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SOLUTION GENERATING TECHNIQUE FOR SELF-DUAL MONOPOLES

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A B S T R A C T

A solution generating method based on the linearization of the self-duality (Bogomolny) equations of $SU(N)$ gauge theories is described in detail. We point out its connection to matrix Riemann-Hilbert problems and to the Atiyah-Ward ansätze. We show how the multi-monopoles can be constructed with our method. We also investigate the energy density of the separated $SU(2)$ two monopole solutions.

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1. - INTRODUCTION

In the last two years we witnessed a breakthrough in the construction of monopole solutions of spontaneously broken $SU(N)$ gauge theories in the special limit when the mass of the Higgs field vanishes. In this case, to find static purely magnetic solutions, one is up against the Bogomolny equations¹⁾ which are equivalent to the Euclidean self-duality equations when all fields are independent of "Euclidean time".

The first successes were scored in the $N = 2$ case where two different lines of attack proved to be useful: the Atiyah-Ward (AW) ansätze²⁾ as initiated by Ward³⁾ and generalized further by Prasad and Rossi as well as by Corrigan and Goddard⁴⁾ and soliton theoretic ideas [Bäcklund transformation (BTs) and "inverse scattering methods"] applied by the present authors⁶⁾.

Our soliton theoretic methods are based on the fact that the Bogomolny (self-duality) equations can be linearized. This linearizability property, which is the existence of an associated linear system, whose integrability conditions are the original non-linear equations (in our case the self-duality equations) lies at the heart of all known two-dimensional completely integrable models. For these models various solution generating techniques have been found, e.g., BTs and an "inverse scattering" method (Riemann-Hilbert problem)⁷⁾.

The BTs that worked so well for generating the axially symmetric $SU(2)$ monopoles were generalized by Bais and Sasaki for an arbitrary gauge group⁸⁾ but nothing is known about BTs for the non-axially symmetric equations yet. The AW ansätze have also been generalized for groups other than $SU(2)$ and new axially symmetric monopole solutions were found in an $SU(3)$ gauge theory which are different from the simple $SU(2)$ embeddings⁹⁾.

The aim of this paper is to give a detailed treatment of our "inverse scattering" method [given in bare outline in Ref. 6)]. We set up a Riemann-Hilbert problem for the self-duality equations and derive a huge family of solutions which is constructed algebraically from a known solution. We work out in detail the method for an $SU(N)$ gauge group. Furthermore, we discuss its relation to the AW ansätze to point out similarities and differences between the two approaches. We argue that our method can be successfully applied to generate all $SU(N)$ monopoles.

The advantage of our method is that the solutions we obtain are completely explicit and the construction is algebraic. The main drawback is that the solutions are local, that is their regularity is not automatically guaranteed. (This applies to the AW ansätze as well.)

The Atiyah-Drinfeld-Hitchin-Manin construction as applied by Nahm¹⁰⁾ (ADHMN) solves this problem as regularity is guaranteed, however, to get explicit solutions it is simpler to use the other methods. An important problem would be to relate these different techniques as they seem to complement each other rather nicely.

We also studied the energy density of two separated SU(2) monopoles using our exact solution. This is important to get a physical picture of what these solutions represent. Our results show that when the monopoles are far enough they slightly distort each other but otherwise can be considered as a "naive" superposition of two monopoles of charge one.

Our paper is organized as follows: we introduce our notations and set up the problem in Section 2. In Section 3 we describe in detail the solution generating method. Its connection to a matrix Riemann-Hilbert problem and to the AW ansätze is elaborated in Section 4. The three subsections of Section 5 contain the detailed application of the method to the construction of various monopoles. We study the energy density of the SU(2) two monopole in Section 6.

2. - STATEMENT OF THE PROBLEM

We consider an SU(N) gauge theory with scalar fields in the adjoint representation in the limit of vanishing Higgs potential. The Lagrangian density is

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} (D_\mu \Phi)^a (D^\mu \Phi)^a \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$$

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a - f^{abc} A_\mu^b \Phi^c$$

(We choose the coupling constant $g = 1$.)

The Hamiltonian for static configurations with no electric fields ($A_0^a = 0$) is

$$H = \frac{1}{4} F_{ij}^a F^{a ij} + \frac{1}{2} (D_i \Phi)^a (D_i \Phi)^a \quad (2)$$

The field equations of this theory are solved by configurations satisfying the Bogomolny equations¹⁾:

$$F_{ij}^a = -\varepsilon_{ijk} (D_k \Phi)^a \quad (3)$$

For solutions of (3) the energy can be written as

$$E = \int H d^3x = \int (D_\mu \Phi^a)(D_\mu \Phi^a) d^3x = \frac{1}{2} \int \Delta (\Phi^a \Phi^a) d^3x = \int \Delta \text{Tr} \Phi^2 d^3x \quad (4)$$

So the energy can be calculated from the Higgs field alone. [In Eq. (4) we introduced $\Phi = \Phi^a (t_a/2)$ where t_a are the $SU(N)$ generators.]

In the absence of a scalar potential in (1), the symmetry breaking is imposed on the solutions of (3) as a boundary condition at infinity. Without loss of generality we may assume that asymptotically (i.e., $r \rightarrow \infty$) $\Phi = \text{diag}(\Lambda_1, \dots, \Lambda_N)$: $\sum_{i=1}^N \Lambda_i = 0$. Furthermore, to obtain monopole solutions we demand that

$$2 \text{Tr} \Phi^2 = 2 \sum_{i=1}^N \Lambda_i^2 - \frac{p}{r} + O(1/r^2) \quad , \quad r \rightarrow \infty \quad (5)$$

as p determines the energy of the solution via (4) - provided that $\text{Tr} \Phi^2$ is regular everywhere. On the other hand, from a study of the topological properties of the solutions of (3)^{11),12)}, we know that p is connected to certain integers describing these properties in the following way: choose a gauge in which the Higgs field on the positive x_3 axis asymptotically has the form:

$$\Phi = \text{diag}(\Lambda_1, \dots, \Lambda_N) - \frac{1}{2x_3} \text{diag}(n_1, n_2 - n_1, n_3 - n_2, \dots, -n_{N-1}) + O\left(\frac{1}{x_3^2}\right) \quad (6)$$

The numbers n_i ($i = 1, \dots, N-1$) are always integers¹²⁾. For the $SU(2)$ case there is just one n and it gives the magnetic charge of the solution. For $N > 2$ the question of what n_i - s give topological charges is slightly more complicated¹¹⁾, however, for our purposes it is enough to note that (6) implies

$$p = 2 \sum_{i=1}^N \Lambda_i (n_i - n_{i-1}) \quad n_0 = n_N = 0 \quad (7)$$

Therefore, in $SU(N)$ a solution of (3) describes a regular monopole if $\text{Tr} \Phi^2$ is free of singularities and Φ satisfies (5) and (7).

Consider a pure $SU(N)$ gauge theory with Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

The Euclidean space field equations are solved by self-dual configurations which satisfy

$$F_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\zeta\sigma} F^{a\zeta\sigma}$$

In the case when all fields A_μ^a are independent of the Euclidean time the self-dual equations (SDE) reduce to

$$F_{ij}^a = -\epsilon_{ijk} (D_k A_0)^a$$

where we recognize the Bogomolny equations reinterpreting A_0^a as the Higgs field.

Introducing now the complex co-ordinates $y_1 = x_1 + ix_2$, $z = x_3 + ix_4$, $\bar{y} = x_1 - ix_2$, $\bar{z} = x_3 - ix_4$ and matrix notation ($F_{\mu\nu} = F_{\mu\nu}^a(t_a/2)$) the SDE take the form

$$F_{yz} = F_{\bar{y}\bar{z}} = 0 \quad ; \quad F_{y\bar{y}} + F_{z\bar{z}} = 0$$

The first two equations are solved by

$$B_y = -D_y D^{-1} \quad B_{\bar{y}} = D^{+ -1} D_{\bar{y}}^+$$

where $D \in SL(N, \mathbb{C})$ and $B_y = \frac{1}{2}(A_1 - iA_2)$, etc. Introducing the gauge invariant quantity

$$g = D^+ D \quad (8)$$

the last equation reduces to

$$(g_{,y} g^{-1})_{,\bar{y}} + (g_{,z} g^{-1})_{,\bar{z}} = 0 \quad (9)$$

It is easy to see that g is a Hermitian matrix with unit determinant. The crucial observation is that Eq. (9) is the compatibility condition for the following linear system⁶⁾:

$$(\lambda \partial_{\bar{z}} + \partial_y) \Psi = A_y \Psi \quad ; \quad (-\lambda \partial_{\bar{y}} + \partial_z) \Psi = A_z \Psi \quad (10)$$

where $A_y = g_y g^{-1}$, $A_z = g_z g^{-1}$ and $\Psi = \Psi(\lambda, y, \bar{y}, z, \bar{z})$ is a $N \times N$ matrix for which we impose the boundary condition $\Psi(\lambda=0, y, \bar{y}, z, \bar{z}) = g$.

3. - A METHOD FOR PRODUCING SOLUTIONS OF THE SDE

In what follows we show in detail how one can construct solutions of the linear system (10) which enables us to generate the multimonopoles. For our construction to work we need the following input: a known solution of (9) denoted by g_0 and the corresponding solution of the linear system (10), $\psi_0(\lambda)$, satisfying $\psi_0(\lambda=0) = g_0$.

We look for solutions of (10) in the following form:

$$\Psi(\lambda) = \chi(\lambda) \Psi_0(\lambda) \quad (11)$$

and the other crucial input is the following ansatz for $\chi(\lambda)$:

$$\chi(\lambda) = I + \sum_{k=1}^n \frac{R_k}{\lambda - \mu_k} \quad (12)$$

where R_k 's are $N \times N$ matrices and μ_k 's are functions independent of λ . In other words $\chi(\lambda)$ is a meromorphic function in the complex λ plane. The ansatz (12) is motivated by the work on two-dimensional completely integrable models (Sine-Gordon, KdV, etc.), for a good review see Ref. 7).

Substituting (11) into (10) we get the following linear system for $\chi(\lambda)$:

$$\begin{aligned} (\lambda \partial_{\bar{z}} + \partial_y) \chi \cdot \chi^{-1} + \chi A_y^0 \chi^{-1} - A_y \\ (-\lambda \partial_{\bar{y}} + \partial_z) \chi \cdot \chi^{-1} + \chi A_z^0 \chi^{-1} - A_z \end{aligned} \quad (13)$$

Here we introduced the obvious notation $A_z^0 = g_{0,z} g_0^{-1}$, $A_y^0 = g_{0,y} g_0^{-1}$. Our ansatz (12) for $\chi(\lambda)$ immediately implies that

$$R_k \chi^{-1}(\mu_k) = 0 \quad (14)$$

from which we see that

$$\begin{aligned} R_k = \sum_{j_k=1}^{s_k} n^{(k,j_k)} o m^{(k,j_k)}; \quad \chi^{-1}(\mu_k) = \sum_{\ell_k=s_k+1}^N q^{(k,\ell_k)} o p^{(k,\ell_k)} \\ m^{(k,j_k)} \cdot q^{(k,\ell_k)} = \sum_{a=1}^N m_a^{(k,j_k)} q_a^{(k,\ell_k)} = 0, \quad \forall j_k, \ell_k \end{aligned} \quad (15)$$

Here the integers s_k ($1 \leq s_k \leq N-1$) denote the dimensions of the subspaces upon which R_k project.

The left-hand side of (13) would contain terms with second and first order poles at $\lambda = \mu_k$ whereas the right-hand side of (13) is analytic in λ . Therefore, we immediately get the following equations (the residues must vanish at the pole sites): absence of second order poles:

$$\mu_k \partial_{\bar{z}} \mu_k + \partial_y \mu_k = 0; \quad -\mu_k \partial_{\bar{y}} \mu_k + \partial_z \mu_k = 0$$

The general solution of this pole equation is

$$h(\mu_k y - \bar{z}, \mu_k \bar{z} + y, \mu_k) = 0 \quad (16)$$

where h is an arbitrary (nice) function.

Absence of first order poles:

$$\begin{aligned} (\mu_k R_{k,\bar{z}} + R_{k,y}) \chi^{-1}(\mu_k) + R_k A_y^0 \chi^{-1}(\mu_k) &= 0 \\ (-\mu_k R_{k,y} + R_{k,\bar{z}}) \chi^{-1}(\mu_k) + R_k A_z^0 \chi^{-1}(\mu_k) &= 0 \end{aligned} \quad (17)$$

Using now Eqs. (15) we obtain that the vectors $m^{(k,j_k)}$ satisfy

$$[(\mu_k \partial_{\bar{z}} + \partial_y) m^{(k,j_k)} + m^{(k,j_k)} A_y^0] q^{(k,l_k)} = 0 \quad (18)$$

plus a similar equation with appropriately changing y and z . Observing that $\psi_0^{-1}(\mu_k)$ satisfies

$$-(\mu_k \partial_{\bar{z}} + \partial_y) \psi_0^{-1}(\mu_k) = \psi_0^{-1}(\mu_k) A_y^0 \quad \text{etc.}$$

a solution of (18) can be constructed using ψ_0^{-1} :

$$m_b^{(k,j_k)} = M_c^{(k,j_k)} \psi_0^{-1}(\mu_k)_{cb} \quad (19)$$

where

$$M^{(k,j_k)} = M^{(k,j_k)}(\mu_k y - \bar{z}, \mu_k \bar{z} + y, \mu_k)$$

but are otherwise arbitrary vectors. The next problem is to determine the $n^{(k,j_k)}_s$. For this we need a new equation, which we obtain exploiting certain properties of (13) and the hermiticity of g and g_0 . Namely we observe that if $\chi(\lambda)$ satisfies Eqs. (13) then

$$\hat{\chi}(\lambda) = g \chi^{+ -1}(-\frac{1}{\bar{\lambda}}) g_0^{-1}$$

satisfies the same equation; therefore we may take $\hat{\chi}(\lambda) = \chi(\lambda)$ yielding

$$g = \chi(\lambda) g_0 \chi^+(-\bar{\lambda}^{-1}) \quad (20)$$

As a consequence, we see that $X(\infty) = I$ and if $X(\lambda)$ has a pole at $\lambda = \mu$ then $X^{-1}(\lambda)$ has a pole at $\lambda = -\bar{\mu}^{-1}$.

The equations determining the $n^{(k,j_k)}_s$ are obtained by taking the residues of Eq. (20) at $\lambda = -\bar{\mu}_k^{-1}$:

$$\sum_{k=1}^n \sum_{j_k=1}^{s_k} n_a^{(k,j_k)} \frac{m_d^{(k,j_k)}(g_0)_{dc} \bar{m}_c(t, l_t)}{1 + \mu_k \bar{\mu}_t} \equiv$$

$$\equiv \sum_{k=1}^n \sum_{j_k=1}^{s_k} n_a^{(k,j_k)} \Gamma^{(k,j_k)}(t, l_t) = \frac{1}{\bar{\mu}_t} (g_0)_{ab} \bar{m}_b(t, l_t) \quad (21)$$

From (21) it is trivial to express $n^{(k,j_k)}_s$:

$$n_a^{(k,j_k)} = \sum_{t, l_t} \frac{1}{\bar{\mu}_t} (g_0)_{ab} \bar{m}_b(t, l_t) \Gamma^{-1}(t, l_t; k, j_k)$$

One should keep in mind, however, that the existence of Γ^{-1} is not completely self-evident. So, apart from this problem, we have now constructed a new solution, $\psi(\lambda)$ of our linear system and the final expression for $g = X(\lambda=0)g_0$ can be written as

$$g_{ab} = (g_0)_{ab} - \sum_{k,j_k,t,l_t} (\mu_k \bar{\mu}_t)^{-1} \Gamma^{-1}(t, l_t; k, j_k) \bar{N}_a^{(t,l_t)} N_b^{(k,j_k)} \quad (22)$$

where $N_a^{(k,j_k)} = m_b^{(k,j_k)}(g_0)_{ba}$. The matrix g we have just constructed indeed solves Eq. (9), still there is a rub. There is nothing that guarantees that $\det g = 1$. In fact

$$\det g = \det g_0 \prod_{k=1}^n (-\mu_k \bar{\mu}_k)^{-s_k} \quad (23)$$

The calculation of $\det g$ is not entirely trivial, so we outline the main steps of the derivation. It relies on the observation⁷⁾ that putting n poles in (12) is formally equivalent to putting the poles successively one by one. Therefore, for the moment consider only the simplest case, when we have only one pole in (12). (We say that we make one step.) Then $\psi(\lambda)$ can be written as

$$\psi(\lambda) = \left(I + \frac{1 + \mu \bar{\mu}}{(\lambda - \mu) \bar{\mu}} P \right) \psi_0(\lambda) \quad (24)$$

where

$$P = \sum_{i,j=1}^s (g_0 \bar{m})^j{}_0 m^i (\gamma^{-1})^j{}_i$$

$$\gamma^i{}_j = m^i{}_a (g_0)_{ab} \bar{m}^j{}_b$$

It is easy to see that P is an s dimensional projector: $PP = P$, $\text{Tr } P = s$, $\det P = 0$. It is not difficult to calculate the determinant of $(1 + \alpha P)$ since we can diagonalize P , so

$$\det(1 + \alpha P) = (1 + \alpha)^s$$

Using this result we get for $\det \psi$:

$$\det \psi(\lambda) = \left(\frac{\lambda + \bar{\mu}^{-1}}{\lambda - \mu} \right)^s \det \psi_0$$

So, indeed, after one step the determinant of $\psi(\lambda)$ changes. Fortunately, if $\psi(\lambda)$ solves (10) then $\tilde{\psi} = (\det \psi)^V \psi(\lambda)$ also solves (10) with $\tilde{g} = (\det g)^V g$. To see this it is sufficient to note

$$(\lambda \partial_{\bar{z}} + \partial_y) \ln \det \psi = (\ln \det g)_{,y}$$

$$(-\lambda \partial_{\bar{y}} + \partial_z) \ln \det \psi = (\ln \det g)_{,z}$$

It is obvious how to extend the above calculation of $\det \psi$ for n steps and then one really gets (23) for $\det g$. Defining the "physical" g, g_p as

$$g_p = g \prod_{k=1}^n |\mu_k|^{-\frac{2s_k}{N}} \quad (25)$$

it still solves (9) and $\det g_p = (-1)^{\sum_k^n s_k} \det g_0$. Therefore, if we start with a g_0 having $\det g_0 = 1$ then if $\sum_k s_k$ is even we get $\det g_p = 1$, however, if $\sum_k s_k$ is odd we get $\det g_p = -1$ which is a problem. One would conclude that in the latter case one should start with a seed solution with $\det g_0 = -1$. For a very important special case there is another possibility that we would need later on. Namely, we generate the monopoles starting with a diagonal g_0 . In this case one can readily check that multiplying g_0 with a suitable (constant) diagonal matrix, C , we can change the sign of $\det g_0$ without changing its hermiticity properties. This trick corresponds to the formal substitution $D_0 \rightarrow D_0, D_0^+ \rightarrow CD_0^+$.

Now it is very easy to adopt the above results for generating solutions of Bogomolny equations, which in our formalism look like:

$$(g_y g^{-1})_{,\bar{y}} + (g_z g^{-1})_{,z} = 0 \quad (26)$$

where $y = \frac{1}{2}(x_1 + ix_2)$, $z = x_3$. In this case the length of the Higgs field is determined by g as $2\text{Tr } \phi^2 = \frac{1}{2} \text{Tr}(g_z g^{-1})^2$.

The linear equations corresponding to (26) are:

$$(\lambda \partial_z + \partial_y) \psi = g_y g^{-1} \psi ; \quad (-\lambda \partial_{\bar{y}} + \partial_z) \psi = g_z g^{-1} \psi \quad (27)$$

The formula for g after n steps is the same as in (22) and (25), only the characteristic variables, and correspondingly the poles change a bit:

$$M^{(k, j_k)} = M^{(k, j_k)}(\gamma(\mu_k), \mu_k); \quad \gamma(\lambda) = y\lambda - \frac{\bar{y}}{2} - z \quad (28)$$

and the μ_k 's now satisfy:

$$h(\gamma(\mu), \mu) = 0 \quad (29)$$

It is a remarkable fact that we are able to construct a new family of solutions of the Bogomolny (SDE) equations purely algebraically from a given solution. The only analytic work to be done is to solve (27) to find $\psi_0(\lambda)$, which is not very difficult for a sufficiently simple seed solution.

At this stage we would like to connect our approach with Riemann-Hilbert problems (RHP) and then with the twistor approach of Atiyah-Ward.

4. - A RIEMANN-HILBERT PROBLEM ASSOCIATED WITH THE SDE

First we define the regular matrix RHP.

Consider a closed curve Γ in the complex λ plane. Define on Γ a matrix $G(\lambda)$ analytic in an annular neighbourhood of Γ and find $X_1(\lambda)$ and $X_2(\lambda)$ analytic inside (respectively outside) Γ satisfying

$$X_1^{-1}(\lambda) X_2(\lambda) = G(\lambda) ; \quad \lambda \in \Gamma \quad (30)$$

We assume that $\det X_i \neq 0$ in their domains of analyticity. A simple way to connect our linear system (10) with the RHP is the following: for any two solutions ψ_1, ψ_2 of (10):

$$(\lambda \partial_{\bar{z}} + \partial_y)(\psi_1^{-1} \psi_2) = 0 ; \quad (-\lambda \partial_{\bar{y}} + \partial_z)(\psi_1^{-1} \psi_2) = 0 \quad (31)$$

that is

$$\psi_2 = \psi_1 G(\lambda, -\lambda y + \bar{z}, \lambda z + \bar{y})$$

Noting that if $\psi(\lambda)$ solves (10) then $g \psi^{+-1}(-\bar{\lambda}^{-1})$ also solves (10), so we have an involution acting in the solution space of (10) (we used the fact that $g = g^+$), thus

$$g \psi^{+-1}(-\bar{\lambda}^{-1}) = \psi(\lambda) G(\lambda, -\lambda y + \bar{z}, \lambda z + \bar{y}) \quad (32)$$

If ψ is analytic around $\lambda = 0$ then $g \psi^{+-1}(-\bar{\lambda}^{-1})$ is analytic near $\lambda = \infty$, therefore, G is analytic in the overlapping region of the two domains.

Let us now suppose that we are given a non-singular $N \times N$ matrix G (i.e., $\det G \neq 0$) satisfying

$$\begin{aligned} (\lambda \partial_{\bar{z}} + \partial_y) G &= \nabla_1 G = 0 & G(\lambda) &= G^+(-\bar{\lambda}^{-1}) \\ (-\lambda \partial_{\bar{y}} + \partial_z) G &= \nabla_2 G = 0 \end{aligned} \quad (33)$$

in an annular neighbourhood of the unit circle. Split G according to (30) into the product of two matrices with non-vanishing determinants analytic inside (respectively outside) the circle, with the additional property that $\chi_2(\lambda)$ is normalized to I at $\lambda = \infty$. This normalization is needed to ensure the uniqueness of the solution. Acting now with ∇_i ($i = 1, 2$) on Eq. (30) we get

$$\nabla_i \chi_1 \cdot \chi_1^{-1} = \nabla_i \chi_2 \cdot \chi_2^{-1} \quad \lambda \in \Gamma \quad (34)$$

Equation (34) defines the analytic continuation of both sides from Γ to the whole complex plane. As χ_2 is normalized to I at infinity, by Liouville's theorem both sides are independent of λ . Therefore

$$\nabla_i \chi_1 \cdot \chi_1^{-1} = \nabla_i \chi_2 \cdot \chi_2^{-1} = A_i \quad (35)$$

where the matrices A_i do not depend on λ . The compatibility of (35) implies $A_y = g_y g^{-1}$, $A_z = g_z g^{-1}$ with g satisfying (9). It is easy to see that $G(\lambda) = G^+(-\bar{\lambda}^{-1})$ guarantees that $\chi_1^{-1}(\lambda) = \chi_2^+(-\bar{\lambda}^{-1}) g^{-1}$ with a Hermitian g and on account of $\chi_2(\infty) = I$ we also have $\chi_1(0) = g$. This way from the RHP we indeed get solutions of (10).

To get all solutions of the SDE one should consider a RHP with zeros, that is one should allow $\det X_i = 0$ in their domains of analyticity. The regular RHP is equivalent to the following linear singular integral equation:

$$\frac{1}{\pi i} \left[P \int_{\Gamma} \frac{\sigma(\xi)}{\xi - \lambda} d\xi + 1 \right] T(\lambda) + \sigma(\lambda) = 0, \quad \lambda \in \Gamma \quad (36)$$

where $T(\lambda) = [G(\lambda) - I][G(\lambda) + I]^{-1}$, and we used the following integral representations for $X_i(\lambda)$'s:

$$X_1(\lambda) = I + \int_{\Gamma} \frac{\sigma(\xi)}{\xi - \lambda} d\xi \quad \text{inside } \Gamma \quad (37)$$

$$X_2(\lambda) = I + \int_{\Gamma} \frac{\sigma(\xi)}{\xi - \lambda} d\xi \quad \text{outside } \Gamma$$

The RHP with zeros is equivalent to a system of linear integral equations coupled to algebraic equations determining the residues of $X_i(\lambda)$'s at the pole sites⁷⁾.

If we take in (30) $G = \psi_0^{-1}(\lambda) G_0 \psi_0(\lambda)$ where $\psi_0(\lambda)$ is a solution of (10) and $G_0 = G_0(\lambda, -\lambda y + \bar{z}, \lambda z + \bar{y})$; $G_0(\lambda) = G_0^+(-\bar{\lambda}^{-1})$; $\det G_0 = 1$ then acting on the solutions of this RHP, $\gamma_1^{-1}(\lambda) \gamma_2(\lambda) = \psi_0^{-1} G_0 \psi_0$ with ∇_i ($i = 1, 2$) we obtain

$$\nabla_i \gamma_1 \cdot \gamma_1^{-1} + \gamma_1 A_i^0 \gamma_1^{-1} = \nabla_i \gamma_2 \cdot \gamma_2^{-1} + \gamma_2 A_i^0 \gamma_2^{-1} \quad \lambda \in \Gamma \quad (38)$$

Using again Liouville's theorem from (38) we conclude that γ_1 solves (13), i.e., from this RHP we obtain a solution of (10) in the form $\psi(\lambda) = \gamma_1(\lambda) \psi_0(\lambda)$.

The construction described in the first part of this paper corresponds to solving the RHP with zeros only. This means that we take $G_0 \equiv 1$, and construct a meromorphic $X(\lambda)$ with simple poles only. These solutions are referred to as "purely solitonic" in analogy with the known two-dimensional models. Here it is just a terminology.

It is not known how to solve the RHP explicitly for an arbitrary matrix G . However, in some special case, when G is diagonal, or has an appropriate triangular form, or $G = \psi_0^{-1} (I + v) \psi_0$ with v being infinitesimal, the splitting can be done explicitly. Clearly a diagonal G leads to several RHP's for ordinary functions that can be tackled by conventional methods¹³⁾. It is straightforward to obtain an infinitesimal solution (i.e., first order in v) of the integral

equation (36) with $G = I + v$. It is extremely useful for studying the group theoretical aspects of Eqs. (9) and (10)^{6),14)}; in particular, it can be used to show the existence of an infinite parameter invariance group of the SDE¹⁵⁾. The splitting of a G having the form

$$G = \begin{pmatrix} \lambda^k & 0 \\ s & \lambda^{-k} \end{pmatrix}; \quad s = s(\lambda, -\lambda y + \bar{z}, \lambda z + \bar{y}); \quad A(\lambda) \cdot B(\lambda) = G$$

was carried out in detail by Corrigan et al.¹⁶⁾ (with a certain boundary condition at $\lambda = \infty$) and later the procedure was generalized⁹⁾ for higher dimensional triangular matrices whose diagonal contains only powers of λ . As a triangular G certainly does not satisfy $G(\lambda) = G^+(-\bar{\lambda}^{-1})$ we can use this method to solve our RHP only in those cases when we can find non-singular matrices M_1, M_2 , analytic inside (respectively outside) Γ satisfying $\nabla_j M_i = 0$ such that $M_1 G M_2 = \tilde{G}$; $\tilde{G}(\lambda) = \tilde{G}^+(-\bar{\lambda}^{-1})$, (i.e., when G is equivalent to a $\tilde{G}$ possessing this property). A further delicate point in the application of the method comes from the difference in the boundary conditions at $\lambda = \infty$; in our case the free functions arising during the splitting procedure are determined from the requirement

$$B M_2 \xrightarrow{\lambda \rightarrow \infty} I \quad (39)$$

Then the solution of Eq. (9) is given as $(M_1(\lambda=0) A(\lambda=0))^{-1}$. For completeness for the $N = 2$ case we give the expressions relevant for this splitting as well as their application for the one monopole solution in an Appendix.

Everything we said so far about the RHP applies to the Bogomolny equations written in the form of (28) and (29) as well if we assume that $G = G(\lambda, \gamma(\lambda))$.

Next we would like to compare our method to the Atiyah-Ward ansätze²⁾. These ansätze also correspond to the splitting of a matrix, H , of the aforementioned triangular form, however, the linear system they correspond to is slightly different from ours:

$$\nabla_1 \phi = -(\lambda B_{\bar{z}} + B_y) \phi, \quad \nabla_2 \phi = -(-\lambda B_{\bar{y}} + B_z) \phi \quad (40)$$

Here ϕ is an $N \times N$ matrix depending on $\lambda, y, \bar{y}, z, \bar{z}$ and the B 's are the gauge vector components thus $\phi(\lambda=0) = D$. The involution acting on the solutions (40) is $\phi^{+-1}(-\bar{\lambda}^{-1}) = \phi(\lambda) \tilde{H}$ so $\tilde{H}^+(-\bar{\lambda}^{-1}) = \tilde{H}(\lambda)$. Therefore again, the only H 's which are interesting are those which are equivalent to a $\tilde{H}$ with this property; however, if this is the case, then the gauge vectors obtained from splitting H or $\tilde{H}$ are related by a gauge transformation.

Using the definition of g it is easy to establish the relation between the solutions of (40) and (10):

$$\phi = D^{+-1} \psi \quad (41)$$

As D^{+-1} is independent of λ clearly the analytic properties of ϕ and ψ must be the same. Since from the method we presented in the previous chapter we always obtain a meromorphic ψ and in the Atiyah-Ward ansätze one always has an analytic ϕ the relation between the two methods is more complicated than the simple equation (41). In fact, in our method we take $H_0 = \psi_0^{-1}(\lambda) g_0 \psi_0^{+-1}(-\bar{\lambda}^{-1})$ fixed and construct a meromorphic function ψ with n simple poles such that $\psi^{-1}(\lambda) g \psi^{+-1}(-\bar{\lambda}^{-1}) = H_0$. The connection to the twistor method is the following: if there exist k_1 and k_2 matrices satisfying $\nabla_j k_i = 0$ such that $k_1 \psi^{-1} (g \psi^{+-1}(-\bar{\lambda}^{-1}) k_2)$ are analytic inside (respectively outside) Γ and their determinants do not vanish in their domains of analyticity and $\tilde{H} = k_1 H_0 k_2 = \tilde{H}^+(-\bar{\lambda}^{-1})$ then $\tilde{H}$ is the patching matrix of a holomorphic vector bundle over CP^3 and the ϕ obtained by splitting $\tilde{H}$ and $(k_1 \psi^{-1})^{-1}$ are indeed related by Eq. (41).

5. - CONSTRUCTING MONOPOLE SOLUTIONS

The method we described in Section 3 has a lot of applications. So far using it we constructed⁶⁾ the 5N parameter multi-instanton solution of 't Hooft and Witten and in a recent paper it enabled us to establish an iteration rule for the superpotential of axially symmetric $SU(N)$ monopoles¹⁷⁾. Quite recently Chakrabarti¹⁸⁾ - using methods similar to ours - described a framework capable of generating "instanton chains" that give the $SU(2)$ monopoles when the number of instantons tends to infinity.

Now we apply our formalism to construct explicitly the monopoles. First we show how one can reduce Eqs. (26) and (27) to the axially symmetric case and construct the $SU(2)$ monopoles with this symmetry. Then we derive expressions for the most general $SU(2)$ monopoles and set the framework for the construction of $SU(N)$ monopoles.

In the case of the axial symmetry $g = g(\rho, z)$ ($\rho = \sqrt{x_1^2 + x_2^2}$), however, imposing it on our linear system is not entirely trivial as $\psi(\lambda)$ depends on the polar angle φ in the following way: $\psi(\lambda, \rho, z, \varphi) = \psi(\lambda e^{\varphi}, \rho, z)$. Now introducing $\tilde{\lambda} = \lambda e^{\varphi}$ we obtain from (27) a linear eigenvalue problem for $\psi(\tilde{\lambda}, \rho, z)$:

$$\begin{aligned}\tilde{\lambda} \psi_{,z} + \psi_{,\bar{s}} + \frac{\tilde{\lambda}}{s} \psi_{,\tilde{\lambda}} &= A_s \psi \\ -\tilde{\lambda} \psi_{,\bar{s}} + \frac{\tilde{\lambda}^2}{s} \psi_{,\tilde{\lambda}} + \psi_{,z} &= A_z \psi\end{aligned}\tag{42}$$

It is easy to show that (42) is equivalent to the linear eigenvalue problem for the Ernst equation of Belinsky and Zakharov¹⁹⁾ in the SU(2) case, when ψ is a 2×2 matrix. It is important to note that in the axially symmetric case the pole equations have a unique solution:

$$\mu = \frac{-w+z - \sqrt{(w-z)^2 + s^2}}{2y} = \frac{-w+z - R(w)}{s} e^{-i\varphi} = \frac{\tilde{\mu}}{s} e^{-i\varphi}\tag{43}$$

where w is an arbitrary constant. Generally, the w -s are determined from imposing regularity. This μ corresponds to the following choice of the function $h(\gamma(\mu), \mu)$ in the pole Eq. (29):

$$h \equiv \gamma + w = 0\tag{44}$$

A further simplification arises from the fact that in the axial symmetric case the $M_a^{(k,j_k)}$ vectors are independent of ρ and z .

5.1 SU(2) monopoles with axial symmetry

If we suppose in this case that g is a real symmetric 2×2 matrix then (9) is nothing but the Ernst equation of general relativity, describing stationary axisymmetric vacuum space-times. As it was shown in Ref. 20), axial symmetry strongly reduces the degrees of freedom for finite energy SU(2) monopoles, so the only possible configurations are superimposed monopoles at a single point. This means that the number of parameters for this case is five. We remark here that when g is Hermitian it is only seemingly more general than the real symmetric case.

To generate the monopoles we now have to specify the seed solution g_0 , and solve (27) to find $\psi_0(\lambda)$. It is natural to choose a Higgs vacuum, which is $\Sigma_a \phi_a^2 \equiv 1$ [$\Lambda_1 = -\Lambda_2 = 1/2$ in (5)] and $F_{ij} \equiv 0$. We made the following choice for g_0 :

$$g_0 = \text{diag} (e^z, e^{-z})\tag{45}$$

However, this choice is not unique, that is, there are other g_0 's which also correspond to the Higgs vacuum but not all of them give the same result when used to construct monopoles. There is no problem in solving (27) for $\psi_0(\lambda)$:

$$\psi_0(\lambda) = \text{diag} \left(e^{z-\lambda y}, e^{-z+\lambda y} \right) \quad (46)$$

Using the fact that μ in (43) satisfies $\mu_{,z} = -\mu/R(w)$ and that for $N = 2$ all s_k 's in (15) and (22) must be one, it is not difficult to see that after applying n steps to the seed solution (45) and (46) asymptotically we have:

$$2T\tau\phi^2 = \frac{1}{2} T\tau (g_{,z} g^{-1})^2 \sim 1 - \frac{2n}{\tau} + \mathcal{O}(\tau^{-2})$$

i.e., the number of steps equals the magnetic change.

Thus, to obtain the one monopole (1MP) we apply one step to ψ_0 . Since in the present case all $s_k = 1$, $\sum_k^n s_k$ is even or odd if n , i.e., the number of steps is even or odd. As we remarked earlier steps with $\sum_k^n s_k$ being odd change the sign of $\det g_0$, so for the 1MP we multiply g_0 with $C = \text{diag}(1, -1)$. It is not difficult to show that g in (22) (with all $s_k = 1$) would depend only on the ratios $M_1^{(k)} M_2^{(k)-1}$ therefore, without loss of generality we can take

$$M_a^{(k)} = \left(\exp[-f_k(t, \lambda)], \exp[f_k(t, \lambda)] \right) \quad (47)$$

In the axially symmetric case these f_k 's are constants. After a trivial calculation we obtain for Γ_{11} :

$$\Gamma_{11} = - \frac{s^2}{z+R} \frac{1}{R} \sinh(R-\alpha)$$

where $R = \sqrt{\rho^2 + z^2}$, $\alpha = f + \bar{f}$ and we assumed $w = 0$. A necessary condition for the regularity of the solution is the regularity of Γ_{11} which implies $\alpha = 0$. Then we get for $g_{(ph)}$:

$$g_{11} = \frac{e^z}{s} \left(\frac{R e^R}{\sinh R} - [z+R] \right) \quad g_{12} = g_{21} = \frac{R}{s \sinh R}$$

$$g_{22} = \frac{e^{-z}}{s} \left(z+R + \frac{R e^{-R}}{\sinh R} \right) \quad (48)$$

One can immediately see that $\det g = 1$. For any Hermitian 2×2 matrix with unit determinant we adopt the following parametrization:

$$g = \frac{1}{f} \begin{pmatrix} 1 & -\psi \\ -\bar{\psi} & f^2 + \psi\bar{\psi} \end{pmatrix}$$

For our g we get

$$f = \frac{s e^{-z}}{R \coth R - z}, \quad \psi = -f \frac{R}{s \sinh R} \quad (49)$$

The result we got using Bäcklund transformations⁵⁾ was somewhat different. One should apply the following Ehlers transformation to (49):

$$f' = \frac{2f}{(1-\psi)^2 + f^2}, \quad \psi' = 2 \frac{\psi - (f^2 + \psi^2)}{(1-\psi)^2 + f^2} + 1$$

to obtain the form of the LMP in Ref. 5).

To emphasize the importance of the representation of the Higgs vacuum, we present the following example: choosing $g = \text{diag}(\rho^k e^z, -\rho^{-k} e^{-z})$ we get for $\psi_0(\lambda)$:

$$\psi_0(\lambda) = \text{diag} \left([-4y\lambda \delta(\lambda)]^{k/2} e^{z-2\lambda y}, -[-4y\lambda \delta(\lambda)]^{-k/2} e^{-z+2\lambda y} \right)$$

The resulting new g after one step is a singular solution, quite different from the monopole.

We now proceed to construct the axially symmetric n monopole configurations. Using (46) and (47) we get for Γ_{ij} :

$$\Gamma_{ij} = (1 + \mu_i \bar{\mu}_j)^{-1} \left[\exp \left[-z - (f_i + \bar{f}_j) + \mu_i y + \bar{\mu}_j \bar{y} \right] + (-1)^n \exp \left[z - \mu_i y - \bar{\mu}_j \bar{y} + f_i + \bar{f}_j \right] \right] \quad (50)$$

where we multiplied g_0 with $C = \text{diag}(1, (-1)^n)$. The poles, μ_k 's are given by (43) with different $\tilde{w}_k$'s, i.e., they satisfy:

$$\prod_{k=1}^n (\delta - \tilde{w}_k) = 0 \quad (51)$$

For regularity we require Γ_{ij} stay finite when $\mu_i = -\bar{\mu}_j$; this - via (43) - in turn implies:

$$\exp[-\tilde{w}_i - (f_i + \bar{f}_j)] + (-1)^n \exp[\tilde{w}_i + f_i + \bar{f}_j] = 0 \quad \forall i, j \quad (52)$$

Equation (52) means that for all i $f_i = f$ and in this case the common $f + \bar{f}$ can be absorbed into the $\tilde{w}_i$'s. (In fact, it turns out that $f + \bar{f}$ determines the location of the superimposed monopoles on the symmetry axis.) To guarantee regularity we must ensure that $\det \Gamma$ is never zero, this seems to fix the $\tilde{w}_i$'s uniquely

$$\tilde{w}_i = i\pi \frac{n+1-2k}{2} - (f + \bar{f}) \quad (53)$$

The above result is in complete agreement with the one obtained earlier by us using Bäcklund transformations and by Prasad and Rossi using the Atiyah-Ward ansätze.

5.2 General SU(2) monopoles

As it was shown in Ref. 21) in SU(2) any n monopole solution belongs to a $4n-1$ parameter class, so the next step in our programme is to look for these more general configurations. It is essential to find the correct pole equation. As in the axially symmetric case $h(\gamma(\mu), \mu)$ was given by Eq. (51) it is natural to take again for h a polynomial of degree n in γ . The most general form of such an h is:

$$h = \sum_{i=0}^n a_i(\mu) \gamma^i \quad (54)$$

As we want to introduce free parameters by a "minimal deformation" of the axially symmetric $h(\gamma, \mu)$ we restrict the degree of the equation for the μ_k 's to $2n$. That is

$$a_i(\mu) = \sum_{j=0}^{n-i} (b_j \mu^j + c_j \mu^{-j}) \quad (55)$$

so h contains $2(n+1)^2 - 2$ free parameters. In Eq. (47) the f_k 's are not constants any more, however, the "minimal deformation" of the axially symmetric $M_a^{(k)}$'s is to assume that all the $M_a^{(k)}$'s are described by just one function $f_k(\gamma, \lambda) = f(\gamma, \lambda)$ $k = 1, \dots, n$. In addition, we make one last assumption which is quite natural, namely that $f(\gamma, \lambda)$ is analytic at $\lambda = 0$, i.e., $f(\gamma, \lambda) = \sum_{i=0}^{\infty} f_i(\lambda) (\lambda \gamma)^i$.

As in (22) and (25) only $\psi_0(\mu_k)$ and $M^k(\mu_k, \gamma(\mu_k))$'s are present. We effectively need the values of the arbitrary function, f , at the pole sites, μ_k 's. Since the algebraic equation defining the poles is of degree n in γ , we can express all γ^m 's when $m > n-1$ in terms of γ^i ($i = 0, \dots, n-1$) at the pole sites, μ_k . Consequently, one can uniquely associate with any power series in γ a polynomial of degree $n-1$ (in γ) yielding the same values at all pole sites, μ_k . Therefore, without any loss of generality we can take f as a polynomial of degree $n-1$ in γ .

Next - just in the axially symmetric case - we investigate what restrictions arise from requiring the non-singularity of Γ_{ij} when $\mu_i = -\bar{\mu}_j^{-1}$. Γ_{ij} in the general case has the form

$$\Gamma_{ij} = (1 + \mu_i \bar{\mu}_j)^{-1} [e^{\alpha_{ij}} + (-1)^n e^{-\alpha_{ij}}] \quad (56)$$

$$\alpha_{ij} = \delta(\mu_i) + \bar{\gamma}(\mu_i^{-1} + \bar{\mu}_j) - \bar{f}(\mu_i, \delta(\mu_i)) - \bar{f}(\mu_j, \delta(\mu_j))$$

The only formal difference compared to (50) is that now the f does depend on μ and γ . Γ_{ij} has no singularity at $\mu_i = -\bar{\mu}_j^{-1}$ if

$$\alpha(\mu_i, \delta(\mu_i)) = i\pi w_i^{(n)} \quad (57)$$

where $\alpha(\lambda, \gamma(\lambda)) = \gamma - f(\lambda, \gamma(\lambda)) - f^*$ with $f^* = \bar{f}(-\bar{\lambda}^{-1}, \gamma(-\bar{\lambda}^{-1}))$ and $w_i^{(n)}$ satisfy

$$e^{i\pi w_i^{(n)}} + (-1)^n e^{-i\pi w_i^{(n)}} = 0 \quad (58)$$

The next step is to compute α explicitly. Since α is a polynomial of degree $n-1$ in γ with its values given at n points [in Eq. (57)], it is determined by Lagrange's interpolation formula:

$$\alpha(\lambda, \delta(\lambda)) = i\pi \sum_{j=1}^n w_j^{(n)} \prod_{i \neq j} \frac{\delta(\lambda) - \delta_i(\lambda)}{\delta_j(\lambda) - \delta_i(\lambda)} \quad (59)$$

In Eq. (59) we wrote h , [(54) and (55)] as $\prod_{i=1}^n (\gamma - \gamma_i(\lambda))$. As $\alpha(\lambda, \gamma(\lambda)) = \bar{\alpha}(-\bar{\lambda}^{-1}, \gamma(-\bar{\lambda}^{-1}))$ by definition, this implies constraints for h , namely h must also satisfy: $h(\mu, \gamma) = h(-\bar{\mu}^{-1}, \gamma(-\bar{\mu}^{-1}))$.

This reduces the number of free b_i and c_i parameters in h to $(n+1)^2 - 1$. Furthermore, in (59) the $w_i^{(n)}$'s should satisfy $w_i^{(n)} = -w_j^{(n)}$ if $\gamma_i(\lambda) = \gamma_j^*(\lambda) = \bar{\gamma}_j(-\bar{\lambda}^{-1})$. We solve this requirement together with Eq. (58) by choosing $w_i^{(n)}$'s to be the same as in the axially symmetric case, Eq. (53) (with $f + \bar{f} = 0$ of course).

Assuming that the coefficient functions, α_i , in $\alpha = \sum_{i=0}^{n-1} \alpha_i(\lambda) \gamma^i$ are analytic in an annular domain we get for $f(\lambda, \gamma)$:

$$f(\lambda, \gamma) = \frac{1}{2\pi i} \oint_{|z| < |\lambda|} \frac{\alpha(z, \gamma)}{\lambda - z} dz + \gamma \quad (60)$$

Using (60) together with the fact that $f(f^*)$ is apolynomial in $\lambda\gamma$ (respectively $\gamma\lambda^{-1}$) we find further constraints for $\alpha_i(\lambda)$ (and therefore for b_i, c_i):

$$\oint \frac{\alpha_i(z)}{z^{j+1}} dz = 0 \quad j = -i+1, \dots, i-1; \quad 2 \leq i \leq n-1$$

$$\frac{1}{2\pi i} \oint \frac{\alpha_1(z)}{z} dz = 1 \quad (61)$$

Thus, the final number of free parameters we can introduce is $(n+1)^2 - 1 - \{\sum_{i=2}^{n-1} (2i-1) + 1\} = 4n-1$. Consequently, our solution depends on the right number of free parameters. It reduces to the axially symmetric case when $a_i(\mu)$'s are constants, independent of μ (uniquely determined by w_i 's). In the light of this, as well as noting that asymptotically $\mu_{i,z} = -r^{-1}\mu_i$, and thus that the asymptotic behaviour of $\text{Tr } \phi^2$ is the expected one, we claim that our solutions describe $SU(2)$ monopoles. Of course, to prove this rigorously would require the demonstration that $\det \Gamma \neq 0$ on the whole R^3 .

5.3 $SU(N)$ monopoles

The application of our method to the construction of $SU(N)$ monopoles is similar in spirit to what we have done for $SU(2)$, but more complicated in details, therefore here we only sketch the main steps.

In the $SU(N)$ case we choose a seed solution g_0 , corresponding to the Higgs vacuum $2 \text{Tr } \phi_0^2 = 2 \sum_{i=1}^N \Lambda_i^2$ ($\sum_{i=1}^N \Lambda_i = 0$), $F_{ij} = 0$, that is a natural extension of the g_0 we used for $N = 2$:

$$g_0 = \text{diag} (e^{2\Lambda_1 z}, \dots, e^{2\Lambda_N z}) \quad (62)$$

The solution of the linear equations (27) with this g_0 is

$$\psi_0(z) = \text{diag} \left(e^{2\lambda_1(z-\lambda_y)}, \dots, e^{2\lambda_N(z-\lambda_y)} \right) \quad (63)$$

The main difference between the $N = 2$ and the $N > 2$ cases arises from the possibility of having subspaces of dimensions $s_k > 1$ associated to the poles μ_k , for $N > 2$. As a consequence, the connection between the $N-1$ integers appearing in the asymptotic ϕ (6) and (7) and the number of steps taken becomes more complicated.

To exhibit this consider making one step on (62) and (63) with a one-dimensional projector P (24) where μ is given by (43). Assuming $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_N$ it is not difficult to see that $g_{(ph)}$, Eq. (25), asymptotically (i.e., $r \rightarrow \infty$) has the form:

$$g_{(ph)}^{as} = \left(\frac{\hat{\mu}}{s} \right)^{1/N} \left[I - \left(1 + \frac{s^2}{\hat{\mu}^2} \right) \delta_{aN} \delta_{bN} \right] g_0 \quad (64)$$

Equation (64) implies that asymptotically

$$2\text{Tr} \Phi^2 = 2 \sum_{i=1}^N \Lambda_i^2 - \frac{1}{r} \left\{ \sum_{i=1}^{N-1} \frac{2\Lambda_i}{N} + 2\Lambda_N \left(\frac{1}{N} - 2 \right) \right\} + O(r^{-2}) \quad (65)$$

Comparing this to (5) and (7) and using that $\Lambda_N = -\sum_{i=1}^{N-1} \Lambda_i$ we obtain $n_j = j \frac{2}{N}$, $j = 1, \dots, N-1$, i.e., all the n_j 's are fractionals. This is a surprising result as for monopoles we know that n_j must be integers. Repeating the calculation with an s ($\leq N-1$) dimensional projector one obtains $n_j = (2s/N)j$, $j = 1, \dots, N-s$; $n_{N-s+k} = 2(N-s)(s-k)/N$, $k = 0, \dots, s-1$; so in general these numbers are again fractionals.

Clearly to construct regular monopoles we must carefully choose the number of poles and the dimension of their associated projectors such that the resulting n_i 's be integers. The next step would be to determine to what extent the constant $M_{a^{(k,jk)}}^{(k,jk)}$ vectors and w_k are determined by imposing regularity. In a recent paper¹⁷⁾, for the $SU(3)$ case - using the superpotential - we successfully accomplished this programme as we constructed regular axially symmetric monopoles depending on a continuous free parameter by assuming two poles with two and one dimensional projectors, respectively.

6. - THE SOLITONIC NATURE OF MULTIMONOPOLES

As we emphasized in Chapter 3, the method we derived is closely related to that of producing solitons in certain two-dimensional theories. Therefore, it is interesting to investigate if the solutions we generate possess solitonic properties. (Here we use the notion "soliton" in a broad sense, implying solutions with well localized energy density.) Therefore, in the following, we inquire the energy density of the $SU(2)$ 2MP solution to see whether it has two well-localized maxima corresponding to the two constituent monopoles.

The length of the Higgs field for the separated 2MP has been investigated previously in three different ways: using the procedure given in Chapter 5.2, we computed it numerically and verified that it is free of singularities for a range of the free parameter⁶⁾; then using the Atiyah-Ward ansätze in Ref. 22), the regularity was proved analytically at least for restricted regions of R and finally using the ADHMN construction¹⁰⁾ in Ref. 23), an analytical expression for the Higgs field on the axis connecting its two zeros was given.

A motivation to study the energy density came from the observation⁵⁾ that for the axially symmetric 2MP the energy is no longer localized near the (single) zero of the Higgs field, but is contained in a torus-like region surrounding the symmetry axis. (See Fig. 1.) This configuration certainly has no similarity to the superposition of two peaks at the origin.

When applying the framework of Chapter 5.2 to the 2MP we take h as

$$h = \gamma^2 + A(\mu^{-2} + \mu^2) + B \quad (66)$$

where A, B are real constants. We note that by appropriate co-ordinate transformations h can always be reduced to this form. Equation (61) implies the following relation between A and B :

$$\sqrt{B} = \frac{1}{\sqrt{1+\beta}} K\left(\frac{2\beta}{1+\beta}\right), \quad A = \frac{1}{2}\beta B; \quad -1 < \beta \leq 0 \quad (67)$$

where $K(m)$ is a complete elliptic integral of the first kind with parameter m ²⁴⁾.

If h is given by (66) then for f we find

$$f(2, \gamma) = \frac{\gamma \lambda}{4i} \oint_{|z| < |\gamma|} \frac{dz}{z - \lambda} [A(\gamma^4 + 1) + B\gamma^2]^{-1/2} \quad (68)$$

For β negative the integrand in (68) has four branch points on the real axis. We deform the contour of integration in (68) to run from $-\infty$ to the smallest branch point and from the largest to ∞ . This way we get

$$f(\lambda, \delta) = \frac{\delta}{\tilde{\beta} \sqrt{-A}} \left[K(\tilde{\beta}^{-1}) - \pi(\lambda^2 \tilde{\beta}^{-2} \setminus \delta) \right] \quad (69)$$

$$\sin \delta = \tilde{\beta}^{-2} \quad \tilde{\beta} = \sqrt{-\beta^{-1} + \sqrt{\beta^{-2} - 1}}$$

and $\pi(n \setminus \delta)$ is the complete elliptic integrand of the third kind²⁴⁾. Thus, our solution contains essentially one free parameter β . When $\beta = 0$ then $A = 0$, $B = (\pi/2)^2$ and we obviously get back the axially symmetric solution with x_3 as symmetry axis. In Ref. 6) we found that up to our numerical accuracy the zeros of the Higgs field are on the x_1 axis at

$$x_0 = \pm \sqrt{-m} K(m) \quad m = \frac{2\beta}{1+\beta} \quad (70)$$

In Ref. 22) it was confirmed that in the β range we investigate ($-0.999 \leq \beta \leq 0$), Eq. (70) is indeed a very good approximation.

For $\beta = 0$ the maximal "ring" of energy density is on the $x_1 x_2$ plane thus we computed numerically the energy density

$$\mathcal{E} = \frac{1}{2} \Delta (\sum \phi_a^2) = \frac{1}{4} \Delta \text{Tr}(g_{,z} g^{-1})^2 \quad (71)$$

on the $x_1 x_2$ plane for various values of β using (65) and (66) as inputs in (22), (25) and (71). As even the separated 2MP has a D_{2h} symmetry²⁵⁾ we made the calculation only for $x_1, x_2 > 0$.

Our results are summarized in the table and on Figs. 1-4. [In the table α_0 is determined from (70).]

β	α_0	$x_{\max}$	$\mathcal{E}_{\max}$
0	0	2.07	0.08
-0.5	1.66	2.42	0.11
-0.99	4.02	4.17	0.19
-0.999	5.19	5.27	0.22

From the figures and the table we conclude that the larger the distance is between the monopoles the more pronounced is the peak in the energy density. The location of the maxima, $x_{\max}$ for large separation seems to coincide with the location of the zeros. We also point out that in the last line of the table $\delta_{\max}$ is practically identical to $\delta_{\max}$ of the 1MP: 2/9. Furthermore, the figures convincingly show the absence of any structure between the individual monopoles for separations ≥ 8 . We also investigated the x_3 dependence on $x_1 = \text{const.}$ planes and we found that δ falls off with increasing x . Based on this, we claim to have established the solitonic nature of the 2MP in the sense that δ for sufficiently well separated monopoles is really the sum of the energy density of the individual monopoles.

7. - CONCLUSIONS

In this paper we described in detail a framework capable of generating infinitely many new solutions - starting from an old one - of the self-duality (Bogomolny) equations of an $SU(N)$ gauge theory. We set up this framework by applying ideas - borrowed from two dimensional completely integrable systems - to the linearized version of the SD (Bogomolny) equations. As a result we obtained an explicit expression for the new solution in terms of the starting one plus some free parameters. In addition, we connected the solutions of these linear equations to the solutions of certain matrix Riemann-Hilbert problems (RHP) and pointed out that in this context our method amounts to solving the RHP with zeros only. By reformulating our method in terms of the RHP we were able to discuss its relation to the Atiyah-Ward ansätze.

To emphasize the usefulness of our method we applied it to the construction of various multimonopoles. We fixed the starting solution of the Bogomolny equation to describe a "Higgs vacuum", i.e., a solution with constant scalar field and vanishing energy density all over R^3 . Proceeding with our method we determined what choice of the free parameters in the expression of the new solution guarantees the regularity and the expected asymptotic behaviour of the multimonopoles. This way, we were able to construct all $SU(2)$ monopoles and set the stage for $SU(N)$ monopoles as well.

Motivated by the apparent lack of any peak in the energy density of the axially symmetric $SU(2)$ 2MP we investigated the energy density of the separated 2MP. From our numerical study we concluded that the sufficiently well-separated 2MP can really be thought of as the superposition of two individual monopoles, as for separations > 8 (in units of $\sum_a \phi_a^2$) we found only two peaks in the energy density.

APPENDIX

In this Appendix we give the formulas needed to carry out the splitting in our framework of a G having the form:

$$G = \begin{pmatrix} \lambda^\ell & 0 \\ g & \lambda^{-\ell} \end{pmatrix} \quad s = s(\lambda, -\lambda y + \bar{z}, \lambda z + \bar{y}) \quad \lambda \in \Gamma \quad (A.1)$$

into

$$A(\lambda) B(\lambda) = G \quad \lambda \in \Gamma \quad (A.2)$$

where $A(\lambda)$ ($B(\lambda)$) is analytic inside (respectively outside) Γ . We also assume $\det A = \det B = 1$. As we mentioned the method is identical to that of Ref. 16) the only difference comes from the boundary conditions at $\lambda = \infty$.

Parametrizing A and B as

$$A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from (A.2) we obtain

$$\alpha = d \lambda^\ell ; \quad b = -\beta \lambda^{-\ell} \quad (A.3)$$

$$a = \delta \lambda^\ell - \beta g ; \quad c = \alpha g - \delta \lambda^\ell \quad (A.4)$$

$$\gamma = d g - c \lambda^{-\ell} ; \quad \delta = a \lambda^{-\ell} - b g \quad (A.5)$$

As $\alpha(\beta)$ is analytic inside Γ and $d(b)$ is analytic outside Γ , from (A.3) we conclude that α and β can be polynomials of degrees at most ℓ in λ :

$$\alpha = \sum_{r=0}^{\ell} \alpha_r \lambda^r ; \quad \beta = \sum_{r=0}^{\ell} \beta_r \lambda^r \quad (A.6)$$

This - together with (A-4) - implies that the coefficients of λ^r in the Laurent expansion of βg vanish for $0 < r < \ell$:

$$\oint \frac{d\lambda}{\lambda^{r+1}} \beta g = 0 = \oint \frac{d\lambda}{\lambda^{r+1}} \alpha g \quad (A.7)$$

Introducing

$$\Delta_m = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda} s \lambda^m$$

(A.7) can be written as

$$\sum_{t=0}^{\ell} \beta_t \Delta_{t-\tau} = \sum_{t=0}^{\ell} \alpha_t \Delta_{t-\tau} = 0 \quad 0 < \tau < \ell \quad (\text{A.8})$$

The functions a , c , γ and δ are expressed in terms of α , β (and $d = \lambda^{-\ell} \alpha$, $b = -\beta \lambda^{-\ell}$) as

$$\begin{aligned} a(\lambda) &= -\frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \frac{\lambda \beta s(\xi)}{\lambda - \xi} & \gamma(\lambda) &= \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi - \lambda} d s(\xi) \\ & & & | \lambda | > 1 & & | \lambda | < 1 \\ c(\lambda) &= \frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi} \frac{\lambda \alpha s(\xi)}{\lambda - \xi} & \delta(\lambda) &= -\frac{1}{2\pi i} \oint_{|\xi|=1} \frac{d\xi}{\xi - \lambda} b s(\xi) \end{aligned} \quad (\text{A.9})$$

Thus the A , B matrices are given in terms of the $2\ell + 2$ functions α_r, β_r . Equation (A.8) represents $2\ell - 2$ conditions for these functions, thus four of them remain free. These four functions are fixed by requiring

$$B M_2 \xrightarrow{\lambda \rightarrow \infty} I \quad (\text{A.10})$$

where M is the matrix which transforms G into $\tilde{G}$ ($M_1 G M_2 = \tilde{G}$) satisfying $G(\lambda) = \tilde{G}^+(-\bar{\lambda}^{-1})$. It is obvious that if $\det M = 1$ then (A.10) guarantees $\det A = \det B = 1$.

Next, as an illustration, we show how the LMP (48) emerges from this splitting procedure. In the light of the relation we pointed out between our RHP and the Atiyah-Ward ansätze it is natural to take $\ell = 1$ and the following ρ :

$$\rho = e^{ix_0} \frac{\exp[(x_1 + ix_2)\lambda - x_3] - \exp[x_3 + (x_1 - ix_2)\lambda^{-1}]}{(x_1 + ix_2)\lambda - 2x_3 - (x_1 - ix_2)\lambda^{-1}} \quad (\text{A.11})$$

In this case we may choose the following M_1 and M_2 matrices:

$$M_1 = \begin{pmatrix} 0 & \exp\left[-\frac{1}{2}([x_1 + ix_2]\lambda + ix_0 - x_3)\right] \\ -\exp\left[\frac{1}{2}(ix_0 - x_3 + [x_1 + ix_2]\lambda)\right] & 2\lambda \gamma \exp\left[-\frac{1}{2}(ix_0 - x_3 + [x_1 + ix_2]\lambda)\right] \end{pmatrix} \quad (A.12)$$

$$M_2 = \text{diag} \left(\exp\left[-\frac{1}{2}\{(x_1 - ix_2)\lambda^{-1} + ix_0 + x_3\}\right], \exp\left[\frac{1}{2}\{(x_1 - ix_2)\lambda^{-1} + ix_0 + x_3\}\right] \right)$$

$$\gamma = (x_1 + ix_2)\lambda - 2x_3 - (x_1 - ix_2)\lambda^{-1}$$

In the $\ell = 1$ case, Eq. (A.8) gives no restriction, thus once we expressed

$$\begin{aligned} \alpha &= \alpha_0 + \alpha_1 \lambda & b &= b_0 + b_1 \lambda^{-1} \\ d &= \alpha_0 \lambda^{-1} + \alpha_1 & \beta &= -b_0 \lambda - b_1 \end{aligned} \quad (A.13)$$

in terms of $\alpha_0, \alpha_1, b_0, b_1$ from (A.9) we immediately obtain the other four functions:

$$\begin{aligned} a &= -\frac{e^{ix_0}}{x_1 + ix_2} (b_0 R_2 + b_1 R_1) & \gamma &= -\frac{e^{ix_0}}{x_1 + ix_2} (\alpha_0 T_2 + \alpha_1 T_1) \\ c &= -\frac{e^{ix_0}}{x_1 + ix_2} (\alpha_1 R_2 + \alpha_0 R_1) & \delta &= \frac{e^{ix_0}}{x_1 + ix_2} (b_1 T_2 + b_0 T_1) \end{aligned} \quad (A.14)$$

where

$$\begin{aligned} R_1 &= \frac{\lambda}{\xi_2 - \xi_1} \left(\frac{e^{-\tau}}{\lambda - \xi_1} + \frac{e^{\tau}}{\xi_2 - \lambda} \right) - \frac{\lambda \exp\{x_3 + \lambda^{-1}(x_1 - ix_2)\}}{(\xi_2 - \lambda)(\lambda - \xi_1)} \\ R_2 &= \frac{\lambda}{\xi_2 - \xi_1} \left(\frac{\xi_1 e^{-\tau}}{\lambda - \xi_1} - \frac{\xi_2 e^{\tau}}{\lambda - \xi_2} \right) + \frac{\lambda^2 \exp\{x_3 + \lambda^{-1}(x_1 - ix_2)\}}{(\lambda - \xi_2)(\lambda - \xi_1)} \\ T_1 &= \frac{1}{\xi_2 - \xi_1} \left(\frac{\xi_1 e^{-\tau}}{\xi_1 - \lambda} - \frac{\xi_2 e^{\tau}}{\xi_2 - \lambda} \right) - \frac{\lambda \exp\{-x_3 + \lambda(x_1 + ix_2)\}}{(\xi_1 - \lambda)(\lambda - \xi_2)} \\ T_2 &= \frac{1}{\xi_2 - \xi_1} \left(\frac{e^{-\tau}}{\xi_1 - \lambda} - \frac{e^{\tau}}{\xi_2 - \lambda} \right) - \frac{\exp\{\lambda(x_1 + ix_2) - x_3\}}{(\xi_1 - \lambda)(\xi_2 - \lambda)} \end{aligned} \quad (A.15)$$

and

$$\xi_{1,2} = \frac{x_3 \mp \tau}{x_1 + ix_2}$$

Using (A.12)-(A.15) in (A.10) we get a linear system of equations for $\alpha_0, \alpha_1, b_0, b_1$, its solution is

$$\begin{aligned} b_0 &= 0 \quad \alpha_0 = - \frac{e^{-\frac{i}{2}x_0}}{e^{-\tau} - e^{\tau}} \left\{ [\xi_1 e^{-\tau} - \xi_2 e^{\tau}] e^{-\frac{x_3}{2}} + (\xi_2 - \xi_1) e^{\frac{x_3}{2}} \right\} \\ \alpha_1 &= e^{-\frac{1}{2}(ix_0 + x_3)} \quad b_1 = - \frac{e^{-\frac{i}{2}x_0 + \frac{1}{2}x_3}}{e^{-\tau} - e^{\tau}} (\xi_2 - \xi_1)(x_1 + ix_2) \end{aligned} \quad (A.16)$$

This then completes the splitting of the G having the ρ given in Eq. (A.11).

To obtain the corresponding solution of Eq. (9) we have to determine $(M_1(0)A(0))^{-1}$. A straightforward calculation yields:

$$(M_1(0)A(0))^{-1} = \frac{\tau}{\sinh \tau} \begin{pmatrix} M & \frac{1}{x_1 - ix_2} (M-1) \\ \frac{1}{x_1 + ix_2} (M-1) & \frac{1}{g^2} (M+N-2) \end{pmatrix} \quad (A.17)$$

where

$$M = \cosh \tau - \frac{x_3}{\tau} \sinh \tau, \quad N = \cosh \tau + \frac{x_3}{\tau} \sinh \tau$$

Equation (A.17) is similar, but not identical, to (48). The main difference is that (A.17) still depends on the azimuthal angle φ . We can get rid of this dependence by carrying out an invariance transformation on (A.17). It is easy to see that if g solves (9) then $\tilde{g} = \Omega(y, z)g\Omega^+$ (with $|\det \Omega| = 1$) is also a (Hermitian) solution of Eq. (9) with $\det \tilde{g} = 1$. Therefore, choosing Ω as $\Omega = \text{diag} ((x_1 + ix_2)^{-1/2}, (x_1 + ix_2)^{1/2})$ from (A.17) we obtain:

$$\tilde{g} = \frac{\tau}{g \sinh \tau} \begin{pmatrix} M & M-1 \\ M-1 & M+N-2 \end{pmatrix} \quad (A.18)$$

Finally, carrying out one more Ω transformation on this $\tilde{g}$, but this time with a constant, triangular Ω : $\Omega = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ one really arrives at (48).

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FIGURE CAPTIONS

- Fig. 1 : Axonometric view of the energy density on the x_1x_2 plane of the 2MP with $\beta = 0$.
- Fig. 2 : Same as Fig. 1 with $\beta = -0.5$.
- Fig. 3 : Same as Fig. 1 with $\beta = -0.99$.
- Fig. 4 : Same as Fig. 1 with $\beta = -0.999$.

ENERGY

Fig. 1

ENERGY

Fig. 2

ENERGY

Fig. 3

ENERGY

Fig. 4