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THE ABELIAN MONOPOLE FERMION SYSTEM AND ITS
SELF-ADJOINT EXTENSIONS AS A LIMIT OF A
NON-ABELIAN SYSTEM

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A B S T R A C T

We construct a non-Abelian $SU(2)$ monopole fermion system which, in the limit of vanishing monopole size and infinite mass for one of the charges, reproduces the Abelian monopole fermion system for a single charge with an arbitrary mass matrix and the boundary condition at the origin which renders the Hamiltonian Hermitian.

It has been pointed out that the fermion Hamiltonian in the presence of an Abelian (Dirac) monopole is not Hermitian when acting on the full function space on which it is naturally defined^{1,2}. This difficulty is related to the singularity of the potential at the location of the monopole. However if the Hilbert space is chosen to consist of wave functions obeying a particular boundary condition at the origin then the Hamiltonian is Hermitian². The choice of Hilbert space along with the Hamiltonian is called a self-adjoint extension and quantum physics can be consistently defined. Recently^{2,3,4}, it has been stressed that the family of such extensions, for the Hamiltonian under consideration, should be labeled by a free angle θ and that the extended system is endowed with the following properties. First, for $\theta \neq 0, \pi$ the extensions are CP non-invariant. Second, the fermion wave functions have a non trivial θ dependence and a monopole-fermion bound state exists for $\cos\theta < 0$. Third^{3,4}, for fermions of charge e and mass M , the fermion vacuum carries an electric charge $-e\theta/2\pi$ spread out over a distance of order $1/M$ around the monopole. It has been claimed that this effect is related to charge fractionalization of the dyon in the presence of an $\tilde{F}\tilde{F}$ term in the Lagrangian⁵.

If monopoles exist, it is widely believed that they must have arisen from the spontaneous breakdown of a non-Abelian gauge symmetry⁶. The size of such monopoles is on the order of the inverse of the symmetry breaking scale and thus very small. The field of the non-Abelian monopole is everywhere regular and the Hamiltonian is Hermitian. No extension is required to define the quantum mechanics problem. As the symmetry breaking mass scale is

increased the field of the non-Abelian monopole tends towards the field of the Dirac monopole. The question that arises is how the self-adjoint extended Abelian system can emerge as the limit of a non-Abelian system. In the remainder of our letter we shall first recall the salient features of the Abelian problem and then construct the desired non-Abelian system.

The Dirac equation for a massive fermion of charge e in the presence of an Abelian monopole field A_k is given by⁷

$$i \frac{\partial}{\partial t} \chi(\vec{x}, t) = H_D \chi(\vec{x}, t) \quad (1)$$

$$H_D = \alpha_k (-i \partial_k - e A_k) + \beta M e^{i \omega \gamma_5}$$

In a partial wave expansion⁸, it is found that H_D is non-Hermitian only when acting on the lowest ($j=0$) wave^{1,2} and from now on we will restrict our attention to this mode. In the $j=0$ sector equation (1) reduces to

$$H_D \chi = E \chi$$

where

$$H_D = -i \frac{q}{|q|} \gamma_5 \frac{d}{dr} + \beta M e^{i \omega \gamma_5} \quad (2)$$

$$\chi(r) = \begin{pmatrix} F(r) \\ G(r) \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $q = eg$, the product of the fermion electric charge and the magnetic charge. The wave function χ can be written as¹

$$\chi(\vec{x}, t) = \frac{\chi(r)}{r} \otimes \eta_m(\theta, \phi) e^{-iEt} + \text{higher partial waves} \quad (3)$$

where η_m is the appropriate two component monopole harmonic. The condition that H_D be Hermitian is

$$(H_D \chi_1, \chi_2) = (\chi_1, H_D \chi_2) \quad (4)$$

and this condition is most generally obtained by requiring

$$\frac{F(0)}{G(0)} = i \tan\left(\frac{\pi}{4} + \frac{\Theta}{2}\right) \quad (5)$$

where Θ is real but arbitrary. Condition (5) is equivalent to the bag boundary condition⁹:

$$i \not{n} \chi(0) = e^{-i\gamma_5 \Theta} \chi(0) \text{ with } n^0 = 0, n^1 = -1 \quad (6)$$

Physically condition (6) means that no current flows into the origin.

The wave functions obtained by solving equation (2) for $q > 0$, satisfying the boundary condition (5) are:

$$F_{\Theta, \omega}(r) = iN \left[\sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) \cos kr + \left\{ \frac{E}{k} \cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) + \frac{M}{k} \cos\left(\frac{\Theta}{2} + \frac{\pi}{4} + \omega\right) \right\} \times \sin kr \right] \quad (7a)$$

$$G_{\Theta, \omega}(r) = N \left[\cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) \cos kr + \left\{ -\frac{E}{k} \sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) + \frac{M}{k} \sin\left(\frac{\Theta}{2} + \frac{\pi}{4} + \omega\right) \right\} \times \sin kr \right] \quad (7b)$$

where N is the normalization and $E = \pm \sqrt{k^2 + M^2}$. There is a bound state with energy $E = M \sin(\Theta + \omega)$ provided $\cos(\Theta + \omega) < 0$:

$$F_{\Theta, \omega}^B = iN^B \sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) e^{-\kappa r} \quad (8)$$

$$G_{\Theta, \omega}^B = N^B \cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) e^{-\kappa r}$$

with $\kappa = -M \cos(\Theta + \omega)$.

We now turn to the non-Abelian monopole. We first consider an isodoublet fermion in the presence of the 't Hooft-Polyakov⁶ SU(2) monopole. The Dirac equation is

$$i \frac{\partial}{\partial t} \chi = H \chi \quad (9)$$

where

$$H = \alpha_k \left(-i \partial_k - e A_k^a T^a \right) + \beta M e^{i\omega \gamma_5}$$

and $\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$.

Now χ can be written in terms of its upper and lower components as

$$\chi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} \quad (10)$$

where $\chi_{im}^\pm$ carry both a spin index $i=1,2$ and a group index $m=1,2$.

Thus $\chi^\pm$ can be regarded as 2×2 matrices. If we restrict χ to be in the $j=0$ mode then the most general form for the matrices $\chi^\pm$ is

$$\chi^\pm(\vec{x}, t) = \frac{1}{r} \left(G^\pm(r, t) + P^\pm(r, t) \sigma^{\hat{a}} \hat{r}^{\hat{a}} \right) \sigma^2 \quad (11)$$

Let us define electric charge projection operators

$$Q_\pm = \frac{1}{2} \pm \frac{1}{2} \sigma^{\hat{a}} \hat{r}^{\hat{a}} \quad (12)$$

which certainly are the proper projectors far from the monopole core. With the help of these operators it is easy to find wave functions which look like + and - charge states far from the monopole core.

$$\chi_{+charge}^\pm = \frac{1}{2r} \left[(G^\pm - P^\pm) + (P^\pm - G^\pm) \sigma^{\hat{a}} \hat{r}^{\hat{a}} \right] \sigma^2 \quad (13)$$

$$\chi_{-charge}^\pm = \frac{1}{2r} \left[(G^\pm + P^\pm) + (P^\pm + G^\pm) \sigma^{\hat{a}} \hat{r}^{\hat{a}} \right] \sigma^2$$

We will now solve equation (9) for the $j=0$ fermion (11). We will work in the Prasad-Sommerfield limit¹⁰ where A_k^a takes the simple form, for a magnetic monopole of negative charge

$$A_k^a = \epsilon_{aki} \frac{r^i}{r^2} [1 - K(r)] \quad (14)$$

and

$$K(r) = \frac{ar}{\sinh ar}$$

with a being the mass scale of the monopole. The Dirac equation (9) with $\omega=0$ is now

$$\begin{aligned}
 i(E \mp M) G^\pm &= \frac{dP^\pm}{dr} + \frac{K}{r} P^\pm \\
 i(E \mp M) P^\pm &= \frac{dG^\pm}{dr} - \frac{K}{r} G^\pm
 \end{aligned}
 \tag{15}$$

The regular solutions to these equations are readily obtained:

$$\begin{aligned}
 G_{\omega=0}^\pm &= N^\pm \left[\tanh \frac{ar}{2} \cos kr + \frac{2k}{a} \sin kr \right] \\
 P_{\omega=0}^\pm &= iN^\pm \frac{k}{E \mp M} \left[\frac{1}{\tanh \frac{ar}{2}} \sin kr - \frac{2k}{a} \cos kr \right]
 \end{aligned}
 \tag{16}$$

$N^\pm$ specify the two independent solutions and all of these solutions are valid for positive and negative energies $E = \pm \sqrt{k^2 + M^2}$. For $a \gg M$ no bound states occur.

The $\omega \neq 0$ solutions are just chiral rotations of the $\omega=0$ solutions and can be written as

$$\begin{aligned}
 \begin{pmatrix} G_\omega^+ \\ G_\omega^- \end{pmatrix} &= \begin{pmatrix} \cos \frac{\omega}{2} & -i \sin \frac{\omega}{2} \\ -i \sin \frac{\omega}{2} & \cos \frac{\omega}{2} \end{pmatrix} \begin{pmatrix} G_{\omega=0}^+ \\ G_{\omega=0}^- \end{pmatrix} \\
 \begin{pmatrix} P_\omega^+ \\ P_\omega^- \end{pmatrix} &= \begin{pmatrix} \cos \frac{\omega}{2} & -i \sin \frac{\omega}{2} \\ -i \sin \frac{\omega}{2} & \cos \frac{\omega}{2} \end{pmatrix} \begin{pmatrix} P_{\omega=0}^+ \\ P_{\omega=0}^- \end{pmatrix}
 \end{aligned}
 \tag{17}$$

We will now argue that in the $j=0$ sector of this theory the + and - electric charge states can never be isolated. First it is easy to see that the solutions (16) make equal contributions to the + and - charge states (13), indicating that the energy eigenstates are of mixed charge. In fact the charge operator $Q = \frac{1}{2} \sigma^a \hat{r}^a$ does not commute with the Hamiltonian H . The commutator vanishes far from the monopole but nearby is not zero. Since $j=0$ modes do not vanish at the monopole core and since inside the monopole electric charge is not defined, we are not

surprised that the fermionic electric charge is not a good quantum number. In order to obtain the Abelian monopole with a single fermion charge as a limit of a non-Abelian system the Hamiltonian (9) will not do.

A single charge fermion theory is constructed by giving the + and - charge states different masses, M_+ and M_- , and letting one of the masses (say M_+) become very heavy while keeping the other light. The limiting Abelian theory of interest is that describing the - charge. It is an accurate description of this charge provided that the monopole mass is much heavier than M_+ which in turn is much greater than M_- as well as the energies of the scattering states entering the problem. A simple generalization of (9) allows us to incorporate these features

$$H = \alpha_k \left(-i\partial_k - eA_k^a T^a \right) + \beta_{M_+} e^{i\omega_+ \gamma_5} \left(\frac{1}{2} + \frac{1}{2} \hat{r}^a \sigma^a \Phi(r) \right) + \beta_{M_-} e^{i\omega_- \gamma_5} \left(\frac{1}{2} - \frac{1}{2} \hat{r}^a \sigma^a \Phi(r) \right) \quad (18)$$

Here $\Phi(r)$ is a function which approaches unity well outside the monopole and vanishes at the origin. It may be taken to be the radial Higgs field of the monopole.

We seek the solutions to $H\chi = E\chi$ in the limit $a \gg M_+ \gg M_-$, E . Unfortunately these solutions can not be presented analytically, however we can make an approximation which maintains all of the essential physics of the monopole system and makes the solutions tractable. Let r_0 be the monopole "radius", some arbitrary distance of order $\frac{1}{a}$. Let

$$\begin{aligned} \Phi(r) &= 0 & r < r_0 \\ &= 1 & r > r_0 \\ K(r) &= \frac{ar}{\sinh ar} & r < r_0 \\ &= 0 & r > r_0 \end{aligned} \quad (19)$$

The equation $H\chi = E\chi$ is now easily solved, first for $r < r_0$ then for $r > r_0$, and the full solution is obtained by matching at the boundary $r = r_0$.

Inside the monopole, $r < r_0$, the equation is just (9) with a mass term

$$M_\Omega e^{i\Omega\gamma_5} = \frac{1}{2}M_+ e^{i\omega_+\gamma_5} + \frac{1}{2}M_- e^{i\omega_-\gamma_5} \quad (20)$$

In the limit $M_+ \gg M_-$, $M_\Omega \approx \frac{M_+}{2}$ and $\Omega \approx \omega_+$. In the solutions (16) the wave number k is imaginary but the solutions are still regular inside the monopole. For $a \gg M_+ \gg E, M_-$ we obtain

$$\begin{aligned} G^+ &\approx \left(N^+ \cos \frac{\Omega}{2} \quad -iN^- \sin \frac{\Omega}{2} \right) \tanh \frac{ar}{2} \\ G^- &\approx \left(N^- \cos \frac{\Omega}{2} \quad -iN^+ \sin \frac{\Omega}{2} \right) \tanh \frac{ar}{2} \\ P^\pm &\approx 0 \end{aligned} \quad (21)$$

and for all solution, i.e. values of $N^\pm$, we get the relation at the boundary

$$\left. \frac{P^+ - G^+}{P^- - G^-} \right|_{r_0} = \left. \frac{P^+ + G^+}{P^- + G^-} \right|_{r_0} \quad (22)$$

On the outside, $r > r_0$, the equations decouple into the Abelian equations for a + and - charge.

$$E \begin{pmatrix} P^+ - G^+ \\ P^- - G^- \end{pmatrix} = \left(+i\gamma_5 \frac{d}{dr} + \beta M_+ e^{i\omega_+\gamma_5} \right) \begin{pmatrix} P^+ - G^+ \\ P^- - G^- \end{pmatrix} \quad (23a)$$

$$\text{and } E \begin{pmatrix} P^+ + G^+ \\ P^- + G^- \end{pmatrix} = \left(-i\gamma_5 \frac{d}{dr} + \beta M_- e^{i\omega_-\gamma_5} \right) \begin{pmatrix} P^+ + G^+ \\ P^- + G^- \end{pmatrix} \quad (23b)$$

These equations are to be compared with (2), keeping in mind that our magnetic charge is negative. For the + charge the energy E is less than the mass M_+ so the wave number is imaginary and there are

exponentially growing and exponentially decaying solutions.

Requiring that the exponentially growing solutions vanish and making use of our limits we obtain:

$$\begin{pmatrix} P^+ - G^+ \\ P^- - G^- \end{pmatrix} = N \begin{pmatrix} \sin\left(\frac{\pi}{4} - \frac{\omega_+}{2}\right) \\ -i \cos\left(\frac{\pi}{4} - \frac{\omega_+}{2}\right) \end{pmatrix} e^{-M_+ r} \quad (24)$$

At the boundary we then get the condition

$$\left. \frac{P^+ - G^+}{P^- - G^-} \right|_{r_0} = i \tan\left(\frac{\pi}{4} - \frac{\omega_+}{2}\right) \quad (25)$$

The - charge is governed by the equation (23b) outside the monopole. This is just the equation for a - charge in the field of an Abelian monopole of negative magnetic charge. Matching the boundary conditions (25) and (22) we get

$$\left. \frac{P^+ + G^+}{P^- + G^-} \right|_{r_0} = i \tan\left(\frac{\pi}{4} - \frac{\omega_+}{2}\right) \quad (26)$$

which is just the Abelian boundary condition (5) with $\Theta = -\omega_+$.

The solutions are given by equations (7) with $\Theta = -\omega_+$ and $\omega = \omega_-$.

The bound state energy is $M_+ \sin(\omega_- - \omega_+)$ provided $\cos(\omega_- - \omega_+) < 0$.

Physically our results suggest the following conclusions.

Letting the masses of the monopole and + charge fermion go to infinity an Abelian monopole fermion system is recovered. The heavy charge sector decouples except for the residual chiral angle ω_+ which is reflected in a boundary condition that renders the Abelian system automatically self-adjoint. For the quantum mechanics problem the physically relevant parameter is $\omega_+ - \omega_-$, the chiral angle which can not be rotated away in the Hamiltonian (18). The angle $\omega_+ + \omega_-$ could be physically relevant when the fermions are

second quantized due to anomalies. The study of these aspects, as well as the question of the electric charge carried by the fermion vacuum, are postponed to a further publication.

Finally, we would like to examine how the limit $M_{\text{GUT}} \rightarrow \infty$ in SU(5) produces a boundary condition on the fermi fields when they interact with the stable monopole of that theory. That monopole lives in an SU(2) subspace and for one family, it interacts with four lefthanded SU(2) doublets

$$\begin{aligned} \chi_{1L} &= \begin{pmatrix} e^- \\ -\bar{d}^3 \end{pmatrix}_L & \chi_{2L} &= \begin{pmatrix} \bar{d}^3 \\ e^+ \end{pmatrix}_L \\ \chi_{3L} &= \begin{pmatrix} \bar{u}^1 \\ u^2 \end{pmatrix}_L & \chi_{4L} &= \begin{pmatrix} \bar{u}^2 \\ -u^1 \end{pmatrix}_L. \end{aligned} \quad (27)$$

The fermion masses in SU(5) are generated by a Higgs $\underline{5}$ whose expectation value is left invariant under the SU(2) of the monopole. Thus we should consider the fermion masses independent of the monopole fields. The coupling of these fermion doublets is most easily expressed in terms of the Dirac spinors

$$\chi_A = \begin{pmatrix} e^- \\ -\bar{d}^3 \end{pmatrix} \quad \chi_B = \begin{pmatrix} \bar{u}^1 \\ u^2 \end{pmatrix} \quad (28)$$

and can be written as

$$\bar{\chi}_A \gamma^\mu T^a \chi_A A_\mu^a + \bar{\chi}_B \gamma^\mu T^a \chi_B A_\mu^a + m_A \bar{\chi}_A \chi_A + m_B \bar{\chi}_B \chi_B. \quad (29)$$

This system consists of two independent copies of the system described by equation (9). The eigenfunctions of the corresponding Hamiltonian are given by eq. (16) and we notice that for $r < r_0$, and for $M_{\text{GUT}} = a \gg E, M_A$ and M_B we have $P^\pm \approx 0$. Thus if one wishes to consider the fermi fields outside the monopole in the Abelian limit, the condition $P^\pm(0) = 0$ should be imposed as has previously been noticed¹¹.

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References and Footnotes

- 1) Y. Kazama, C.N. Yang, A.S. Goldhaber, Phys. Rev. D15 (1977) 2287.
- 2) A.S. Goldhaber, Phys. Rev. D16 (1977) 1815.
- C. Callias, ibid. 3068.
- 3) B. Grossman, Phys. Rev. Lett. 50 (1983) 440.
- 4) H. Yamagishi, Princeton University preprint (1982).
- 5) E. Witten, Phys. Lett. 86B (1979) 283.
- 6) G.'t Hooft, Nucl. Phys. B79 (1974) 276.
- A.M. Polyakov, JETP Lett. 20 (1974) 194.
- 7) The magnetic potential and the wave functions can be described in a formalism in which strings do not appear. (See 8)
- 8) T.T. Wu and C.N. Yang, Nucl. Phys. B107 (1976) 365.
- 9) R.L. Jaffe, Pointlike Structures Inside and Outside Hadrons (ed., A. A. Zichichi, Plenum Press, N.Y., 1982).
- 10) M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.
- 11) C.G. Callan Jr., Phys. Rev. D25 (1982) 2141.