namely

$$\begin{aligned} A = & B_{ab} \wedge V^a \wedge V^b + \frac{1}{2} \bar{\psi} \Gamma^a \gamma_2 \wedge V_a + \bar{\psi} \Gamma^{ab} \left(\frac{1}{4} \gamma_1 + \frac{1}{10} \gamma_2 \right) \wedge B^{ab} \\ & + i \bar{\psi} \wedge \Gamma^{a_1 \dots a_5} \left(\frac{1}{288} \gamma_1 + \frac{1}{480} \gamma_2 \right) \wedge B^{a_1 \dots a_5} \end{aligned} \quad (4.6)$$

which corresponds to the complete superalgebra (4.3f). With (4.6) the theory can now be completely rewritten in terms of the hidden potentials and the explicit form of the transformation laws under the new symmetries can be found. The algebra involved is, however, cumbersome and we prefer to illustrate the relevant conceptual points on the much simpler case of the Sohnius-West model.

B) Sohnius-West New minimal model

In section 3, we analyzed the Sohnius-West model as the "gauge theory" of the free differential algebra (Cartan integrable system) (3.1). Similarly to what we did in the previous subsection, we can try to find "hidden potentials" for this model. Our goal is that of taking the square root of $T_{\mu\nu}(x)$ by writing

$$T_{\mu\nu}(x) = \phi_{\mu}^A(x) \phi_{\nu}^B(x) C_{AB} \quad (4.7)$$

where $\phi_\mu^A(x)$ are one-form potentials of a convenient superalgebra, and C_{AB} is a constant matrix which plays an analogous rôle to the flat Minkowskian metric η^{ab} in the case of general relativity [Eq. (4.5)].

What we will require is that the uninterpretable "gauge transformation" of $T_{\mu\nu}$

$$\delta T_{\mu\nu}(x) = \partial_{[\mu} \Lambda_{\nu]}(x) \quad (4.8)$$

be the outcome of an "interpretable" group transformation on the hidden potentials $\phi_\mu^A(x)$:

$$\begin{aligned} \delta T_{\mu\nu}(x) &= 2\delta\phi_\mu^A(x)\phi_\nu^B C_{AB} = \partial_{[\mu} \Lambda_{\nu]}(x) \\ \Rightarrow \Lambda_\nu(x) &= \epsilon^A(x)\phi_\nu^B \epsilon_{AB} \end{aligned} \quad (4.9)$$

where ϵ_{AB} is a new convenient constant matrix. Obviously, not only the gauge transformation (4.8) but also all the other symmetries acting on $T_{\mu\nu}(x)$ (super-symmetry in this case) should be originated by convenient transformations on the hidden potentials $\phi_\mu^A(x)$: moreover, these transformations on $\phi_\mu^A(x)$ should just be the standard soft group transformations:

$$\delta\phi_\mu^A(x) = \nabla_\mu \epsilon^A + 2\epsilon^F R_{FG}^A \phi_\mu^G(x) \quad (4.10)$$

where R_{FG}^A are curvature components.

In the case of gravity, the general co-ordinate transformation on $g_{\mu\nu}(x)$ is, in this way, replaced by the soft translation transformation on the vierbein:

$$\delta V_\mu^a = \nabla_\mu \epsilon^a + 2\epsilon^b R_b^a{}^c V_\mu^c \quad (4.11)$$

where R_{bc}^a are the intrinsic torsion components. The case of gravity also teaches us another lesson. There are transformations on the "hidden potentials" which leave $g_{\mu\nu}(x)$ invariant (Lorentz rotations). Therefore, we expect that there will be transformations on $\phi_\mu^A(x)$ which will leave $T_{\mu\nu}(x)$ invariant. It is interesting that in our case, as in $d = 11$ supergravity, these special transformations will include a second supersymmetry, not visible in the $T_{\mu\nu}$ formulation. In any case, the model rewritten in terms of the hidden potentials will have more fields and a larger symmetry, although it has the same particle spectrum.

The aforementioned programme can be carried out to the very end in the following way. First we look into the problem of replacing the Cartan integrable system (3.1) with an ordinary superalgebra. This is done by representing the two-form T as the following polynomial:

$$T = K^a \wedge V_a + \bar{\Psi} \wedge \gamma \quad (4.12)$$

where K^a is a new vector one-form and η a new Majorana spinor one-form. If we postulate the following curvatures for the system $(\omega^{ab}, V^a, \psi, A, K^a, \eta)$:

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega_c^b \quad (4.13a)$$

$$R^a \equiv DV^a - \frac{i}{2} \bar{\Psi} \wedge \gamma^a \Psi \quad (4.13b)$$

$$R^\square \equiv dA \quad (4.13c)$$

$$\rho \equiv D\psi - i\gamma_5 \psi \wedge A \quad (4.13d)$$

$$R^{\otimes a} \equiv DK^a - \frac{i}{2} \bar{\Psi} \wedge \gamma^a \Psi \quad (4.13e)$$

$$\sigma^\otimes \equiv D\gamma + i\gamma_5 \gamma \wedge A + \frac{i}{2} \gamma^a \Psi \wedge K_a \quad (4.13f)$$

we can first prove that they do indeed define a (super) Lie algebra, because the Maurer-Cartan equations $R^{ab} = R^a = R^\square = \rho = \overset{\otimes}{R}^a = \overset{\otimes}{\sigma} = 0$ are self-consistent (= closed). This is the same as checking Jacobi identities. In this proof, one just needs the following identity:

$$\gamma_m \psi \wedge \bar{\psi} \gamma^m \psi = 0 \quad (4.14)$$

The closure of (4.13) results in the following Bianchi identities:

$$D R^{ab} = 0 \quad (4.15a)$$

$$D R^a + R^{ab} \wedge V_b - i \bar{\psi} \wedge \gamma^a \psi = 0 \quad (4.15b)$$

$$D \rho - \frac{i}{2} R^{ab} \wedge \Sigma_{ab} \psi - i R^\square \wedge \gamma_5 \psi - i \gamma_5 \rho \wedge A = 0 \quad (4.15c)$$

$$d R^\square = 0 \quad (4.15d)$$

$$D R^{\otimes a} + R^{ab} \wedge K^b - i \bar{\psi} \wedge \gamma^a \psi = 0 \quad (4.15e)$$

$$\begin{aligned} D R^\otimes + i \gamma_5 \gamma \wedge R^\square - \frac{i}{2} \Sigma_{ab} \gamma \wedge R^{ab} + \frac{i}{2} \gamma^a \rho \wedge K_a \\ + i \gamma_5 \sigma^\otimes \wedge A - i \gamma^a \psi \wedge R^{\otimes a} = 0 \end{aligned} \quad (4.15f)$$

Secondly, one can easily see that at curvatures (4.13) equal to zero:

$$\begin{aligned} d\tau &= dK^a \wedge V_a - K^a \wedge dV_a + d\bar{\psi} \wedge \gamma - \bar{\psi} \wedge d\gamma \\ &= \frac{i}{2} \bar{\psi} \wedge \gamma^a \psi \wedge V_a \end{aligned} \quad (4.16)$$

which proves that (4.12) equipped with (4.13) is a representation of the original differential algebra (3.1).

Then $K_\mu^a(x)$ and $\eta_\mu(x)$ can be taken as our hidden potentials $\Phi_\mu^A(x)$ and (4.12) yields the explicit form of (4.7), namely,

$$T_{\mu\nu}(x) = K_{\mu}^a(x) V_{\nu}^a(x) + \bar{\Psi}_{\mu}(x) \Psi_{\nu}(x) \quad (4.17)$$

We note that, in the same way as in gravity, the introduction of the hidden potentials enlarges rather than reduces the number of degrees of freedom. Indeed, $K_\mu^a(x)$ contains both an antisymmetric part $K_\mu^a(x) - K_a^\mu(x)$ which represents the degrees of freedom of $T_{\mu\nu}$, and a symmetric one; besides that, one also has new fermionic degrees of freedom. We can now consider the action of the Sohnius-West model given in the table, and consistently replace T by (4.12) everywhere. In this way, we obtain a theory in terms of ordinary group potentials only. This Lagrangian has obviously the same physical content as the original one with respect to propagating modes. It has, however, a larger symmetry: there is a symmetry transformation for each generator in the superalgebra (4.13); in particular, there is a new supersymmetry associated to η .

To find them we proceed as follows. First we consider the equations⁴⁾ of the table, giving the structure of the curvatures which correspond to the off-shell transformation rules. If we insert the explicit form of $R^\otimes$ in terms of hidden potentials, we obtain the following equation:

$$\begin{aligned} R^\otimes &= R^{\otimes a} \wedge V_a - K^a \wedge R_a + \bar{\psi} \wedge \eta - \bar{\psi} \wedge \sigma^\otimes \\ &= f^a \wedge V^b \wedge V^c \wedge V^d \epsilon_{abcd} \end{aligned} \quad (4.18)$$

Using (4.18) and the remaining equations of the table, we can determine the structure of $R^{\otimes a}$ and $\sigma^\otimes$. The solution of this equation is not unique and there are arbitrary outer terms in the parametrization of $R^{\otimes a}$ and $\sigma^\otimes$. They would correspond to transformation rules including fields not originally appearing in the Lagrangian. If we decide to put these arbitrary outer components equal to zero, we obtain:

$$\begin{aligned}
 R^{\otimes a} = & t \begin{array}{|c|c|} \hline m & a \\ \hline n & \\ \hline \end{array} V_m^\wedge V_n - \epsilon^{amnp} V_m^\wedge V_n f_p \\
 & + (3+a) \bar{\eta} \gamma_5 \Sigma^{ab} \psi f_b - (3+b) i \bar{\eta} \gamma_5 \psi f^a \\
 & + \bar{\eta} \sigma^{\otimes ba} V_b - \bar{\eta} \rho^{ba} V_b
 \end{aligned} \tag{4.19a}$$

$$\sigma^{\otimes} = \sigma^{\otimes ab} V_a^\wedge V_b + a \gamma_5 \Sigma_{ab} V^a f^b + i b \gamma_5 \gamma f_m V^m \tag{4.19b}$$

where $t \begin{array}{|c|c|} \hline m & a \\ \hline n & \\ \hline \end{array}$ denotes the VV part of $R^{\otimes a}$ symmetrized according to the written Young tableau and a and b are free parameters.

Given Eqs. (4.19), one can use them to define the transformation rules of the hidden potentials K^a and η under all transformations. Calling k^a and ξ the parameters of the new symmetries associated to K^a and η , and ϵ the old supersymmetry parameter, we obtain the following transformations for the complete system:

$$\delta V^a = i \bar{\epsilon} \gamma^a \psi \tag{4.20a}$$

$$\delta \psi = \not{\partial} \epsilon - i \gamma_5 \not{\epsilon} A + 3i \gamma_5 \not{\epsilon} f_m V^m - 3 \gamma_5 \Sigma_{ab} \not{\epsilon} V^a f^b \tag{4.20b}$$

$$\begin{aligned}
 \delta K^a = & i \bar{\epsilon} \gamma^a \psi + (a+3) \bar{\epsilon} \gamma_5 \Sigma^{ab} \gamma f_b + i(3+b) \bar{\epsilon} \gamma_5 \gamma f^a \\
 & + \bar{\epsilon} \sigma^{ba} V_b + (a+3) \bar{\epsilon} \gamma_5 \Sigma^{ab} \psi f_b - (3+b) i \bar{\epsilon} \gamma_5 \psi f^a \\
 & + \not{\partial} k^a - \not{\epsilon} \rho^{ba} V_b
 \end{aligned} \tag{4.20c}$$

$$\begin{aligned} \delta\gamma = & B\{ + i\gamma_5\{ A - \frac{i}{2} \gamma^a \epsilon K_a + \frac{i}{2} \gamma^a \psi k_a \\ & + a\gamma_5 \Sigma_{ab} \{ V^a \rho^b + ib\gamma_5 \{ f^m V_m \end{aligned} \quad (4.20d)$$

We can check explicitly that the action is invariant under (4.20). No ΔR are needed. It is particularly interesting to compute the variation of T using the chain rule. One obtains:

$$\delta T = i\bar{\epsilon} \gamma^a \psi \wedge V_a + d(\bar{\gamma} \epsilon - \bar{\Psi} \{ + V^a k_a) \quad (4.21)$$

which is the explicit version of Eq. (4.9) in the present case. Indeed:

$$\Lambda_\mu = \bar{\gamma}_\mu \epsilon - \bar{\Psi}_\mu \{ + V_\mu^a k_a \quad (4.22)$$

In this way we have completed the programme of reformulating the model in terms of hidden potentials. As a final remark, we point out an unsolved problem. The curvatures of Eq. (4.19) do not satisfy the Bianchi identities (4.15) even on-shell (using all equations of motion). This means that the algebra (4.20) is not closed even on-shell.

5. - CONCLUSIONS

In this article we have considered the problem of the geometrical meaning of supersymmetry invariance of an action functional.

First we recalled that, differently from what is done in the Noether method, no reference to the invariance of the action is needed to determine the Lagrangian if one works in the geometrical formalism. Instead, the Lagrangian is determined by a set of requirements which is equivalent to demanding on-shell supersymmetry invariance, namely supersymmetry invariance of the x space field equations. In the case of a theory without auxiliary fields, off-shell invariance of the action is simply an afterthought. The action is given by general principles, and in

this respect, the group manifold approach is superior to component approaches, but the transformation rules (better: the additions proportional to x space field equations) must be found, just as in the component approach, by requiring invariance of the action. The off-shell algebra has no good geometrical interpretation related to the fact that it is off-shell open. Indeed, the on-shell x space algebra is interpreted as the projection on x space solutions of the algebra of diffeomorphisms in superspace (see the Figure), the extension from x space to superspace being consistent because the Bianchi identities are on-shell satisfied. On the other hand, the off-shell algebra cannot be interpreted as the projection on x space configurations of the algebra of superspace diffeomorphisms, because Bianchi identities are not off-shell satisfied.

This situation changes in the case of theories with auxiliary fields, as exemplified in our discussion of the Sohnius-West model. We may summarize the whole discussion of that model in the following way:

- a) the Sohnius-West model is the "local theory" of an appropriate free differential algebra described by Eqs. (3.1). Relations between $R_{V\psi}$ and R_{VV} mean relations between the x and θ derivatives of the potentials, hence they allow a solution of the Cauchy problem. We stress at this point that this way of defining rheonomy is independent of the particular form of the action. Until now, rheonomy was defined in terms of field equations; the new definition of this paper is thus kinematical rather than dynamical.
- b) The action is of the form $\int_M \mathcal{L}$ where $\mathcal{L}$ depends on the p forms of the differential algebra, and M_4 is a four-dimensional surface immersed in superspace. The p -forms are not arbitrary, however, but are rheonomic, meaning that the θ dependence of the potentials can be computed once their values at $\theta = 0$ are known. The implementation proceeds as follows: one writes all curvatures as

$$R = R_{VV} V \wedge V + R_{V\psi} V \wedge \psi + \dots$$

and requires two things: (i) that Bianchi identities be satisfied, (ii) that $R_{V\psi}$ etc., be functions of R_{VV} .

- c) The action is completely determined by the following principles:
 - i) the vacuum $R^A = 0$ is a solution;
 - ii) there are also non-trivial solutions;
 - iii) the solutions of the field equations fall into the class of superfields of (b), namely those which are rheonomic.

- iv) the action is invariant under diffeomorphisms in superspace. This is the same as demanding that the value of the action does not depend on the specific choice of M^4 : this is guaranteed by the condition $d\mathbb{L} = 0$. The transformation rules are now completely geometrical; they are diffeomorphisms in superspace, with the curvature determined under (b).

In the last section, we considered another problem, namely that of reducing the local theory of a free differential algebra to the local theory of an ordinary Lie (super) algebra. We emphasized the analogy between this programme and the transition from metric gravity to vierbein gravity. It appears that in general, the p -form potentials can be replaced by composites of ordinary Lie algebra valued potentials, obtaining a Lagrangian which has a larger set of symmetries. However, it seems to be a general feature that the algebra of symmetry transformations associated with the "hidden potentials" is open in a very severe way; namely, it does not close even on-shell. The interpretation of this phenomenon is a challenging problem which requires further investigation.

We conclude this article by drawing an analogy between the concept of rheonomy of superfield potentials on superspace and the concept of analyticity of functions of a complex variable.

The analyticity of a function allows one to determine it completely in the whole complex plane, starting from its boundary value on any line. This is possible because the Cauchy-Riemann conditions relate derivatives along the line to derivatives away from it. In the same way, the reconstruction of the superfield potential from its boundary value in x space is allowed by the rheonomic conditions which relate θ derivatives to x derivatives. We are thus led to regard the rheonomy of a set of potentials as holomorphicity in superspace. Furthermore, we recall that while considering the Hilbert space of holomorphic functions, the norm in this space is defined as an integral over a line of the boundary value of $f(z)$, for instance

$$\|f(z)\| = \int_{-\infty}^{\infty} |f(z)|^2 \operatorname{Im} f(z) dx$$

This is the analogue of the action as an integral only over the surface M immersed in superspace. The independence of the integral of an analytic function on the contour is the analogue of the independence of the value of the action from the particular surface M . Thus, analytic functions are like field theories

with auxiliary fields. Non-analytic functions are like theories without auxiliary fields. One cannot, in this case, extend the function to the complex plane and at the same time satisfy the Cauchy-Riemann conditions (= rheonomy), unless the initial data on the line (the x space fields) are analytic (on-shell). In one respect, this analogy is not complete; the notion of an analytic function corresponds both to theories with auxiliary fields and to theories without auxiliary fields but which are on-shell.

TABLE : SOHNIUS-WEST NEW MINIMAL SUPERGRAVITY

1) Cartan Integrable System:

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}$$

$$R^a = DV^a - \frac{i}{2} \bar{\Psi} \wedge \gamma^a \Psi$$

$$\mathcal{G} = D\Psi - \frac{i}{2} \gamma_5 \Psi \wedge A$$

$$R^\square = dA$$

$$R^\otimes = dT - \frac{i}{2} \bar{\Psi} \wedge \gamma^m \Psi \wedge V_m$$

2) Bianchi Identities:

$$DR^{ab} = 0$$

$$DR^a + R^{ab} \wedge V_b - i \bar{\Psi} \wedge \gamma^a \mathcal{G} = 0$$

$$D\mathcal{G} - \frac{i}{2} \gamma_5 \mathcal{G} \wedge A - \frac{i}{2} R^{mn} \wedge \Sigma'_{mn} \Psi - i \gamma_5 \Psi \wedge R^\square = 0$$

$$dR^\square = 0$$

$$dR^\otimes - i \bar{\Psi} \wedge \gamma_m \mathcal{G} \wedge V^m + \frac{i}{2} \bar{\Psi} \wedge \gamma_m \Psi \wedge R^m = 0$$

3) Off-shell Rheonomic Curvatures:

$$R^a = 0$$

$$\mathcal{G} = \mathcal{G}_{ab} V^a \wedge V^b + 3i \gamma_5 \Psi f_a V^a - 3 \gamma_5 \Sigma'_{ab} \Psi V^a f^b$$

$$R^\square = F_{mn} V^m \wedge V^n - i \bar{\Psi} \wedge \gamma_5 \gamma_a \Sigma'^{mn} S_{mn} V^a$$

$$R^{ab} = R^{ab}_{mn} V^m \wedge V^n - 2i \bar{\Psi} \wedge \gamma^{[a} \gamma^{b]} \mathcal{G} V_m - i \bar{\Psi} \wedge \gamma_m \mathcal{G} V^{ab} - \frac{3i}{2} \epsilon^{abmn} f_{mn} \bar{\Psi} \wedge \gamma_m \Psi$$

$$R^\otimes = f_a V_b \wedge V_c \wedge V_d \epsilon^{abcd}$$

4) Action (Invariant under Off-shell Transformations)

$$\begin{aligned}
 I = & \int \left\{ R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} + 4 \bar{\Psi} \wedge \gamma_5 \gamma_a \Psi \wedge V^a \right. \\
 & \left. - 8 R^a \wedge T + \right. \\
 & \left. + 36 \left[f_a R^a \wedge V + \frac{1}{8} f_a f^a \epsilon_{ijkl} V^i \wedge V^j \wedge V^k \wedge V^l \right] \right\}
 \end{aligned}$$

REFERENCES

- 1) R. D'Auria, P. Fré and T. Regge in "Supergravity '81", Eds. S. Ferrara and J.C. Taylor, Cambridge University Press (1982) and references therein.
- 2) P. van Nieuwenhuizen, Erice Lectures 1982.
- 3) This article extends an Ecole Normale preprint (unpublished) by the last two authors, in which $N = 1$ new minimal supergravity was studied.
- 4) R. D'Auria, P. Fré, E. Maina and T. Regge, Ann. Phys. 139 (1982) 93.
- 5) M. Sohnius and P. West, Phys. Lett. 105B (1981) 353.
- 6) R. D'Auria and P. Fré, Nucl. Phys. B201 (1982) 101.
- 7) L. Castellani, P. Fré, F. Giani, K. Pilch and P. van Nieuwenhuizen, ITP/SB 82-18, to appear in Annals of Physics.
- 8) D. Sullivan, Bulletin de l'Institut des Hautes Etudes Scientifiques, Publications Mathématiques No. 47.
- 9) P. van Nieuwenhuizen, Proceedings of the 1982 Istanbul Meeting on Group-Theoretical Methods in Physics.
- 10) P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 351.
- 11) E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 79B (1978) 231.

FIG. 1